

Existence of Solutions for Some $p(x)$ -polyharmonic Elliptic Kirchhoff Equations

Yijian Ge

School of Science, University of Shanghai for Science and Technology, Shanghai, China

Email: yjGeusst@163.com

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Abstract

In this paper, we study the existence of solution for some $p(x)$ -polyharmonic Kirchhoff equations. The latter is allowed to vanish at the origin (degenerate case). Firstly, we study the existence of solutions of approximate equations. Secondly, we prove the existence of the solutions of the original equation. The main tool is the Schauder's Theorem.

Keywords

$p(x)$ -polyharmonic Kirchhoff Equations, Existence of Solutions, Schauder's Fixed Point Theorem

1. Introduction

In this paper, we prove the existence of solution of Dirichlet problems involving the p -polyharmonic operators Δ_p^s . We consider

$$\begin{cases} M(\|u\|^p) \Delta_p^s u + a(x) g(u) = f(x) & \text{in } \Omega, \\ D^\alpha u(x)|_{\partial\Omega} = 0 & \forall \alpha, \text{ with } |\alpha| \leq s-1, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p \geq 2$, $s = 1, 2, \dots$, $\|\cdot\|$ is denoted in section 2, and $f(x) \in L^1(\Omega)$, $0 \leq a(x) \in L^1(\Omega)$. Here, the p -polyharmonic operator is defined by

$$\Delta_p^s u = \begin{cases} -\operatorname{div} \Delta^{j-1} \left(|D \Delta^{j-1} u|^{p-2} \right) D \Delta^{j-1} u, & s = 2j-1, \\ \Delta^j \left(| \Delta^j u |^{p-2} \Delta^j u \right), & s = 2j, \end{cases} \quad j = 1, 2, \dots, \quad (2)$$

which becomes the usual p -Laplacian for $s = 1$. Kratochvíl and Nečás introduced the p -biharmonic operator in [1] [2] [3] to study the physical equations, the p -biharmonic operator for $s = 2$ and the polyharmonic operator for

$p = 2$, which reduces to the more approximate case

$$\begin{cases} M(\|u\|^2)(-\Delta u)^s = f(x, u) & \text{in } \Omega, \\ D^\alpha u(x)|_{\partial\Omega} = 0 & \forall \alpha, \text{ with } |\alpha| \leq s-1 \end{cases} \quad (3)$$

We introduce for $s = 1, 2, \dots$, the main s -order differential operator

$$\mathcal{D}_s u = \begin{cases} D\Delta^{j-1}u & \text{if } s = 2j-1, \\ \Delta^j u & \text{if } s = 2j \end{cases} \quad j = 1, 2, \dots \quad (4)$$

Note that \mathcal{D}_s is an n -vectorial operator when s is odd and $n > 1$, while it is a scalar operator when s is even.

In our hypothesis, the Kirchhoff function $M : R_0^+ \rightarrow R_0^+$ is assumed to be continuous and to verify the structural assumptions (M):

(M₁) M is non-decreasing;

(M₂) there exists a number $\gamma \in [1, p_s)$ such that for all $t \in R_0^+$;

$$tM(t) \leq \gamma \hat{M}(t), \quad \text{where } \hat{M}(t) = \int_0^t M(\theta) d\theta;$$

(M₃) for all $t \geq \sigma$, there exists $m_0 = m_0(\sigma) > 0$ such that $M(t) \geq m_0$ for all $\sigma \geq 0$.

We introduce the Sobolev critical exponent p_s^* and the number p_s defined by following

$$p_s^* = \begin{cases} \frac{np}{n-sp} & \text{if } n > sp, \\ \infty & \text{if } n \leq sp. \end{cases} \quad p_s = \frac{p_s^*}{p} = \begin{cases} \frac{n}{n-sp} & \text{if } n > sp, \\ \infty & \text{if } n \leq sp. \end{cases} \quad (5)$$

A very special Kirchhoff function verifying (M) is denoted by

$$M(t) = a + bt^{\gamma-1}, \quad a, b \geq 0, \quad a + b > 0, \quad \gamma \begin{cases} \in (1, p_s) & \text{if } b > 0, \\ = 1 & \text{if } b = 0. \end{cases} \quad (6)$$

when M is of the type (6) and $a > 0, b \geq 0$, problem (1) is said to be non-degenerate, while it is called degenerate if $a = 0$. Besides, problem (2) reduces to the usual well-known quasilinear elliptic equation while $a > 0, b = 0$. The existence of positive solutions of non-degenerate Kirchhoff-type problems has been proved in [4] [5] for $L = 1$. The novelty of this paper is to treat the degenerate case with allowing Kirchhoff function to take the zero value. Several authors have considered fourth order problems with nonlinear boundary conditions involving third order derivatives, see [6]. The classical counterpart of our problem models containing several interesting phenomena were deeply studied in physics even in the one-dimensional case. It dates back to 1883 when Kirchhoff proposed his celebrated equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

as a nonlinear extension of D’Alambert’s wave equation for free vibrations for elastic strings.

Here we study a stationary version of Kirchhoff-type problems, where $u = u(x)$ is the lateral displacement at the space coordinate x and M is typically a line with positive slope. Our result allows M to have this property. The classical Kirchhoff theory described further details and physical models, which can be found in [7] [8]. In the standard case $L = 2$, problem of type (2) arise in the theory of bending extensible elastic beams. There $u = u(x)$ denotes a thin extensible elastic beam. The function f models a small changes with effect in the length of beam but acts as a force exerted on the beam. We read to [6] and the references therein for a discussion about modelling of Kirchhoff-type strings and beams. We cite the wide literature on the subject, the works [9] [10] [11] [12], where Kirchhoff-type problems new studied by exploiting different methods.

