

A Comparative Study of Analytical Solutions to the Coupled Van-der-Pol's Non-linear Circuits Using the He's Method (HPEM) and (BPES)

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Abstract

In this paper, the He's parameter-expanding method (HPEM) and the 4q-Boubaker Polynomials Expansion Scheme (BPES) are used in order to obtain analytical solutions to the non-linear modified Van der Pol's oscillating circuit equation. The resolution protocols are applied to the ordinary Van der Pol equation, which annexed to conjoint delayed feedback and delay-related damping terms. The results are plotted, and compared with exact solutions proposed elsewhere, in order to evaluate accuracy.

Keywords: Van-der-Pol's Oscillating Circuit, Delayed Feedback, Damping, BPES, HPEM, Exact Solutions, Electrical Triode-Valve Circuit

1. Introduction

Originally, the Van der Pol's equation was associated, in the 1920s, with an electrical triode-valve circuit (Figure 1). In the last decades' literature, it was the subject of several investigations due to the panoply of dynamical oddness as relaxation oscillations, elementary bifurcations, quasiperiodicity, and chaos. Its application has already reached nerve pulse propagation and electric potential evolution across neural membranes.

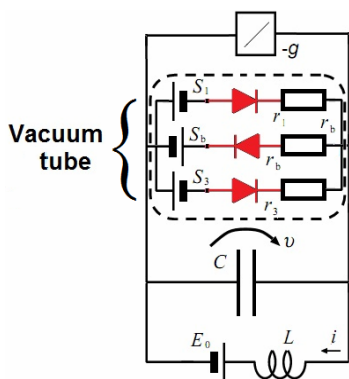


Figure 1. Van der Pol oscillator synoptic scheme.

The actual study tries to give a theoretical supply to the recent attempts to yield analytical solutions to this equation, like the studies of D. D. Ganji *et al.* [1,2] and A. Rajabi *et al.* [3] in the heat transfer domain, the investigations of L. Cveticanin [4] and J. H. He [5-7] on non-linear mechanics, fluid dynamics and oscillating systems modelling (Figure 2).

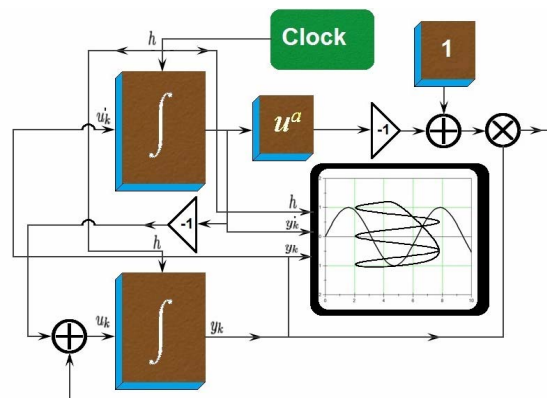


Figure 2. A prototype of Van der Pol oscillating systems modelling (The two integrators are Trapezoidal-type $\{y_k = h/2(u_k + u_{k-1}) + y_{k-1}\}$).

Among the different formulations, the well-known standard boundary value-free Van der Pol oscillator problem (BVFP) is given by F. M. Atay [8] by the following system (1):

$$\begin{cases} \dot{x}''(t) + x(t) = \varepsilon \times f(x(t), x'(t), x(t-\tau)) \\ f(x(t), x'(t), x(t-\tau)) = (1-x(t)^2)x'(t) + k \times x(t-\tau) \end{cases} \quad (1)$$

where τ is a positive parameter representing the delay, $\varepsilon > 0$ and k is the feedback gain.

A simpler formulation is that of W. Jiang *et al.* [9]:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -x(t) + kf(x(t-\tau)) - \varepsilon \times (x(t)^2 - 1)y(t) \end{cases} \quad (2)$$

In this study, an attempt to give analytical solution to the nonlinear second-order Van der Pol equation annexed to conjoint delayed feedback and delay-related damping terms as presented by A. Kimiaefar *et al.* [10]:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -x(t) - \varepsilon \times (1-x(t)^2)y(t) + k \times x(t-\tau) \\ x(0) = x_0 = H \neq 0 \\ x'(0) = x'_0 = 0 \end{cases} \quad (3)$$

2. Analytical Solutions Derivation

2.1. The Enhanced He's Parameter-Expanding Method (HPEM) Solution

The resolution protocol based on the enhanced He's parameter-expanding method (HPEM) is founded on the infinite serial expansions:

$$\begin{cases} x(t) = \sum_{n=0}^{+\infty} \varepsilon^n x_n(t) \\ x(t-\tau) = \sum_{n=0}^{+\infty} \varepsilon^n x_n(t-\tau) \end{cases} \quad (4a)$$

