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Transcendental Meromorphic Functions Whose First Order Derivatives Have Finitely Many Zeros

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Abstract

Let f be a meromorphic function in $\mathbb{C}$. If the order of f is greater than 2, f' has finitely many zeros and f takes a non-zero finite value finitely times, and then $f'\left(f^{-1}(0)\right)$ is unbounded.

Keywords

Transcendental Meromorphic Functions, Derivatives, Zeros

1. Introduction and Main Result

Let f be a meromorphic function in $\mathbb{C}$, define $M_f = f'\left(f^{-1}(0)\right) = \left\{f'(z) : z \in \mathbb{C} \text{ and } f(z) = 0\right\}$. W. Bergweiler [1] gave a conjecture in 2001 as follow: Conjecture 1: Let f be a transcendental meromorphic function in $\mathbb{C}$. If $f'(z) \neq 1$ for all $z \in \mathbb{C}$, then M_f is unbounded.

Bergweiler pointed that let $g(z) = z - f(z)$, Conjecture 1 is equivalent to the following one.

Conjecture 2: Let g be a transcendental meromorphic function in $\mathbb{C}$. Suppose that g' does not have zeros. Then there exist a sequence $\{z_n\}_{n=1}^{\infty}$ of fixed points of g such that $\left|g'(z_n)\right| \to \infty$.

Bergweiler [1] has separately proved Conjecture 1 is affirmative for finite order meromorphic functions and entire functions; Jianming Chang [2] has confirmed the conjecture for infinite order meromorphic functions for the first time, which is based on theory of normal and quasinormal families.

For the conjecture, $f' \neq 1$ and $f' \neq c\,(c \neq 0)$ are essentially equivalent. In fact, if $f' \neq c\,(c \neq 0)$, then $\frac{f'}{c} \neq 1$ and $\frac{f}{c}$ is also transcendental meromorphic function; the zeros of *f* and the zeros of $\frac{f}{c}$ are the same and M_f is unbounded if and only if $M_{\frac{f}{c}}$ is unbounded.

Considering the discussion above, it's natural to research the problem that whether the conclusion is true if $f' \neq 0$ but not $f' \neq c(c \neq 0)$, the problem is radically different to the conjecture and gives a important supplement. We can give a example to show that the problem is significant. Let $f = e^{e^z} - 1$, it's obvious that $f' \neq 0$ and M_f is unbounded. In details, we have

Theorem 1. *Let f be meromorphic in* $\mathbb{C}$ *and the order is greater than* 2. *If* f' *has finitely many zeros and f takes a finite non-zero value finitely many times, then* M_f *is unbounded.*

Theorem 2. *Let f be entire in* $\mathbb{C}$ *and the order is greater than* 1. *If* f' *has finitely many zeros and f takes a finite non-zero value finitely many times, then* M_f *is unbounded.*

2. Preliminary Lemmas

Lemma 1. *Let f be a meromorphic function. If the spherical derivative* $f^{\#}(z)$ *of* $f(z)$ *is bounded. Then the order of* $f(z)$ *is at most* 2.

For details of lemma 1, can see [3]

Remark: Let $f_n(z)$ be a sequence of meromorphic functions, $f_n(z) \overset{loc.}{\rightrightarrows} g(z)$ means $f_n(z)$ locally uniformly convergence to meromorphic function $g(z)$; for a meromorphic function *f* in $\mathbb{C}$, let $D(z_0, M) = \{z : |z - z_0| \leq M\}$.