We recall that study of semilinear case with datum $f(x) \in L^1(\Omega)$ in [13] [14] [15] [16], with respect to (1), we assume that the coefficient $a(x)$ of the zero order term and to the datum $f(x)$, in addition to imposing that

$$f(x), a(x) \in L^1(\Omega), \tag{7}$$

and there exists $Q > 0$ such that, for $x \in \Omega$ a.e.,

$$|f(x)| \leq Qa(x). \tag{8}$$

There is assumption that $g(s)$ is continuous function satisfies

$$\lim_{s \rightarrow -\infty} g(s) = -\infty \quad \text{and} \quad \lim_{s \rightarrow +\infty} g(s) = \infty. \tag{9}$$

There has been an increasing interest in studying equations involving $p(x)$ -Laplace operators over the last few decades. Motivated by theoretical research in the regularizing effect of the interaction between the coefficient of the zero order term and the datum $f(x) \in L^1(\Omega)$ in some nonlinear Dirichlet problems, we pay attention to the existence of solutions for $p(x)$ -polyharmonic Kirchhoff equations. Now we consider the problems

$$\begin{cases} M(\varphi(u)) \Delta_{p(x)}^s u + a(x)g(u) = f(x) & \text{in } \Omega, \\ D^\alpha u(x)|_{\partial\Omega} = 0 & \forall \alpha, \text{ with } |\alpha| \leq s-1, \end{cases} \tag{10}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain Lipschitz boundary, M is a degenerate Kirchhoff function and $p \in C(\bar{\Omega})$. More details and conditions are given in section 4. The $p(x)$ -polyharmonic operator is given by

$$\Delta_{p(x)}^s u = \begin{cases} -\text{div} \Delta^{j-1} \left(|D \Delta^{j-1} u|^{p(x)-2} \right) D \Delta^{j-1} u, & s = 2j-1, \\ \Delta^j \left(|\Delta^j u|^{p(x)-2} \Delta^j u \right), & s = 2j, \end{cases} \quad j = 1, 2, \dots. \tag{11}$$

The author exploits the symmetric mountain pass theorem to proves the multiplicity of solutions for $p(x)$ -polyharmonic elliptic Kirchhoff equations in [17]. In contrast, in this paper, the keystone of the proofs them is the deduction, by condition (7), (8), of the L^∞ -estimate of the approximated solutions, we prove the problem existing a solution $u \in W_0^{s,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

This paper is organized as follows. In Section 2, we introduce some basic no-

tation and properties in variable exponent Sobolev spaces. In Section 3, we prove the problem (1) ($p \equiv \text{Const}$) existing a solution $u \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$. In Section 4, we treat the more delicate case $p = p(x)$.

2. Notations and Preliminaries

In this section, we briefly introduce some basic results and notations. Let Ω be a bounded domain in \mathbb{R}^n , we denote a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$, with length $|\alpha| = \sum_{i=1}^n \alpha_i \leq s$, such that the corresponding partial differentiation:

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

Write:

$$\|u\|_{W^{s,p}(\Omega)} = \left(\sum_{|\alpha| \leq s} \|D^\alpha u\|^p \right)^{\frac{1}{p}}, \tag{12}$$

where $\|\cdot\|_p$ denotes the standard L^p -norm. See [18], we denote the space $W_0^{s,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the standard norm of $W^{s,p}(\Omega)$. Moreover, denote $\mathcal{D}^{s,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$, with respect to the norm:

$$\|u\|_{\mathcal{D}^{s,p}(\Omega)} = \left(\sum_{|\alpha|=s} \|D^\alpha u\|^p \right)^{\frac{1}{p}}. \tag{13}$$

By the poncaré inequality, there exists a positive constant $\mathcal{K} = \mathcal{K}(n, p, \Omega)$, with $m = s, p \geq 1$, such that

$$\|u\|_{W^{s,p}(\Omega)} \leq \mathcal{K} \|u\|_{\mathcal{D}^{s,p}(\Omega)}, \quad \forall u \in W_0^{s,p}(\Omega). \tag{14}$$

Hence, we obtain that the norms $\|\cdot\|_{W^{s,p}(\Omega)}$ are equivalent, so that the two completions of $C_0^\infty(\Omega)$, with corresponding these norms, namely

$$W_0^{s,p}(\Omega) = \mathcal{D}^{s,p}(\Omega).$$

We endow the vectorial space $[L^p(\Omega)]^n$, with respect to the norm

$$\|v\|_p = \left(\sum_{i=1}^n \|\beta_i\|_p^p \right)^{\frac{1}{p}}, \tag{15}$$

where $v = (\beta_1, \beta_2, \dots, \beta_n)$ and $n > 1$, we still use the same symbol $\|\cdot\|_p$ to denote both the standard L^p -norm in the scalar space $L^p(\Omega)$ and the norm define in (15), in the vectorial space $[L^p(\Omega)]^n$.

For $s = 2, 1 < p < \infty$, by the Caldéron-Zygmund inequality, see details in [19] [20], there exists a constant $k_2 = k_2(n, p) > 0$ such that:

$$\|u\|_{\mathcal{D}^{2,p}(\Omega)} \leq k_2 \|D_2 u\|_p, \quad \forall u \in W_0^{2,p}(\Omega). \tag{16}$$

Proposition 2.1. *If $p \in (1, \infty)$ and $s = 1, 2, \dots$, then there exists a positive constant $k_s = k_s(n, p)$ such that:*

$$\|u\|_{\mathcal{D}^{s,p}(\Omega)} \leq k_s \|D_s u\|_p, \quad \forall u \in W_0^{s,p}(\Omega). \tag{17}$$

where \mathcal{D}_s is denoted in (4), see also in [17].

Hence, from now on we endow $W_0^{s,p}(\Omega)$ with the norm $\|\cdot\| = \|\mathcal{D}_s \cdot\|_p$, which is equivalent to the standard Sobolev norm.

Remark 2.1. For all $s = 1, 2, \dots$, $1 < p < \infty$. $W_0^{s,p}(\Omega)$ is a separable, uniformly convex, reflexive, real Banach space.

