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A Solution to the Famous "Twin's Problem"

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Abstract

In the following pages I will try to give a solution to this very known unsolved problem of theory of numbers. The solution is given here with an important analysis of the proof of formula (4.18), with the introduction of special intervals between square of prime numbers that I call silver intervals δ_M. And I make introduction of another also new mathematic phenomenon of logical proposition "In mathematics nothing happens without reason" for which I use the ancient Greek term "catholic information". From the theorem of prime numbers we know that the expected multitude of prime numbers in an interval $[x, x+dx]$ is given by formula $\pi(x) \simeq dx/\ln(x)$ considering that interval as a continuous distribution of real numbers that represents an elementary natural numbers interval. From that we find that in the elementary interval $[\nu, \nu+1)$ around of a natural number ν we easily get by $dx = 1$ the probability $p(\nu) \simeq 1/\ln(\nu)$ that has the ν to be a prime number. From the last formula one can see that the second part $p(\nu) > 1/(8\sqrt{\nu})$ of formula (4.18) is absolutely in agreement with the above theorem of prime numbers. But the benefit of the (4.18) is that this formula enables correct calculations in set N on finding the multitude of twin prime numbers, in contrary of the above logarithmic relation which is an approximation and must tend to be correct as ν tends to infinity. Using the relationship (4.18) we calculate here the multitude of twins in N, concluding that this multitude tends to infinite. But for the validity of the computation, the distribution of the primes in a random silver interval δ_M is examined, proving on the basis of catholic information that the density of primes in the same random silver interval δ_M is statistically constant. Below, in introduction, we will define this concept of "catholic information" stems of "information theory" [1] and it is defined to use only general forms in set N, because these represent the set N and not finite parts of it. This concept must be correlated to Riemann Hypothesis.

Keywords

Twin Problem, Twin's Problem, Unsolved Mathematical Problems, Prime

Number Problems, Millennium Problems, Riemann Hypothesis, Riemann's Hypothesis, Number Theory, Information Theory, Probabilities, Statistics

Stay away from infinity
Never look it in the eye

Friedrich Gauss

1. Introduction

The symbol q_ν (with $\nu \in N$) from now on will symbolize the prime numbers. There are the definitions of the endless sequence of the prime numbers that will be symbolized as follows: $q_0 = 1, q_1 = 2, q_2 = 3, q_3 = 5, \cdots$.

Let a random natural number $a_\nu = \nu$, $\nu \in N$ and let two more prime numbers q_α, q_β that are not equal to each other, $q_\alpha \neq q_\beta$. At first, it will be shown the independency of the fact that "the random Natural number a_ν, can be divided by the prime number q_α" from the fact that "the random Natural number a_ν can be divided by the prime number q_β". Let, also, without impairment of the generality of this proof, that $q_\alpha = 7$ and $q_\beta = 5$. Obviously, per 35 successive Natural numbers the 5 Natural numbers will be multiple of 7 and the 7 Natural numbers will be multiples of 5 and only <u>one</u> Natural number will be multiple of both 7 and 5. So, when selecting an Integer number a_ν from the infinite multitude of Natural numbers, the information of the fact that "a_ν is multiple of 7" does not interfere with the probability of the fact that "a_ν is multiple of 5" because every <u>five</u> multiples of 7 there will be only one that can be divided once again by the prime number 5, regardless of the information of the first fact. The Natural number a_ν will indeed belong to a group of thirty-five, if the multitude of Natural number *N* is divided into groups of thirty-five successive Natural numbers. Therefore, if the first fact, "a_ν is multiple of 7", is valid then a_ν will belong to the group of 5 (of a group of 35) that are multiples of 7. This group of 5, however will include only one multiple of 5, therefore a_ν will again have 1/5 probability of being multiple of 5, regardless of the information of the first fact that "it is multiple of 7". The proof is obviously generalized with the same methodology for any of the prime number q_α, q_β, not equal to each other.