Lemma 2. *Let* $\mathcal{F}$ *be a family of functions meromorphic in a domain D. Suppose that there exist* $K > 0$ *such that* $M_g \subset \overline{D(0,K)}$ *for all* $g \in \mathcal{F}$. *For any given* α *satisfying* $-1 < \alpha \leq 1$, *if* $\mathcal{F}$ *is not normal, then there exist a sequence* $\{f_n\}_{n=1}^{\infty}$ *in* $\mathcal{F}$, *a sequence* $\{z_n\}_{n=1}^{\infty}$ *in D, a sequence* $\{\rho_n\}_{n=1}^{\infty}$ *of positive real numbers and a non-constant finite order function f which is meromorphic in* $\mathbb{C}$ *such that* $z_n \to z_0$ *for some* $z_0 \in D, \rho_n \to 0$ *and*

$$\frac{f(z_n + \rho_n z)}{\rho_n^{\alpha}} \overset{loc.}{\rightrightarrows} f(z) \quad (z \in \mathbb{C}; n \to \infty).$$

Moreover, the spherical derivative $f^{\#}(z)$ of *f* satisfies

$f^{\#}(z) \leq f^{\#}(0) = K + 1$ for all $z \in \mathbb{C}$.

For details of lemma 2, can see [4]

In the case no hypothesis on M_g is required, the case $\alpha = 0$ is due to Zalcman [5], and the case $-1 < \alpha < 1$ is due to Pang [6] [7]. In the hypothesis of Lemma 2, if all $g \in \mathcal{F}$ have no zero in *D*, then $M_g = \varnothing$ and the conclusion is still right according to the proof of Lemma 2 in which *K* can take 0.

Lemma 3. *Let f be a meromorphic function, D be an bounded domain and c be constant in* $\mathbb{C}$, *if* $f(z) - c$ *has* $l(l \geq 2)$ *zeros in D and* $f'(z)$ *has* $l - 1$ *zeros in D which are all the zeros of* $f(z) - c$. *Then each discriminating zero of* $f(z) - c$ *and* $f'(z)$ *in D is the same one.*

Proof. Let $f(z)-c=\prod_{j=1}^{l}(z-z_j)g(z)=R(z)g(z)$ with $g(z)$ be a meromorphic function which have no zero in D satisfying $g(z_j)\neq\infty\,(j=1,\cdots,l)$ and $R(z)=\prod_{j=1}^{l}(z-z_j)$ in which $z_j\subset D\,(j=1,\cdots,l)$.

Because $f'(z)$ has $l-1$ zeros in D which are all the zeros of $f(z)-c$, there exist $l-1$ points in $z_j\,(j=1,\cdots,l)$ be zeros of $f'(z)$ and without loss of generality we may assume $f'(z_j)=0\,(j=1,\cdots,l-1)$.

As $f'(z)=R'(z)g(z)+R(z)g'(z)$, we can deduce that

$$R'(z_j)=0\,(j=1,\cdots,l-1),\ R'(z)=l\cdot\prod_{j=1}^{l}(z-z_j)\text{ and }\frac{R'(z)}{R(z)}=\frac{l}{z-z_l}$$

from the above it follows that $R(z)=(z-z_l)^l$ and the proof of Lemma 3 is complete.

Lemma 4. *Let f be a holomorphic function, if the spherical derivative of f is bounded. Then the order of f is at most* 1.

For details of lemma 4, can see [8].

3. Proof of Theorem 1

Proof. we apply Lemma 1 to obtain a sequence $\{\omega_n\}_{n=1}^{\infty},\omega_n\to\infty(n\to\infty)$ such that $f^{\#}(\omega_n)\to\infty,(n\to\infty)$. $\forall n\in\mathbb{N}$, let $f_n(z)=f(z+\omega_n)$, it's easy to apply Marty's theorem to know $\{f_n(z)\}_{n=1}^{\infty}$ is not normal at 0. Suppose $f(z)-\lambda$ only have finitely many zeros ($\lambda\neq0$), there exist a subsequence of $\{f_n(z)\}_{n=1}^{\infty}$ we still suppose it's $\{f_n(z)\}_{n=1}^{\infty}$ such that $f(z)-\lambda$ have no zero in $\mathbb{C}$, thus according to Lemma 2, there exist a sequence $\{z_n\}_{n=1}^{\infty}$, a sequence $\{\rho_n\}_{n=1}^{\infty}$ of positive real numbers and a non-constant finite order function $g(z)$ such that when $n\to\infty$, $z_n\to0,\rho_n\to0$ and