Note that, when $p = 2$, this norm is introduced by the inner product

$$\langle u, v \rangle = \int_{\Omega} \mathcal{D}_s u \mathcal{D}_s v dx, \quad \forall u, v \in H_0^s(\Omega), \tag{18}$$

when s is even the operation between $\mathcal{D}_s u$ and $\mathcal{D}_s v$ is scalar multiplication, while s is odd, it is the n -Euclidean scalar product.

Lemma 2.1. See [21] (Schuader's theorem) Let F be a completely continuous map and let K be a convex, bounded, closed and invariant subset of X . Then F has a fixed point in K .

F is completely continuous map:

1) F is continuous.

2) For every B is bounded subset of X , then $\overline{F(B)}$ is compact.

Proposition 2.2. See [18], for $1 \leq h < p_s^*$, the embedding $W_0^{s,p}(\Omega), L^h(\Omega)$ is compact and continuous, there exists $\delta_h = \delta_h(n, p, s, \Omega) > 0$, such that:

$$\|u\|_h \leq \delta_h \|u\|, \quad \forall u \in W_0^{s,p}(\Omega). \tag{19}$$

We study problem (1) for a solution, we understand:

$$\begin{cases} u \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega), \\ M(\|u\|^p) \int_{\Omega} |\mathcal{D}_s u|^{p-2} \mathcal{D}_s u \mathcal{D}_s v dx + \int_{\Omega} a(x) g(u) \varphi = \int_{\Omega} f(x) \varphi, \\ \varphi \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{20}$$

where \mathcal{D}_s is the operator in (4) and $\int_{\Omega} |\mathcal{D}_s u|^{p-2} \mathcal{D}_s u \mathcal{D}_s v dx$ is the p -polyharmonic operator Δ_p^s in weak sense.

3. Existence and Uniqueness of Solution for (1)

In order to study the solution of problem (1), we consider problems:

$$\begin{cases} M(\|u_n\|^p) \Delta_p^s u_n + a_n(x) g(u_n) = f_n & \text{in } \Omega, \\ D^\alpha u_n(x)|_{\partial\Omega} = 0 & \forall \alpha, \text{ with } |\alpha| \leq s-1. \end{cases} \tag{21}$$

where Ω is a bounded domain in \mathbb{R}^N , $p \in [2, \infty)$, and $s = 1, 2, \dots$. Indeed, suppose $f(x) \in L^1(\Omega)$, $0 \leq a(x) \leq L^1(\Omega)$, and exist $h(x) \in L^{q'}(\Omega)$, then $|g(s)| \leq h(x)$.

Let us define:

$$a_n(x) = \frac{a(x)}{1 + \frac{Q}{n}|a(x)|}, \quad f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|}. \tag{22}$$

and that we choose $k_0 > 0$, such that

$$g(t)t \geq 0, \quad |g(t)| \geq Q. \tag{23}$$

for every $t \geq k_0$.

Theorem 3.1. *There is a solution $u_n \in W_0^{s,p}(\Omega)$ to the problem (21).*

Proof. Since $\varphi = s\left(1 + \frac{s}{n}\right)^{-1}$ is increasing, we deduced by (8) that,

$$|f_n(x)| = \frac{|f(x)|}{1 + \frac{1}{n}|f(x)|} \leq \frac{Qa(x)}{1 + \frac{Q}{n}a(x)} = Qa_n(x). \tag{24}$$

We define:

$$J(\omega) = \frac{1}{p} \hat{M}(\|\omega\|^p) + \int_{\Omega} a_n(x)g(v)\omega - \int_{\Omega} f_n(x)\omega,$$

where $v \in W_0^{s,p}(\Omega)$, by M_2 and (24), we can get

$$J(\omega) \geq \frac{b}{p} \|\omega\|^{p\gamma} - \int_{\Omega} a_n(x)|g(v) - Q|\omega|,$$

by the Hölder's inequality and the Poincaré equality, then

$$J(\omega) \geq \frac{b}{p} \|\omega\|^{p\gamma} - Ca_n(x)\|g(v) - Q\|_{L^{q'}} \|\omega\|,$$

since $p \geq 2$ and $\gamma \in (1, p_s)$, then $J(\omega)$ is bounded, coercive and weakly lower semicontinuous, such that $J(\omega)$ has a minimizer and the Euler equation is:

$$M(\|\omega\|^p)\Delta_p^s \omega + a_n(x)g(v) = f_n.$$

Moreover, such a minimizer is unique, by the strict convexity of J .

Fixed $n \in N$, let $v \in W_0^{s,p}(\Omega)$, define $\omega = S(v)$ to be the unique solution of the problem:

$$\begin{cases} M(\|\omega\|^p)\Delta_p^s \omega + a_n(x)g(v) = f_n & \text{in } \Omega, \\ D^\alpha u(x)|_{\partial\Omega} = 0. & \forall \alpha, \text{ with } |\alpha| \leq s-1. \end{cases} \tag{25}$$

We will use the ω as a test in (25), we get

$$M(\|\omega\|^p) \int_{\Omega} |\mathcal{D}_s \omega|^p = \int_{\Omega} f_n \omega - \int_{\Omega} a_n(x)g(v)\omega,$$

by (M_1, M_3) , (22), (24) and Hölder's inequality, we obtain

$$m_0 \|\omega\|^p \leq n \|\omega\|_{L^1} + \frac{n}{Q} \|g(v)\|_{L^{q'}} \|\omega\|_{L^{q'}}.$$

where $q \in [1, p_s^*)$, and q' is the conjugate exponent of q , by (23) and the Poincaré equality, it follows that

$$\|\omega\| \leq \sqrt[p-1]{\frac{2nC}{m_0}}$$

We take $R_0 = \min \left\{ \sqrt[p-1]{\frac{2nC}{m_0}}, k_0 \right\}$, so that the ball of radius $R_0 > 0$ is invariant

under s in $W_0^{s,p}(\Omega)$. In order to apply the Schauder's Fixed Point Theorem, apart from the invariant, we need to check the continuity and compactness of S as an operator from $W_0^{s,p}(\Omega)$ to $W_0^{s,p}(\Omega)$. So, the proof will be divided into two steps.