Substituting these expansions in the main equation Equation (3) and processing with the standard perturbation method, it has been demonstrated [10] that a solution of the kind:

$$x_0(t) = H \cos(\omega \times t + \phi) \quad (4b)$$

where H , ω and ϕ are constant, gives:

$$\begin{aligned} &\ddot{x}_1(t) + H\omega \sin(\omega \times t + \phi) + \omega^2 x_1(t) + v_1 H \cos(\omega \times t + \phi) \\ &- H^3 \omega \cos^2(\omega \times t + \phi) \sin(\omega \times t + \phi) \\ &- kH \cos(\omega \times t + \phi) \cos(\omega \times \tau + \phi) \\ &- kH \sin(\omega \times t + \phi) \sin(\omega \times \tau + \phi) = 0 \end{aligned} \quad (4c)$$

with, as a final solution (Equation (4d)):

2.2. The Boubaker Polynomials Expansion Scheme (BPES)-Related Solution

The resolution protocol is based on the Boubaker polynomials expansion scheme (BPES) [11-23]. The first step of this scheme starts by applying the expressions:

$$x(t) = \frac{1}{2N_0} \sum_{k=1}^{N_0} \lambda_k \times B_{4k}(t \times r_k) \quad (4e)$$

where B_{4k} are the $4k$ -order Boubaker polynomials, is the normalized time ($t \in [0, 1]$), r_k are B_{4k} minimal positive roots, N_0 is a prefixed integer, and $\lambda_k|_{k=1..N_0}$ are unknown pondering real coefficients.

Consequently, it comes that:

$$y(t) = \dot{x}(t) = \frac{1}{2N_0} \sum_{k=1}^{N_0} \lambda_k r_k \times \frac{dB_{4k}(t \times r_k)}{dt} \quad (5)$$

The main advantage of these formulations (Equations (4) and (5)) is the fact of verifying the boundary conditions in Equation (3), at the earliest stage of resolution protocol. In fact, due to the properties of the Boubaker polynomials [12-18], and since $r_k|_{k=1..N_0}$ are the roots of $B_{4k}|_{k=1..N_0}$, the following conditions stand :

$$\begin{cases} x(t)|_{t=0} = -\frac{1}{N_0} \sum_{k=1}^{N_0} \lambda_k = A = x_0 \neq 0 \\ \left. \frac{dx(t)}{dt} \right|_{t=0} = \frac{1}{2N_0} \sum_{k=1}^{N_0} \lambda_k r_k \times \left. \frac{dB_{4k}(t \times r_k)}{dt} \right|_{t=0} = 0 \end{cases} \quad (6)$$

By introducing expressions (4) and (6) in the system (3), and by majoring and integrating along the interval $[0, 1]$, $x(t)$ is confined, through the coefficients $\lambda_k|_{k=1..N_0}$, to be a weak solution of the system:

$$\begin{cases} \sum_{k=1}^{N_0} r_k^2 \lambda_k M_k + \varepsilon \sum_{k=1}^{N_0} r_k \lambda_k P_k - \sum_{k=1}^{N_0} \lambda_k (Q_k - k \times R_k(\tau)) = 0 \\ M_k = \int_0^1 \frac{d^2 B_{4k}(t \times r_k)}{dt^2} dt \\ P_k = \int_0^1 \frac{dB_{4k}(t \times r_k)}{dt} dt \\ Q_k = \int_0^1 B_{4k}(t \times r_k) dt \\ R_k(\tau) = \int_0^1 B_{4k}((t-\tau) \times r_k) dt \\ \sum_{k=1}^{N_0} \lambda_k = -N_0 x_0 = -N_0 H \end{cases} \quad (7)$$

$$x(t) = H \cos \left(\pm \frac{\sqrt{2k\varepsilon \left(\sqrt{6k\varepsilon\tau^4 + 3k^2\varepsilon^2\tau^4 - 36k\varepsilon\tau^2 + 36 + 3k\varepsilon\tau^2 - 6} \right)}}{k\varepsilon\tau^2} t + \phi \right) \quad (4d)$$