$$\frac{f_n(z_n+\rho_nz)-\lambda}{\rho_n}=\frac{f(\omega_n+z_n+\rho_nz)-\lambda}{\rho_n}\overset{loc.}{\rightrightarrows}g(z)$$

in $\mathbb{C}$ and $g(z)$ satisfies $g^{\#}(z)\le g^{\#}(0)=1$ for all $z\in\mathbb{C}$.

$\forall n\in\mathbb{N}$, let $\tau_n=\omega_n+z_n$, there exist entire functions $F(z)$ and $H(z)$ such that $F(z)$ and $H(z)$ have no common non-trivial divisor and $f(z)=\dfrac{H(z)}{F(z)}$, then

$$\frac{f_n(z_n+\rho_nz)-\lambda}{\rho_n}=\frac{f(\tau_n+\rho_nz)-\lambda}{\rho_n}=\frac{\frac{H(\tau_n+\rho_nz)}{F(\tau_n+\rho_nz)}-\lambda}{\rho_n}\overset{loc.}{\rightrightarrows}g(z)$$

$$\frac{\rho_n}{f(\tau_n+\rho_nz)-\lambda}\overset{loc.}{\rightrightarrows}\frac{1}{g(z)}\quad(z\in\mathbb{C})\tag{1}$$

For $f(\tau_n+\rho_n z)-\lambda\neq 0$ and $g(z)\neq 0$, the derivative of (1) is

$$\frac{\rho_n^2 f'(\tau_n+\rho_n z)}{\{f(\tau_n+\rho_n z)-\lambda\}^2}\overset{loc.}{\rightrightarrows}\frac{g'(z)}{g^2(z)}\ (z\in\mathbb{C}) \tag{2}$$

here we divide two cases:

Case 1: $\frac{g'(z)}{g^2(z)}$ have no zero in $\mathbb{C}$.

Because $\left(\frac{1}{g(z)}\right)^{\#}=g^{\#}(z)$ is bounded, then we apply Lemma 4 to have that the order of $\frac{1}{g(z)}$ and $\frac{g'(z)}{g^2(z)}$ are at most 1. On the other hand, $\frac{g'(z)}{g^2(z)}\neq 0$, we can deduce $\frac{g'(z)}{g^2(z)}=\mathrm{e}^{Az+B}\ (A,B\in\mathbb{C},A\neq 0)$ or constant $\alpha(\alpha\neq 0)$ and $\frac{1}{g(z)}=\frac{\mathrm{e}^{Az+B}}{A}+d$ or $\frac{1}{g(z)}=\alpha z+\beta\ (d,\beta\in\mathbb{C})$.

here we first proof that $d\neq 0$, if $d=0$, we have

$$\frac{f(\tau_n+\rho_n z)-\lambda}{\rho_n}\overset{loc.}{\rightrightarrows}\frac{A}{\mathrm{e}^{Az+B}}\ (z\in\mathbb{C})$$

then we have that

$$f(\tau_n+\rho_n z)=\frac{H(\tau_n+\rho_n z)}{F(\tau_n+\rho_n z)}\overset{loc.}{\rightrightarrows}\lambda\ (z\in\mathbb{C}) \tag{3}$$

$$\frac{H(\tau_n+\rho_n z)-\lambda F(\tau_n+\rho_n z)}{F(\tau_n+\rho_n z)}\overset{loc.}{\rightrightarrows}0\ (z\in\mathbb{C}) \tag{4}$$

From (4) it can be deduced that because $H(\tau_n+\rho_n z)-\lambda F(\tau_n+\rho_n z)$ have no zero, $F(\tau_n+\rho_n z)$ have no zero in any bounded domain when n is large enough,. Then so are $H(\tau_n+\rho_n z)$ unite (3). What' more, $f(\tau_n+\rho_n z)$ has no zero and pole and cannot take λ in any bounded domain, which contradict with Picard's Theorem, therefore $d\neq 0$.