Step 1: We prove the continuity. In order to do this, we define $\omega_k = S(v_k)$ and $\omega = S(v)$ then:

$$\lim_{k \rightarrow \infty} \|\omega_k - \omega\|_{W_0^{s,p}(\Omega)} = 0 \quad \text{where} \quad \lim_{k \rightarrow \infty} \|v_k - v\|_{W_0^{s,p}(\Omega)} = 0. \tag{26}$$

Since the convergence of v_k in $W_0^{s,p}(\Omega)$, by (26) we obtain:

$$\begin{aligned} v_k &\rightarrow v \text{ in } L^q(\Omega), \\ v_k &\rightarrow v \text{ a.e in } \Omega, \quad q \in [1, p_s^*]. \end{aligned} \tag{27}$$

In fact, let $\{v_k\}_{k \in \mathbb{N}}$ be a sequence in $W_0^{s,p}(\Omega)$ converging to $v \in W_0^{s,p}$. To this end, by choosing $\omega_k \rightarrow \omega$ as a test function, we have

$$\begin{aligned} M(\|\omega_k\|^p) \int_{\Omega} |\mathcal{D}_s \omega_k|^{p-2} \mathcal{D}_s(\omega_k - \omega) &= \int_{\Omega} f_n(\omega_k - \omega) - \int_{\Omega} a_n(x) g(v_k)(\omega_k - \omega), \\ M(\|\omega\|^p) \int_{\Omega} |\mathcal{D}_s \omega|^{p-2} \mathcal{D}_s(\omega_k - \omega) &= \int_{\Omega} f_n(\omega_k - \omega) - \int_{\Omega} a_n(x) g(v)(\omega_k - \omega), \end{aligned}$$

by the inequality $(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq |x - y|^p$, for $x, y \in \mathbb{R}$, and $p \geq 2$, by Hölder's inequality, we obtain

$$\begin{aligned} m_0 \|\omega_k - \omega\|^p + (M \|\omega_k\|^p - M \|\omega\|^p) \int_{\Omega} |\mathcal{D}_s \omega|^{p-2} \mathcal{D}_s \omega \mathcal{D}(\omega_k - \omega) \\ \leq \frac{n}{Q} \|g(v_k) - g(v)\|_{L^{q'}} \|\omega_k - \omega\|_{L^q}, \end{aligned} \tag{28}$$

as $k \rightarrow \infty$,

$$\begin{aligned} (M \|\omega_k\|^p - M \|\omega\|^p) \int_{\Omega} |\mathcal{D}_s \omega|^{p-2} \mathcal{D}_s \omega \mathcal{D}(\omega_k - \omega) \\ \leq A \int_{\Omega} |\mathcal{D}_s \omega|^{p-2} \mathcal{D}_s \omega \mathcal{D}(\omega_k - \omega) \rightarrow 0 \end{aligned}$$

where $A = b\gamma(\|\omega_k\|^{\gamma-1} + \|\omega\|^{\gamma-1})$ is bounded, then by (27), and $g(v) \in L^{q'}(\Omega)$, there is $\|\omega_k - \omega\| \rightarrow 0$, hence S is continuous from $W_0^{s,p}(\Omega)$ to $W_0^{s,p}(\Omega)$.

Step 2. We prove S is compact, first we take a sequence $\{v_k\}_{k \in \mathbb{N}}$ that $\|v_k\| \leq C$, therefore by Rellich-Kondrachov Theorem, we obtain

$$\begin{aligned} v_k &\rightharpoonup v \text{ in } W_0^{s,p}(\Omega) \\ v_k &\rightarrow v \text{ in } L^q(\Omega), \quad 1 \leq q < p_s^* \end{aligned} \tag{29}$$

Since S is continuous, $\|S(v_k)\| \leq C$ with S is a positive constant, independent of k , such that,

$$\begin{aligned} S(v_k) &\rightharpoonup \tilde{\omega} \text{ in } W_0^{s,p}(\Omega) \\ S(v_k) &\rightarrow \tilde{\omega} \text{ in } L^q(\Omega), \quad 1 \leq q < p_s^* \end{aligned} \tag{30}$$

Because of the continuity of S , necessarily $\tilde{\omega} = S(v)$, so that proceeding as in (28), we can get:

$$\begin{aligned}
 & m_0 \|S(v_k) - S(v)\|^p + (M \|S(v_k)\|^p - M \|S(v)\|^p) \\
 & \times \int_{\Omega} |\mathcal{D}_s S(v)|^{p-2} \mathcal{D}_s S(v) \mathcal{D}(S(v_k) - S(v)) \\
 & \leq \frac{n}{Q} \|g(v_k) - g(v)\|_{L^{q'}} \|S(v_k) - S(v)\|_{L^q},
 \end{aligned}$$

as $k \rightarrow \infty$, the second term of left hand is vanished by (29) and (30). we can conclude

$$\lim_{k \rightarrow \infty} \|S(v_k) - S(v)\| = 0.$$

and therefore S is compact.

Hence, by lemma 2.1, there exists a solution $u_n \in W_0^{s,p}(\Omega)$ of problem (21), next we will prove the problem (1).