It should be underlined that the Natural number 0 is divided by every Natural number, even when selecting a random integer a_ν, which will obviously have a probability equal to $1/q_k$ of being divided by the prime number q_k. Here, the probability has the meaning of the appearance frequency of a subset of Natural numbers, so when stating that fact Γ is independent of the probability-frequency that is referring to the subset of these Natural numbers, defined; based on an activity-criterion of their selection, it is meant that fact Γ is independent from the activity-criterion of their selection.

Opposite to that now, the information "a Natural number a_ν is divided by one non-prime number (compound) *i.e.* 18" immediately gives the information that a_ν will be divided by all the prime numbers that divide 18, therefore by 2 and 3, since $18 = 2^1 \cdot 3^2$, (or generally $18 = 2^1 \cdot 3^2 \cdot 5^0 \cdot 7^0 \cdot 11^0 \cdot 13^0 \cdots q_k^0 \cdots$). The sentence that was just shown is directly understood by writing the general form of a random Natural number $a_\nu = \nu$:

$$a_\nu = q_{i_1}^{j_1} q_{i_2}^{j_2} q_{i_3}^{j_3} \cdots q_{i_k}^{j_k}$$

where $j_1, j_2, \cdots, j_k$ are Natural, non-zero numbers.

Therefore, the fact "a_ν is divided by the prime number $q_n, n \in \{i_1, i_2, i_3, \cdots, i_k\}$" will be independent from the fact "the Natural number a_ν is divided by the prime number $q_m, m \in \{i_1, i_2, i_3, \cdots, i_k\}$" if $m \neq n$. This will be name Proposition of Divisibility Independence (PDI), which as shown is valid for the set N of Natural numbers.

Every random element selection from a given set A will be called Catholic Selection (CS), a term from ancient Greek language. In addition, as Catholic information will be defined a set K of catholic (logical) propositions that will be valid for infinity CS elements from different appropriate subsets (of finite multitude) of a set A. For example, a set K consisting of finite multitude of relations (written in general form) among infinite elements of another set A. An example of a logical proposition (or simply a proposition) that is a Catholic Proposition (CP), due to the fact that it is valid for infinite elements in N, specifically for infinite pairs of multiples of two prime numbers each time, is the PDI. The set of all the CP, meaning the propositions of catholic cardinality in N, that can be proved using PDI will now defined as catholic information of PDI for N. Owing that in mathematics nothing happens without a reason, it is concluded that if an algorithm of creation of a set A with infinite multitude is proven that does not create a property (proposition) P that will be catholically valid in A, therefore not implied by this algorithm (a set of finite multitude propositions) that the proposition P (e.g. a non-random statistical distribution) is valid in A, then P will not be valid in A. This last sentence will be named Proposition of Catholic Information.

1.1. The Fundamental Principles of This Research

They are: 1) The introduction of "catholic information" that we introduced above. 2) The extraction of all possible catholic or general relationships for which we define to must being them valid in all parts of set of natural numbers N and not only for special parts [2]. So the catholic formulas must use alphanumerical (by general expression) symbols for their catholic variables. 3) The definition of two kinds of intervals which we here call silvers and darks respectively. 4) The statistical calculation of catholic multitude of twin prime numbers in set N that is a calculation until infinity.

1.2. About the Study

After the definitions mentioned above, we pay attention on analyze the proof of catholic relationships (2.1) and (4.18). In the basis of "catholic information" concept and by use of (2.1), (4.18) we solve the problem in two ways: First by compact calculation using the appearance frequencies of twins in *N*, and second by using the dark intervals of *N*, which are increasing their sizes by a monster rate and however they are have infinite multitude. In this last process we initially proof that if some intervals not includes twin primes then this hypothesis drives to the existence of one twin prime on every top of these intervals. Thus we again arrive on the same conclusion. In pages before the relation (4.20), we examine the stability of frequency of prime numbers appearance in a random "silver interval", which is, a condition useful of validity of statistical calculations bellow.

1.3. Conclusions

Our conclusions from the below are: 1) The hypothesis of twin prime numbers is correct. 2) Maybe the concept of "catholic information" can be used as well in other Mathematical investigations. This concept is the other expression of fundamental proposition that "In mathematics nothing happens without reason". This concept of "catholic information" maybe could be connected by Riemann hypothesis [3] [4].