From (1) we have

$$\frac{\rho_n}{f(\tau_n+\rho_n z)-\lambda}-\frac{\rho_n}{-\lambda}\overset{loc.}{\rightrightarrows}\frac{1}{g(z)}\ (z\in\mathbb{C})$$

Because $\frac{1}{g}$ have zero in $\mathbb{C}$, from above it can be deduced that $f(\tau_n+\rho_n z)$ have to have zeros which convergence to the zeros of $\frac{1}{g(z)}$. From (2) when $f(\tau_n+\rho_n z)=0$ then $\frac{\rho_n f'(\tau_n+\rho_n z)}{\lambda^2}\to -Ad$ or α and $f'(\tau_n+\rho_n z)\ (n\in\mathbb{N})$ have to be unbounded.

Case 2: $\frac{g'(z)}{g^2(z)}$ have zero in $\mathbb{C}$. We will proof the case is impossible.

Take a finite zero c of $\frac{g'(z)}{g^2(z)}$ and it's multiple is $k(k\in\mathbb{N},k\geq 1)$ then there exist some sufficiently small neighborhood D_c of c such that D_c only have one zero of $\frac{g'(z)}{g^2(z)}$. (2) can be expressed as

$$\rho_n^2\cdot\frac{F'(\tau_n+\rho_n z)H(\tau_n+\rho_n z)-H'(\tau_n+\rho_n z)F(\tau_n+\rho_n z)}{\{H(\tau_n+\rho_n z)-\lambda F(\tau_n+\rho_n z)\}^2}\overset{loc.}{\rightrightarrows}\frac{g'(z)}{g^2(z)}\ (z\in\mathbb{C}).$$

Because $f(z)$ take λ finitely many times and from (2) $H(\tau_n+\rho_n z)-\lambda F(\tau_n+\rho_n z)$ have no zero in D_c then when n is large enough, $F'(\tau_n+\rho_n z)H(\tau_n+\rho_n z)-H'(\tau_n+\rho_n z)F(\tau_n+\rho_n z)$ have k zeros in D_c, which are all the zeros of $F(\tau_n+\rho_n z)$ due to that $f'=\frac{H'(z)F(z)-H(z)F'(z)}{F^2(z)}$ only has finitely many zeros; therefore, c is the zero of $\frac{1}{g(z)}$ with $k+1$ multiple and $F(\tau_n+\rho_n z)$ have $k+1$ zeros in D_c.

(1) can be expressed as

$$\frac{\rho_n\frac{F(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}}{1-\lambda\frac{F(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}}\overset{loc.}{\rightrightarrows}\frac{1}{g(z)}\ (z\in\mathbb{C}).$$

From (1) we have that

$$\lambda\frac{F(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}\overset{loc.}{\rightrightarrows}1\ (z\in D_c\setminus\{c\}). \tag{5}$$

Here we divide two cases for (5):

Subcase 2.1: If $\left\{\lambda\frac{F(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}\right\}_{n=1}^{\infty}$ is normal in D_c.

From (5) we have

$$\frac{\lambda F(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}\overset{loc.}{\rightrightarrows}1\ (z\in D_c). \tag{6}$$

$$\frac{\lambda F(\tau_n+\rho_n z)-H(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}\overset{loc.}{\rightrightarrows}0\ (z\in D_c).$$

when n is large enough, notice that $H(\tau_n+\rho_n z)-\lambda F(\tau_n+\rho_n z)$ have no zero in D_c, therefore $H(\tau_n+\rho_n z)$ have no zero in D_c and according to (6), $F(\tau_n+\rho_n z)$ have no zero in D_c contradict with $F(\tau_n+\rho_n z)$ have $k+1$ zeros in D_c.