We will use the following function defined for $t \in R$, by

$$G_k(t) = \begin{cases} 0 & \text{if } |t| \leq k \\ t - k & \text{if } t > k \\ t + k & \text{if } t < -k \end{cases} \tag{31}$$

We use $G_{k_0}(u_n)$ as a test function in approximate problem (21), then

$$M(\|u_n\|^p) \int_{\Omega} |\mathcal{D}_s G_{k_0}(u_n)|^p + \int_{\Omega} a_n(x) g(u_n) G_{k_0}(u_n) = \int_{\Omega} f_n G_{k_0}(u_n)$$

by (M_1, M_3) , and (24) we obtain:

$$\begin{aligned}
 & m_0 \int_{\Omega} |\mathcal{D}_s G_{k_0}(u_n)|^p + \int_{\Omega} a_n(x) g(u_n) G_{k_0}(u_n) \\
 & \leq \int_{\Omega} f_n |G_{k_0}(u_n)| \leq Q \int_{\Omega} a_n(x) |G_{k_0}(u_n)|,
 \end{aligned} \tag{32}$$

by (9) and (23), this means:

$$m_0 \int_{\Omega} |\mathcal{D}_s G_{k_0}(u_n)|^p + \int_{\Omega} a_n(x) (|g(u_n)| - Q) |G_{k_0}(u_n)| \leq 0, \tag{33}$$

which by (33) implies that $\|u_n\|_{L^\infty(\Omega)} \leq k_0$ and the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

Next, we use u_n as a test function to deduce, such that,

$$M(\|u_n\|^p) \int_{\Omega} |\mathcal{D}_s u_n|^p + \int_{\Omega} a_n(x) g(u_n) u_n = \int_{\Omega} f_n u_n$$

by $\|u_n\|_{L^\infty(\Omega)} \leq k_0$ and (23) we can get

$$m_0 \int_{\Omega} |\mathcal{D}_s u_n|^p - \max_{|t| \leq k_0} |g(t)t| \int_{\Omega} a_n(x) \leq k_0 \int_{\Omega} |f_n|, \tag{34}$$

then,

$$m_0 \int_{\Omega} |\mathcal{D}_s u_n|^p \leq \max_{|t| \leq k_0} |g(t)t| \int_{\Omega} a(x) + k_0 \int_{\Omega} |f|, \tag{35}$$

and we obtain that $\{u_n\}$ is bounded in $W_0^{s,p}(\Omega)$ and a sequence, still denoted $\{u_n\}$, which converges weakly in $W_0^{s,p}(\Omega)$ and a.e. to u with $\|u\|_{L^\infty(\Omega)} \leq k_0$.

Moreover, using that $|a_n(x)g(v_n)| \leq a(x) \max_{|t| \leq k_0} |g(t)|$, we obtain by the dominated convergence theorem, the convergence of the sequence $\{a_n(x)g(v_n)\}$ to

$\{a(x)g(u)\}$, which together with the $L^1(\Omega)$ convergence of $f_n(x)$ to $f(x)$, we pass to the limit in the problem (21), we prove that u satisfies (1), with $u \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$.

4. A $p(x)$ -polyharmonic Kirchhoff Equation

In this section, we begin by recalling some basic results on the variable exponent Lebesgue and Sobolev spaces, see details in [22] [23].

As before, we define:

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : h_- > 1\} \tag{36}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $h_+ = \max_{x \in \Omega} h(x)$, $h_- = \min_{x \in \Omega} h(x)$. Let h be the function in $C(\bar{\Omega})$, an important role in manipulating the generalized Lebesgue-Sobolev spaces is played by $L^{h(\cdot)}(\Omega)$ spaces, which is the convex function: $\rho_{h(\cdot)} : L^{h(\cdot)}(\Omega) \rightarrow R$ defined by:

$$\rho_{h(\cdot)}(u) = \int_{\Omega} |u(x)|^{h(x)} dx. \tag{37}$$

Let p be a fixed function in $C_+(\bar{\Omega})$. We endow the Luxemburg norm:

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \tag{38}$$

by variable exponent Lebesgue space, it is a separable, reflexive Banach space. For $h \in C(\bar{\Omega})$, $1 \leq h \leq p$ in Ω , then the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is continuous and the norm of the embedding operator does not exceed $|\Omega|+1$. see [23].

p' be the function obtained by conjugating the exponent p pointwise, so that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \bar{\Omega}$, the p' belongs to $C_+(\bar{\Omega})$.

Note that, by Hölder-type inequality is valid:

$$\int_{\Omega} |u(x)v(x)| dx \leq C_{p(\cdot)} \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \text{ for all } u \in L^{p(\cdot)}(\Omega), v \in L^{p'(\cdot)}(\Omega). \tag{39}$$

with $C_{p(\cdot)} = 1 + \frac{1}{p^-} - \frac{1}{p^+}$ as proved in [23].

For $s = 1, 2, \dots$, we introduced the variable exponent Sobolev space $W^{s,p(\cdot)}(\Omega)$ defined by:

$$W^{s,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n, \text{ with } |\alpha| \leq s \right\}, \tag{40}$$

and endow the standard norm:

$$\|u\|_{W^{s,p(\cdot)}(\Omega)} = \sum_{|\alpha| \leq s} \|D^\alpha u\|_{p(\cdot)}.$$

We point out that the nonstandard growth condition of (p_-, p_+) type.

Lemma 4.1. (Theorems 1.3 of [24]) *If $(u_k)_k, u \in L^{h(\cdot)}(\Omega)$, with $1 \leq h_- \leq h_+ \leq \infty$, then the following relations hold:*

$$\begin{cases} \|u\|_{h(\cdot)} < 1 (=1, >1) \Leftrightarrow \rho_{h(\cdot)}(u) < 1 (=1, >1) \\ \|u\|_{h(\cdot)} \geq 1 \Rightarrow \|u\|_{h(\cdot)}^{h_+} \leq \|u\|_{h(\cdot)} \leq \|u\|_{h(\cdot)}^{h_-} \\ \|u\|_{h(\cdot)} \leq 1 \Rightarrow \|u\|_{h(\cdot)}^{h_+} \leq \|u\|_{h(\cdot)} \leq \|u\|_{h(\cdot)}^{h_-} \end{cases}$$

and $\|u_k - u\| \rightarrow 0 \Leftrightarrow \rho_{h(\cdot)}(u_k - u) \rightarrow 0 \Leftrightarrow u_k - u$ in measure in Ω and $\rho_{h(\cdot)}(u_k) \rightarrow \rho_{h(\cdot)}(u)$. In particular, $\rho_{h(\cdot)}$ is continuous in $L^{h(\cdot)}(\Omega)$.