1.4. Twin Pairs

Here will be studied the twin pair problem. Let that an "honest" dice (in the shape of a normal hexagon) is thrown three consecutive times and the three consecutive positions are noted respectively *A*/*B*/Γ. Which twin pairs (meaning repetitions) of a particular number, *i.e.* of 5, are expected?

Answer [5]:

According to the sample space 6(6)6 = 216 facts there will be five cases of the form 5/5/*C*, where Γ is one of the results {1, 2, 3, 4, 6}, which has multitude of five. Similarly, there are five cases of the form *A*/5/5, where *A* is one of the results {1, 2, 3, 4, 6}, and only one is the 5/5/5. Due to the 10 first having 1 twin pair of 5, meaning one boundary "/" for the 5, and the last (one) case having 2 twin pairs, there will be altogether 10(1) + 1(2) = 12 total twin pairs in the 216 cases, and therefore the probability of twin pairs in *A*, *B*, Γ facts, (which is "the 5" in each one of the ordered throw) will be 12/216 = 1/18. On the other aspect of the counting method based on the probability $p = 1/6$ to get number 5 in a throw, there will be probability $p^2 = (1/6)(1/6) = 1/36$ for the twin pair of *A*, *B* and similarly 1/36 for the twin pair of *B*, Γ, so expected probability 1/36 + 1/36 = 2/36 = 1/18, which shows that in this example the two non-independent and at the same time non-incompatible facts $X = A \cap B$ and $Y = B \cap \Gamma$ will be counting their respective percentages without considering their dependency and compatibility. The condition, however, for the proper counting is the independency of *A*, *B*, Γ which is true. Therefore, the expected multitude of twin pairs of 5 will be 216·(1/18) = 12 cases of twin pairs, as found above. Generalizing the above problem for *N* successive throws of a (fair) dice the expected percentage of the twin pairs of the number 5, of the "fair" dice, for the $N - 1$ multitude of the boundaries "/" of the facts of the throws $A_1/A_2/\cdots/A_N$ would be:

$$P=(N-1)p^2=(N-1)/36 \tag{1.1}$$

One could try to prove this last relation in the case of a/b/c/d using the first method with the sample space. However, in this case the counting of the probability P_0 will be completely different in order for at least one of the above facts X and Y to occur. This probability will be counted as follows:

$$P_0(X\cup Y)=P_0(X)+P_0(Y)-P_0(X)P_0(Y\mid X)=\frac{1}{36}+\frac{1}{36}-\frac{1}{36}\frac{1}{6}=\frac{11}{216}$$

The probability $P_0(Y\mid X)$ above is 1/6, because when fact X occurred the information that the second pair has already given 5 is provided, so the $P_0(Y\mid X)$ will correspond only to the probability "the third dice will be again 5", and it is obviously 1/6. In the sample space of the 216 facts there will indeed be 5 + 5 + 1 = 11 of these cases (and not 12 as before), since 5 cases will be of the form 5/5/Γ, the other 5 of the form *A*/5/5 and only 1 will be 5/5/5. The reader perceives that the differentiator is the key phrase in the above sentence: at least one.

Coming to an end, by proving the independency of the events; the "divisibility of the random natural number a_ν by the random prime number q_α (of its sub-sequence)" from the "divisibility of the random natural number a_μ by the random prime number q_β (of its sub-sequence)" the definition of these two events independency will be repeated: Any two events Γ_1 and Γ_2 will be considered independent from each other in a set (their range) *A*, "if and only if the frequency-probability of the elements in A where Γ_1 appears, is the same as the frequency-probability of the elements in A where Γ_1 and Γ_2 appear together (once each)" and additionally the last sentence ("…") is valid if C_1 and Γ_2 are interchanged in it.

2. Specification of the Indefinite Frequency-Probability Appearance of Prime Numbers

A set $Y_M=\{q_1,q_2,q_3,\cdots,q_M\}$ is taken as a sub-sequence of the interval $\delta_M=\left[q_M^2,q_{M+1}^2\right)$. The internal dm will be named Silver interval. It should also be clarified the reason why for the study of the natural numbers. The interval δ_M will be named Silver Interval.