Subcase 2.2: If $\left\{\lambda\frac{F(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}\right\}_{n=1}^{\infty}$ is not normal in D_c. Let

$$\varphi_n(z)=\frac{\lambda F(\tau_n+\rho_n z)-H(\tau_n+\rho_n z)}{H(\tau_n+\rho_n z)}.$$

Then $\{\varphi_n(z)\}_{n=1}^{\infty}$ is not normal and have no zero in D_c, we apply Lemma 2 to obtain $\{\nu_n\}_{n=1}^{\infty}\in\mathbb{C}$ and $\{\rho_n^*\}_{n=1}^{\infty}$ of positive real numbers and a non-constant finite order function $\psi(\xi)$ such that $\nu_n\to c,\rho_n^*\to 0$, and

$$\psi_n(\xi)=\frac{\varphi_n(\nu_n+\rho_n^*\xi)}{\rho_n^*}\overset{loc.}{\rightrightarrows}\psi(\xi)(n\to\infty,\xi\in\mathbb{C}) \text{ with } \psi^{\#}(\xi)\le\psi^{\#}(0)=1.$$

here we will prove that $\psi(\xi)$ has no simple pole if it exist; let ξ_0 be the pole of $\psi(\xi)$, for $\psi(\xi)$ cannot always be ∞, there exist closed disc $\bar{D}(\xi_0,\delta)$ such that $1/\psi(\xi)$ and $1/\psi_n(\xi)$ are holomorphic in $\bar{D}(\xi_0,\delta)$ and $1/\psi_n(\xi)\rightrightarrows 1/\psi(\xi)$ uniformly in $\bar{D}(\xi_0,\delta)$ and so are $1/\psi_n(\xi)+\rho_n^*$.

Notice that $1/\psi_n(\xi)$ cannot be constant, there exist $\{\xi_n\}_{n=1}^{\infty},\xi_n\to\xi_0(n\to\infty)$ such that

$$\frac{1}{\psi_n(\xi_n)}+\rho_n^*=\frac{\rho_n^*}{\varphi_n(\nu_n+\rho_n^*\xi_n)}+\rho_n^*=0,\quad \varphi_n(\nu_n+\rho_n^*\xi_n)+1=0.$$

We firstly show that the discriminating zeros of $\varphi_n(z)+1$ in D_c are all the zeros of $\varphi_n'(z)$ in D_c when n is large enough. In fact, we have that the k zeros of $F'(\tau_n+\rho_n z)H(\tau_n+\rho_n z)-H'(\tau_n+\rho_n z)F(\tau_n+\rho_n z)$ as same as $\varphi_n'(z)$, which are all belong to the $k+1$ zeros of $F(\tau_n+\rho_n z)$ and $\varphi_n(z)+1$ in D_c, then Lemma 3 can be used to prove the conclusion and we further have

$$\left.\left(\frac{1}{\psi(\xi)}\right)'\right|_{\xi=\xi_0}=-\frac{\psi'(\xi_0)}{\{\psi(\xi_0)\}^2}=-\lim_{n\to\infty}\frac{\psi_n'(\xi_n)}{\{\psi_n(\xi_n)\}^2}=-\lim_{n\to\infty}\{\rho_n^*\}^2\cdot 0=0$$

which means $\psi(\xi)$ has to have multiple pole if it exist.

Notice $\dfrac{H'(z)F(z)-H(z)F'(z)}{F^2(z)}$ only has finitely many zeros, then $H(\tau_n+\rho_n z)$ have no multiple zero in $\mathbb{C}$ when n is large enough.

Considering $1/\psi_n(\xi)\overset{loc.}{\rightrightarrows}1/\psi(\xi)(\xi\in\mathbb{C})$, since $\lambda F(\tau_n+\rho_n z)-H(\tau_n+\rho_n z)$ have no zero in D_c when n is large enough, $1/\psi_n(\xi)$ are analytic in D_c and according to Hurwitz's Theorem, $\psi(\xi)$ has no multiple pole. With the assert above, $\psi(\xi)$ have no pole and be entire.