From now on we also assume that $p \in C_+^{\log}(\bar{\Omega})$, where $C_+^{\log}(\bar{\Omega})$ is the space of all the functions of $C_+(\bar{\Omega})$, which are logarithmic Hölder continuous, there exists $\mathcal{R} > 0$, such that:

$$|p(x) - p(y)| \leq \frac{\mathcal{R}}{\log|x - y|}, \text{ for all } x, y \in \Omega.$$

With $0 < |x - y| \leq \frac{1}{2}$, the space $W_0^{s,p}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$

with respect to the norm $\|\cdot\|_{W^{s,p(\cdot)}(\Omega)}$.

Lemma 4.2. $W_0^{s,p(\cdot)}(\Omega)$ is a separable, uniformly convex, Banach space, see details in [22].

By the Poincaré inequality, see [25] [26], the equivalent norm for the space $W_0^{s,p(\cdot)}(\Omega)$ is given by:

$$\|u\|_{\mathcal{D}^{s,p(\cdot)}(\Omega)} = \sum_{|\alpha|=s} \|\mathcal{D}^\alpha u\|_{p(\cdot)}. \tag{41}$$

under this assumption, when $s = 2$, as a consequence for the main Coldéron-Zygmund results, there exists a constant $k_2 = k_2(n, p) > 0$ such that:

$$\|u\|_{\mathcal{D}^{2,p(\cdot)}(\Omega)} \leq k_2 \|\Delta u\|_{p(\cdot)} = k_2 \|\mathcal{D}_2 u\|_{p(\cdot)} \text{ for all } u \in W_0^{s,p(\cdot)}(\Omega). \tag{42}$$

We recall that the operator \mathcal{D}_s is defined in (4) is vectorial, when s is odd, we endow $[L^{p(\cdot)}(\Omega)]^n$ space with the norm

$$\|V\|_{p(\cdot)} = \sum_{i=1}^n \|\beta_i\|_{p(\cdot)}. \tag{43}$$

where $V = (\beta_1, \beta_2, \dots, \beta_n)$ with abuse of the notation we use the same symbol $\|\cdot\|_{p(\cdot)}$ to denote both the standard Luxemburg norm in the scalar space $L^{p(\cdot)}(\Omega)$ and the norm defined in (43) for the vectorial space $[L^{p(\cdot)}(\Omega)]^n$.

Proposition 4.1. See [17] for all $s = 1, 2, \dots$ there exists $k_s = k_s(n, p) > 0$, such that

$$\|u\|_{\mathcal{D}^{s,p(\cdot)}(\Omega)} \leq k_s \|\mathcal{D}_s u\|_{p(\cdot)} \text{ for all } u \in W_0^{s,p(\cdot)}(\Omega). \tag{44}$$

We endow the space $W_0^{s,p(\cdot)}$ with the norm $\|\cdot\|$, such that

$$\|\cdot\| = \|\mathcal{D}_s \cdot\|_{p(\cdot)}$$

Let p_s^* denote the critical variable exponent related to p defined for all $x \in \bar{\Omega}$, by the pointwise relation:

$$p_s^*(x) = \begin{cases} \frac{np(x)}{n-sp(x)} & \text{if } n > sp(x) \\ \infty & \text{if } n \leq sp(x) \end{cases} \tag{45}$$

If $s \geq n, p_- > 1$ for all $h \in C(\bar{\Omega})$, the Sobolev embedding $W_0^{s,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is continuous and compact. If $p_+ < \frac{n}{s}$ and $h \in C(\bar{\Omega})$, the embedding $W_0^{s,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is continuous whenever $1 \leq h(x) \leq p_s^*(x)$ for all $x \in \Omega$, there exists $\delta_{h^+} = \delta_{h^+}(n, p, s, \Omega) > 0$ such that:

$$\|u\|_{h(\cdot)} \leq \delta_{h^+} \|u\| \quad \text{for all } u \in W_0^{s,p(\cdot)}(\Omega)$$

Moreover, for $1 \leq h_- \leq h_+ < (p_s^*)_+$ for all $x \in \Omega$ (or equivalent $1 \leq h(x) < p_s^*$), then $W_0^{s,p(\cdot)}(\Omega)$ is compactly embedded in $L^{h(\cdot)}$, see details in [17] [22] [24] [27].

Consider problem (10) with $s = 1, 2, \dots$ and $p \in C_+^{log}(\Omega)$, such that either $p_+ < \frac{n}{s}$ or $p_- \geq \frac{n}{s}$. The Kirchhoff function $M : R_0^+ \rightarrow R_0^+$ is assumed to be continuous and to verify condition (M) given in introduction, where $p_s = \frac{(p_s^*)_+}{p_+}$.

We denote the Dirchlet function $\varphi : W_0^{s,p(\cdot)}(\Omega) \rightarrow R_0^+$:

$$\varphi(u) = \int_{\Omega} \frac{|\mathcal{D}_s u|^{p(x)}}{p(x)} dx, \tag{46}$$

where \mathcal{D}_s is given by (4).

We study problem (10) for a solution we understand:

$$\begin{cases} u \in W_0^{s,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \\ M(\varphi(u)) \int_{\Omega} |\mathcal{D}_s u|^{p(x)-2} \mathcal{D}_s u \mathcal{D}_s \varphi dx + \int_{\Omega} a(x) g(u) \varphi = \int_{\Omega} f(x) \varphi dx, \\ \varphi \in W_0^{s,p(\cdot)}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{47}$$

where \mathcal{D}_s is the operator in (4), and $\int_{\Omega} |\mathcal{D}_s u|^{p(x)-2} \mathcal{D}_s u \mathcal{D}_s \varphi dx$ is the $p(x)$ -polyharmonic operator, $\Delta_{p(x)}^s$ in weak sense.