It should also be clarified the reason why in the study of natural number $a_\nu=\nu\in\delta_M$ the prime numbers q_λ were chosen as elements of Y_M the natural numbers with the characteristic: It is noticed that if the random natural number a_ν is divided by another positive natural number $\kappa>\sqrt{a_\nu}$, then the quotient of this division, let natural number μ, will satisfy the relation $\mu<\sqrt{a_\nu}$. It is obvious since $\mu\cdot\kappa=a_\nu$. In other words, if $\mu>\sqrt{a_\nu}$ was true, then it would be $a_\nu=\mu\cdot\kappa>\sqrt{a_\nu}\sqrt{a_\nu}=a_\nu$, which is absurd. Therefore, if a_ν has a divisor greater than $\sqrt{a_\nu}$ then it will also have a divisor smaller than $\sqrt{a_\nu}$, which is followed by the fact that is a natural number a_ν is not divided by none of the other prime numbers $q_\lambda<\sqrt{a_\nu}$, then it will not be divided by any other prime number greater than $\sqrt{a_\nu}$. Because if this last statement were to be true, then according to the aforementioned facts there would be a divisor smaller than $\sqrt{a_\nu}$, which would either be prime or it would be analyzed in product of prime numbers that are for sure smaller than $\sqrt{a_\nu}$. The conclusion drawn is that in the case where a_ν, does not have as a divisor a prime number smaller than $\sqrt{a_\nu}$, then $\sqrt{a_\nu}$ is the prime number. Therefore, the prime numbers that define as possible divisors of a_ν being prime, are only the prime numbers that are all smaller than its square root, which is the sub-sequence of prime numbers of a_ν, that was defined above.

The probability $\bar{P}_\nu$, that $a_\nu = \nu$ is not divided by any of the elements of the sub-sequence of $2,3,5,7,\cdots,q_{M\nu}$ (defining $q_0 = 1, q_1 = 2, q_2 = 3, q_3 = 5, \cdots$) will be equal to the products of the probabilities not to be divided by $2,3,5,7,\cdots,q_{M\nu}$. These probabilities will respectively be $1-\frac{1}{2},1-\frac{1}{3},1-\frac{1}{5},1-\frac{1}{7},\cdots,1-\frac{1}{q_{M\nu}}$, due to the fact that M_ν multitude events $A_1, A_2, A_3, \cdots, A_{M\nu}$ that state respectively that the natural number $a_\nu = \nu$ is divided by the prime numbers $2,3,5,7,\cdots,q_{M\nu}$ of its sub-sequence, which according to PDI that was previously proven, per two events that are independent from each other. It is obvious that 1/2 is the probability of the natural number a_ν, to be divided by 2, that is to be an even number with complimentary probability the 1 − (1/2) not to be divided by 2. Similarly, 1/3 is the probability of a_ν to be divided by 3, while 1 − (1/3) is the complimentary probability to not be divided by 3 and so on for every term of the sub-sequence. The $A_1, A_2, A_3, \cdots, A_{M\nu}$, however are not every two exclusive events from each other, owning to the fact that the divisibility of the natural number $a_\nu = \nu$ by a number of its sub-sequence do not exclude its ability to be divided by another term of that sub-sequence. For example, the natural number $a_{30} = 30$ has as a sub-sequence of prime numbers 2, 3, 5 and the fact that it can be divided by another of these three terms. It is indeed divided by the term 3. The probability $\bar{P}_\nu$ of the following Equation (2.1) is a unique enumerate of prime numbers, but (initially) in not-well-defined intervals. The following definition is derived from the available information of the production of infinite element of set *N*, provided that according to the definition of Shannon the probability $\bar{P}_\nu$ is another way of expressing information. Based on the fact that the events $A_1, A_2, A_3, \cdots, A_{M\nu}$ are every two independent from each other, one concludes that the probability-frequency of appearance of all the events above will simply be the product of all their individual probabilities therefore one will obtain the relation.

$$\bar{P}_\nu = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots\left(1-\frac{1}{q_{M\nu}}\right) \tag{2.1}$$