Notice that $\psi^{\#}(\xi)\le\psi^{\#}(0)=1(\xi\in\mathbb{C})$ and Lemma 4, the order of $\psi(\xi)$ is at most 1. Since $\varphi_n(z)$ have no zero in D_c when n is large enough, then $\varphi_n(\nu_n+\rho_n^*\xi)$ have no zero in any bounded domain, from the above it follows that $\psi(\xi)\ne 0(\xi\in\mathbb{C})$ and $\psi(\xi)=\mathrm{e}^{A\xi+B}(A,B\in\mathbb{C},A\ne 0)$.

$1/\psi_n(\xi)\overset{loc.}{\rightrightarrows}1/\psi(\xi)(\xi\in\mathbb{C})$ is

$$\frac{\rho_n^*H(\tau_n+\rho_n\nu_n+\rho_n\rho_n^*\xi)}{\lambda F(\tau_n+\rho_n\nu_n+\rho_n\rho_n^*\xi)-H(\tau_n+\rho_n\nu_n+\rho_n\rho_n^*\xi)}\overset{loc.}{\rightrightarrows}\mathrm{e}^{-A\xi-B}\quad(\xi\in\mathbb{C}). \tag{7}$$

Let

$$\eta_n(z)=\frac{F(\tau_n+\rho_n z)H'(\tau_n+\rho_n z)-F'(\tau_n+\rho_n z)H(\tau_n+\rho_n z)}{\{\lambda F(\tau_n+\rho_n z)-H(\tau_n+\rho_n z)\}^2}$$

then the derivative of (7) is

$$\left(\rho_n^*\right)^2 \rho_n \lambda \eta_n\left(v_n+\rho_n^* \xi\right) \overset{loc.}{\rightrightarrows} -A\mathrm{e}^{-A\xi-B} \left(n\to\infty, \xi\in\mathbb{C}\right). \tag{8}$$

(2) can be expressed as

$$\rho_n^2 \eta_n(z) \overset{loc.}{\rightrightarrows} \frac{g'(z)}{g^2(z)} \left(n\to\infty, z\in\mathbb{C}\right).$$

$\forall n\in\mathbb{N}$, let $h_n(z)$ be the k order derivative of $\eta_n(z)$, then the k order derivative of (2) is

$$\rho_n^2 h_n(z) \overset{loc.}{\rightrightarrows} \left(\frac{1}{g(z)}\right)^{(k+1)} \left(n\to\infty, z\in\mathbb{C}\right).$$

with $\left(\frac{1}{g(z)}\right)^{(k+1)}$ have no zero in D_c. Let $\left.\left(\frac{1}{g(z)}\right)^{(k+1)}\right|_{z=c} = G_c(\neq 0)$ then we have

$$\rho_n^2 h_n\left(v_n+\rho_n^* \xi\right) \overset{loc.}{\rightrightarrows} G_c \left(n\to\infty, \xi\in\mathbb{C}\right). \tag{9}$$

The k order derivative of (8) is

$$\left(\rho_n^*\right)^{k+2} \rho_n \lambda h_n\left(v_n+\rho_n^* \xi\right) \overset{loc.}{\rightrightarrows} (-A)^{k+1} \mathrm{e}^{-A\xi-B} \left(n\to\infty, \xi\in\mathbb{C}\right), \tag{10}$$

(9) + (10) is

$$\left\{\rho_n+\lambda\left(\rho_n^*\right)^{k+2}\right\} \rho_n h_n\left(v_n+\rho_n^* \xi\right) \overset{loc.}{\rightrightarrows} G_c+(-A)^{k+1} \mathrm{e}^{-A\xi-B} \left(n\to\infty, \xi\in\mathbb{C}\right).$$

It shows that $h_n\left(v_n+\rho_n^* \xi\right)$ have to have zeros in $\mathbb{C}$ when n is large enough, however, from (10) and Hurwitz's theorem, it is impossible; this gives a contradiction and the proof of Theorem 1 is complete.