In order to study the solvability of problem (10), we will analyze the associated approximate problem.

$$\begin{cases} M(\varphi(u_n)) \Delta_{p(x)}^s u_n + a_n(x) g(u_n) = f_n & \text{in } \Omega, \\ \mathcal{D}^\alpha u_n(x)|_{\partial\Omega} = 0 & \forall \alpha, \text{ with } |\alpha| \leq |s| - 1. \end{cases} \tag{48}$$

We recall some basic conditions and hypothesis by (22)-(24), and exist $h(x) \in L^{p(\cdot)}(\Omega)$, then $|g(s)| \leq h(x)$.

Theorem 4.1. *There is a solution $u_n \in W_0^{s,p}(\Omega)$ to the problem (48).*

Proof. We define

$$J(\omega) = \hat{M}(\varphi(\omega)) + \int_{\Omega} a_n(x) g(v) \omega - \int_{\Omega} f_n \omega.$$

Observe that, by (46), and lemma 4.1 we get:

$$\varphi(\omega) = \int_{\Omega} \frac{|\mathcal{D}_s \omega|^{p(x)}}{p(x)} dx \geq \frac{1}{p_+} \rho_{p(\cdot)}(\mathcal{D}_s \omega) \geq \frac{1}{p_+} \min \{ \|\omega\|^{p_+}, \|\omega\|^{p_-} \}. \quad (49)$$

we take $\delta = \frac{1}{p_+} \min \{ \|\omega\|^{p_+}, \|\omega\|^{p_-} \} > 0$, so that by (M_1, M_3) , there exists $m_0(\delta)$,

$$M(\varphi(\omega)) \geq m_0. \quad (50)$$

by M_2 and (49), we can get:

$$J(\omega) \geq b\delta^\gamma - \int_{\Omega} a_n(x) |g(v) - \mathcal{Q}| \omega.$$

by the Hölder's inequality and the Poincaré equality, then

$$J(\omega) \geq b\delta^\gamma - C a_n(x) \|g(v) - \mathcal{Q}\|_{L^{p(\cdot)}} \|\omega\|,$$

since $a(x) \geq 0$ and $g(v) \in L^{p(\cdot)}(\Omega)$, then $J(\omega)$ is bounded, coercive and weakly lower semicontinuous, such that $J(\omega)$ has a minimizer and the Euler equation is

$$M(\varphi(\omega)) \Delta_{p(x)}^s \omega + a_n(x) g(v) = f_n.$$

Moreover, such a minimizer is unique, by the strict convexity of J .

Fix $n \in N$, let $v \in W_0^{s,p(\cdot)}(\Omega)$, define $\omega = s(v)$ to be the unique solution of the problem:

$$\begin{cases} M(\varphi(\omega)) \Delta_{p(x)}^s \omega = f_n - a_n(x) g(v) & \text{in } \Omega, \\ D^\alpha u(x)|_{\partial\Omega} = 0. & \forall \alpha, \text{ with } |\alpha| \leq s-1. \end{cases} \quad (51)$$

We will use the ω as a test in (51) we get:

$$M(\varphi(\omega)) \int_{\Omega} |\mathcal{D}_s \omega|^{p(x)} = \int_{\Omega} f_n \omega - \int_{\Omega} a_n(x) g(v) \omega,$$

by (22), (24), (50) and Hölder's inequality, thus,

$$m_0 \rho_{p(\cdot)}(\mathcal{D}_s \omega) \leq n \|\omega\|_{L^1} + C_{p(\cdot)} \frac{n}{\mathcal{Q}} \|g(v)\|_{p(\cdot)} \|\omega\|_{p(\cdot)}.$$

1) If $\|\omega\| \geq 1$, such that

$$m_0 \|\omega\|^{p_-} \leq n C_1 \|\omega\| + C_{p(\cdot)} n C_2 \|\omega\|,$$

$$\|\omega\| \leq \sqrt[p_- - 1]{\frac{n(C_1 + C_{p(\cdot)} C_2)}{m_0}}.$$

2) If $\|\omega\| \leq 1$, such that

$$m_0 \|\omega\|^{p_+} \leq n C_1 \|\omega\| + C_{p(\cdot)} n C_2 \|\omega\|,$$

$$\|\omega\| \leq \sqrt[p_+ - 1]{\frac{n(C_1 + C_{p(\cdot)} C_2)}{m_0}}.$$

We take $R_0 = \min \left\{ \sqrt[p_- - 1]{\frac{n(C_1 + C_{p(\cdot)}C_2)}{m_0}}, \sqrt[p_+ - 1]{\frac{n(C_1 + C_{p(\cdot)}C_2)}{m_0}}, k_0 \right\}$, so that the

ball of radius $R_0 > 0$ is invariant under s in $W_0^{s,p(\cdot)}(\Omega)$. In order to apply the Schauder's Fixed Point Theorem, apart from the invariant, we need to check the continuity and compactness of s as an operator from $W_0^{s,p(\cdot)}(\Omega)$ to $W_0^{s,p(\cdot)}(\Omega)$. So, the proof will be divided into two steps.

Step 1: We prove the continuity. In order to do this, we define $\omega_k = S(v_k)$ and $\omega = S(v)$ then:

$$\lim_{k \rightarrow \infty} \|\omega_k - \omega\|_{W_0^{s,p(\cdot)}(\Omega)} = 0 \quad \text{where} \quad \lim_{k \rightarrow \infty} \|v_k - v\|_{W_0^{s,p(\cdot)}(\Omega)} = 0. \tag{52}$$

Since the convergence of v_k in $W_0^{s,p(\cdot)}(\Omega)$, by (52) we obtain:

$$v_k \rightarrow v \text{ in } L^{h(\cdot)}(\Omega), \quad v_k \rightarrow v \text{ a.e in } \Omega \quad h(\cdot) \in [1, p_s^*]. \tag{53}$$

In fact, let $\{v_k\}_{k \in \mathbb{N}}$, be a sequence in $W_0^{s,p(\cdot)}(\Omega)$ converging to $v \in W_0^{s,p(\cdot)}(\Omega)$.