It will, however be proven and in another way the relation (1.2) [5]. Let $\bar{P}_\nu$ the probability that the natural number a_ν is divided with at least one term of the sub-sequence of its prime numbers. Then the probability not to be divided by any of its terms will obviously be complimentary of the probability:

$$\overline{P}_\nu = 1 - \overline{P}'_\nu \tag{2.2}$$

The probability $\overline{P}_\nu$, for the events $A_1, A_2, A_3, \cdots, A_{M\nu}$, that are per two independent to each other, which state that the given natural number a_ν is divided, respectively, by the prime natural numbers $2,3,5,7,\cdots,q_{M\nu}$ of its sub-sequence, is:

$$\begin{aligned}\overline{P}'_\nu &= \overline{P}'_\nu\left(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{M\nu}\right)\\ &= \sum \overline{P}'_\nu\left(A_j\right) - \sum \overline{P}'_\nu\left(A_{j_1} \cap A_{j_2}\right) + \sum \overline{P}'_\nu\left(A_{j_1} \cap A_{j_2} \cap A_{j_3}\right)\\ &\quad - \cdots + \left(-1\right)^{M\nu-1} . \sum\left(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{M_\nu}\right)\end{aligned}$$

and because the facts $A_1, A_2, A_3, \cdots, A_{M\nu}$ are as previously mentioned per two independent from each other, the relation above becomes

$$\overline{P}'_\nu\left(A_{j_1} \cap A_{j_2}\right) = \overline{P}'_\nu\left(A_{j_1}\right) \cdot \overline{P}'_\nu\left(A_{j_2} \mid A_{j_1}\right) = \overline{P}'_\nu\left(A_{j_1}\right) \cdot \overline{P}'_\nu\left(A_{j_2}\right)$$

The second part of the equation in the last relation is due to the independency of the fact A_{j_1}, A_{j_2}. Similarly there is:

$$\begin{aligned}\overline{P}'_\nu\left(A_{j_1} \cap A_{j_2} \cap A_{j_3}\right) &= \overline{P}'_\nu\left(A_{j_1}\right) \cdot \overline{P}'_\nu\left(A_{j_2} \mid A_{j_1}\right) \overline{P}'_\nu\left(A_{j_3} \mid A_{j_2} \cap A_{j_1}\right)\\ &= \overline{P}'_\nu\left(A_{j_1}\right) \cdot \overline{P}'_\nu\left(A_{j_2}\right) \overline{P}'_\nu\left(A_{j_3}\right)\end{aligned} \quad \text{etc.}$$

So, the probability $\overline{P}_\nu$ results to expression

$$\overline{P}'_\nu = \sum \frac{1}{q_j} - \sum \frac{1}{q_{j_1}} \frac{1}{q_{j_2}} + \sum \frac{1}{q_{j_1}} \frac{1}{q_{j_2}} \frac{1}{q_{j_3}} - \cdots + \left(-1\right)^{M\nu-1} \cdot \sum \frac{1}{q_1} \frac{1}{q_2} \cdots \frac{1}{q_{M_\nu}} \tag{2.3}$$

In the above sums the indicators $j_1, j_2, j_3, \cdots$ are as known, per two different from each other and obviously $\overline{P}'_\nu\left(A_{j_\kappa}\right) = 1/q_{j_k}$ is the probability, of the event A_{j_k}, where the natural number a_ν is divided by a prime number $q_{j_\kappa} = q_\lambda$ of its sub-sequence.

Let now be the Polynomial

$$f\left(x\right) = x^{M\nu} + a_1 x^{M\nu-1} + a_2 x^{M\nu-2} + \cdots + a_{M\nu-1} x^1 + a_{M\nu} x^0 \tag{2.4}$$

with roots $x_1, x_2, x_3, x_4, x_5, \cdots, x_{M\nu}$ respectively the fractions

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \cdots, \frac{1}{q_{M\nu}}$$

Therefore one has

$$f\left(1\right) = 1 + \left(a_1 + a_2 + a_3 + \cdots + a_{M\nu}\right) \tag{2.5}$$

and now the known polynomials give

$$a_1 = -\sum x_j = -\sum \frac{1}{q_j}, \quad a_2 = +\sum x_{j_1} x_{j_2} = \sum \frac{1}{q_{j_1}} \frac{1}{q_{j_2}} \tag{2.6}$$

and also

$$f\left(x\right) = \left(x - x_1\right)\left(x - x_2\right) \cdots \left(x - x_{M\nu}\right) = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right) \cdots \left(x - \frac{1}{q_{M\nu}}\right) \tag{2.7}$$

The relations (2.3), (2.5), (2.6) result in $f\left(1\right) = 1 - \overline{P}'_\nu$ and due to relations

(2.2) and (2.7) where $x = 1$ one concludes in (2.1).