4. Remarks

It follows from the proof of Theorem 1 that the hypothesis for order can be replaced by greater than 1 for entire functions. In fact, from Lemma 4, we can obtain a sequence $\{\omega_n\}_{n=1}^{\infty}, \omega_n\to\infty(n\to\infty)$ such that $f^{\#}(\omega_n)\to\infty, (n\to\infty)$. Then using the start point of proof of Theorem 1, $\forall n\in\mathbb{N}$, let $f_n(z)=f(z+\omega_n)$, it's easy to apply Marty's theorem to know $\{f_n(z)\}_{n=1}^{\infty}$ is not normal at 0. Suppose $f(z)-\lambda$ only has finitely many zeros ($\lambda\neq 0$), there exist a subsequence of $\{f_n(z)\}_{n=1}^{\infty}$. We still suppose it's $\{f_n(z)\}_{n=1}^{\infty}$ such that $f_n(z)-\lambda$ has no zero in $\mathbb{C}$. Thus according to Lemma 2, there exist a sequence $\{z_n\}_{n=1}^{\infty}$, a sequence $\{\rho_n\}_{n=1}^{\infty}$ of positive real numbers and a non-constant finite order function $g(z)$ such that when $n\to\infty$, $z_n\to 0, \rho_n\to 0$ and

$$f_n(z_n+\rho_n z)-\lambda = f(\omega_n+z_n+\rho_n z)-\lambda \overset{loc.}{\rightrightarrows} g(z)$$

in $\mathbb{C}$ and $g(z)$ satisfies $g^{\#}(z)\le g^{\#}(0)=1$ for all $z\in\mathbb{C}$. For $g'\neq 0$ and $g\neq 0$ we apply Lemma 6 to have the order of g at most 1, and

$g = \mathrm{e}^{Az+B}\left(A \neq 0\right)$ and we have

$$f\left(\omega_n + z_n + \rho_n z\right) \overset{loc.}{\rightrightarrows} \mathrm{e}^{Az+B} + \lambda \ \left(n \to \infty, z \in \mathbb{C}\right).$$

and the first order derivative is

$$\rho_n f'\left(\omega_n + z_n + \rho_n z\right) \overset{loc.}{\rightrightarrows} A\mathrm{e}^{Az+B} \ \left(n \to \infty, z \in \mathbb{C}\right).$$

If $f\left(\omega_n + z_n + \rho_n z\right) = 0$, then $\rho_n f'\left(\omega_n + z_n + \rho_n z\right) \to -A\lambda$ and $f'\left(\omega_n + z_n + \rho_n z\right)\left(n \in \mathbb{N}\right)$ is unbounded. Then the proof of Theorem 2 is complete.

The requirement for order in Theorem 2 is sharp, let $f = \mathrm{e}^z - 1$, then $M_f = \{1\}$.

By the equivalence between the conjecture 1 and 2, we can have two corollaries from Theorem 1 and 2.

Corollary 1. *Let g be meromorphic in* $\mathbb{C}$ *and the order is greater than* 2. *If* $g' - 1$ *has finitely many zeros and* $g - z$ *takes a finite non-zero value finitely many times, then* g *has a sequence* $\{z_n\}_{n=1}^{\infty}$ *of fixed points such that* $g'(z_n) \to \infty, (n \to \infty)$.

Corollary 2. *Let g be entire in* $\mathbb{C}$ *and the order is greater than* 1. *If* $g' - 1$ *has finitely many zeros and* $g - z$ *takes a finite non-zero value finitely many times, then* g *has a sequence* $\{z_n\}_{n=1}^{\infty}$ *of fixed points such that* $g'(z_n) \to \infty, (n \to \infty)$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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