To this end, by choosing $\omega_k \rightarrow \omega$ as a test function, we have

$$\begin{aligned} & M(\varphi(\omega_k)) \int_{\Omega} |\mathcal{D}_s \omega_k|^{p(x)-2} \mathcal{D}_s(\omega_k - \omega) \\ &= \int_{\Omega} f_n(\omega_k - \omega) - \int_{\Omega} a_n(x) g(v_k)(\omega_k - \omega), \\ & M(\varphi(\omega)) \int_{\Omega} |\mathcal{D}_s \omega|^{p(x)-2} \mathcal{D}_s(\omega_k - \omega) \\ &= \int_{\Omega} f_n(\omega_k - \omega) - \int_{\Omega} a_n(x) g(v_k)(\omega_k - \omega), \end{aligned}$$

by the inequality $(|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq |x - y|^p$ for $x, y \in \mathbb{R}$, $p \geq 2$, (22), (38) and Hölder's inequality, we obtain:

$$\begin{aligned} & m_0 \rho_{p(\cdot)}(\mathcal{D}_s(\omega_k - \omega)) + (M(\varphi(\omega_k)) - M(\varphi(\omega))) \int_{\Omega} |\mathcal{D}_s \omega|^{p(x)-2} \mathcal{D}_s \omega \mathcal{D}_s(\omega_k - \omega) \\ & \leq C_{p(\cdot)} \frac{n}{Q} \|g(v_k) - g(v)\|_{p'(\cdot)} \|\omega_k - \omega\|_{p(\cdot)}, \end{aligned} \tag{54}$$

as $k \rightarrow \infty$,

$$(M(\varphi(\omega_k)) - M(\varphi(\omega))) \leq \frac{1}{p_-} A \int_{\Omega} |\mathcal{D}_s \omega|^{p(x)-2} \mathcal{D}_s \omega \mathcal{D}_s(\omega_k - \omega) \rightarrow 0.$$

where $A = \max \{\|\omega_k\|^{p_-}, \|\omega_k\|^{p_+}\} + \max \{\|\omega\|^{p_-}, \|\omega\|^{p_+}\}$ is bounded, since (53) and g is continuous, such that

$$\rho_{p(\cdot)}(\mathcal{D}(\omega_k - \omega)) \rightarrow 0 \Rightarrow \|\omega_k - \omega\| \rightarrow 0,$$

hence s is continuous from $W_0^{s,p(\cdot)}(\Omega)$ to $W_0^{s,p(\cdot)}(\Omega)$.

Step 2. We prove S is compact, first we take a sequence $\{v_k\}_{k \in \mathbb{N}}$ that $\|v_k\| \leq C$, therefore by Rellich-Kondrachov Theorem, we obtain

$$\begin{aligned} & v_k \rightharpoonup v \text{ in } W_0^{s,p(\cdot)}(\Omega) \\ & v_k \rightarrow v \text{ in } L^{h(\cdot)}(\Omega), \quad 1 \leq h(\cdot) < p_s^* \end{aligned} \tag{55}$$

Since S is continuous, $\|S(v_k)\| \leq C$.

with C is a positive constant, independent of k , such that,

$$\begin{aligned} S(v_k) &\rightharpoonup \tilde{w} \text{ in } W_0^{s,p(\cdot)}(\Omega) \\ S(v_k) &\rightarrow \tilde{w} \text{ in } L^{h(\cdot)}(\Omega), \quad 1 \leq h(\cdot) < p_s^* \end{aligned} \tag{56}$$

Because of the continuity of S , necessarily $\tilde{w} = S(v)$, so that proceeding as in (54), we can get

$$\begin{aligned} &\left(M(\varphi(S(v_k))) - M(\varphi(S(v))) \right) \int_{\Omega} |\mathcal{D}_s S(v)|^{p(x)-2} \mathcal{D}_s S(v) \mathcal{D}_s (S(v_k) - S(v)) \\ &+ m_0 \rho_{p(\cdot)} \mathcal{D}_s (S(v_k) - S(v)) \\ &\leq C_{p(\cdot)} \frac{n}{Q} \|g(v_k) - g(v)\|_{p'(\cdot)} \|S(v_k) - S(v)\|_{p(\cdot)}, \end{aligned}$$

as $k \rightarrow \infty$, the first term of the left hand is vanished, then by (55) and (56)

$$\rho_{p(\cdot)} \mathcal{D}(S(v_k) - S(v)) \rightarrow 0 \Rightarrow \|S(v_k) - S(v)\| \rightarrow 0,$$

therefore, S is compact.

Given these conditions on S , Schauder's Fixed Point Theorem provides the existence of $u_n \in W_0^{s,p(\cdot)}(\Omega)$, such that $u_n = S(u_n)$, i.e., u_n solves:

$$\begin{cases} M(\varphi(u_n)) \Delta_{p(x)}^s u_n + a_n(x) g(u_n) = f_n & \text{in } \Omega, \\ D^\alpha u_n(x)|_{\partial\Omega} = 0 & \forall \alpha, \text{ with } |\alpha| \leq |s| - 1. \end{cases} \tag{57}$$

By Section 3, we also use $G_k(u_n)$ as a test function, then we can obtain the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, next we will use u_n as a test function, we can get a sequence u_n , which converges weakly in $W_0^{s,p(\cdot)}(\Omega)$ and a.e. to u with $\|u\|_{L^\infty} \leq k_0$. Finally, by the dominated convergence theorem and L^1 convergence, we prove that u satisfies (10) with $u \in W_0^{s,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Thus, we can learn some Kirchhoff equations by the above method.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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