The set of solutions $\hat{\lambda}_k \Big|_{k=1, \dots, N_0}$ is the one which minimizes, for given values of ε and k the Minimum Square function $\Psi_{MS}(\varepsilon, k)$:

$$\Psi_{MS}(\varepsilon, k) = \left(\sum_{k=1}^{N_0} r_k^2 \hat{\lambda}_k M_k + \varepsilon \sum_{k=1}^{N_0} r_k \hat{\lambda}_k P_k - \sum_{k=1}^{N_0} \hat{\lambda}_k (Q_k - k \times R_k(\tau)) \right)^2 \quad (8)$$

under the intrinsic condition:

$$\sum_{k=1}^{N_0} \hat{\lambda}_k = -N_0 H \quad (9)$$

The condition expressed by Equation (9) ensures a non-zero solution to the system (8). The convergence of the algorithm is tested relatively to increasing values of N_0 .

The correspondent solutions are represented in **Figure 3** for the data gathered in **Table 1**, along with the exact solutions given by F. M. Atay [8] and A. Kimiaeifar *et al.* [10]. It is noted that F. M. Atay [8] demonstrated that the presence of delay can change the amplitude of limit cycle oscillations, or suppress them altogether through derivative-like effects, while A. Kimiaeifar *et al.* [10] yielded a highly accurate solution to the same classical Van der Pol equation with delayed feedback and a modified equation where a delayed term provides the damping. The features of the proposed solutions [8-10] (namely behavior at starting phase, first derivatives at limit time, etc.) are concordant with the actually proposed results.

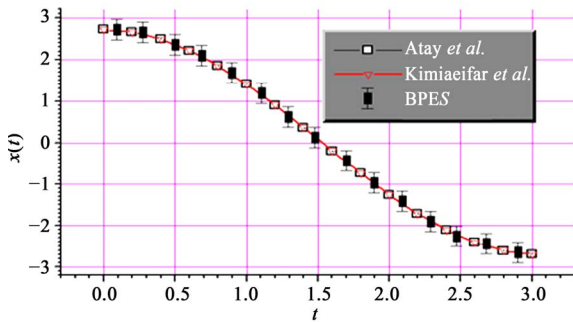


Figure 3. Analytical solutions plots.

Table 1. Solution parameters values.

Parameter	Value
ε	0.1
k	-1.0
τ	1.0
x_0	2.75
\dot{x}_0	0.0
N_0	31
ϕ	0.0

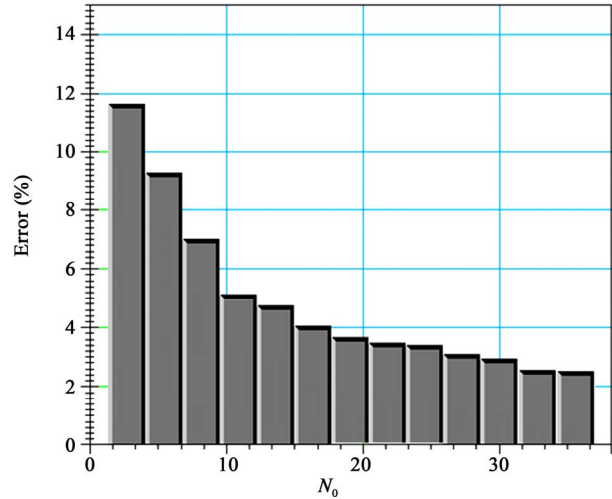


Figure 4. Mean absolute error versus N_0 .

3. Results and Discussions

The results show a good agreement between the proposed analytical solutions (**Figure 3**) and those of the recent studies published elsewhere. The mean absolute error (for $N_0 > 30$) was less than 3.33% (**Figure 4**). The convergence of the *BPES*-related protocol has been recorded for the values of N_0 superior to 30.

4. Conclusions

In this paper, we have used the enhanced He’s parameter-expanding Method (HPEM) along with the Boubaker Polynomials Expansion Scheme (BPES) in order to obtain the Van der Pol’s characteristic periodical solutions.

The obtained solutions were in acceptable agreement with those obtained from values of similarly performed methods. The typical periodical aspect of the oscillations, already yielded [2,10,24-27] by the enhanced He’s parameter-expanding method (HPEM) could be reproduced using a simple and convergent polynomial approximation. This method was based on an original protocol which reduces the stochastic nonlinear system into an equivalent deterministic nonlinear system. This simple and controllable reduction is carried out through the verification of the initial conditions, in the solution basic expression, prime to launching the resolution process.

The results show that the methods are very promising ones and might find wide applications, particularly when exact solutions expressions are difficult to establish [28-34].

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