The relation above (2.1) indefinitely gives the probability to be equal to the positive natural number a_ν, because it cannot be in a defined set $\delta = [\nu_1, \nu_2)$ where the probability $\overline{P}_\nu$ is counting the exact multitude Q_δ of the prime numbers in it: $Q_\delta = \sum_{\nu_1}^{\nu_2} \overline{P}_\nu$. The exact counting as shown below, will be done in appropriate intervals, that have already been named silver intervals δ_M, and with the use of an unknown probability $\overline{P}_\nu$, that will be proven to be greater than a useful expression, that will be related to (2.1). From the above it is becoming clear that all the natural number that have the same sub-sequence of prime numbers should constitute an interval such as δ_M, *i.e.* the intervals:

$$\delta_0 = (1^2, 2^2),\ \delta_1 = [2^2, 3^2),\ \delta_2 = [3^2, 5^2),\ \delta_3 = [5^2, 7^2)$$

respectively correspond in the sub-sequences of the prime number {(1)}, {(1), 2}, {(1), 2, 3}, and these are defined as the four prime silver intervals that clearly include only the natural numbers. For example the interval δ_1 includes a multitude of five numbers. Number one (1) was in purpose set in bracket above so as to declare that number one is not included in the elements of these subsets, because number 1 is not a prime number. It should be clarified that a definition of prime numbers is that prime numbers are all the multiples of number one (therefore they are natural numbers) that have the attribute to not be divided by one another. So the prime numbers define the set of all the possible independent repetitions of number one, since none of them is the repetition of the other. It is noticed that the first of the above silver intervals, that is δ_0, has as a sub-sequence the empty set and includes two prime numbers which are 2 and 3, the second δ_1 has as its sub-sequence the unit-set with 2 as an element and includes two prime numbers, 5 and 7, while the third one includes five prime numbers, the fourth includes sixteen prime numbers and so on. Furthermore, the enumerators-probabilities that were mentioned, $\overline{P}_\nu$ and P_ν, will have constant value in every specific silver interval, which will be explained in details, and be proven in Section 4. In this section it will be defined that these values will be dependent, according to relation (2.1), only on the sub-sequence of prime number, which is the same for all natural numbers and only of the specific silver interval:

$$\overline{P}_\nu = \text{constant}, P_\nu = \text{constant}, \forall\, \nu \in \delta_\kappa,$$

with κ function of ν.

Now certainly the definition of silver intervals is justified

$$\delta_\kappa = [q_M^2, q_{M+1}^2)$$

And seeing that $M = M_\nu = \kappa$, $\forall\, \nu \in N$ one obtains

$$\delta_M = [q_M^2, q_{M+1}^2) \tag{2.8}$$

with $M \in N$ and $Y_M = \{q_1, q_2, q_3, \cdots, q_M\}$ its respective sub-sequence.

A check of $\bar{P}_\nu$ by calculating the counting of $\bar{Q}_\delta = \sum_{\nu=\nu_1}^{\nu_2} \bar{P}_\nu$ and with the use of a computers, via (2.1) with $\nu_1 = 4$ and ν_2 an arbitrarily large natural number, it is shown that the countable multitude of prime numbers, whilst at the beginning coincides with the real, it becomes more and more larger than that of the real multitude of prime numbers, as ν_2 is increased. The reason why this is happening will be explained below and will be proven that the new precise probability $p(\nu) = P_\nu$, that was mentioned before will tally the precise multitude of prime numbers: $\bar{Q}_\delta = \sum_{\nu=\nu_1}^{\nu_2} \bar{P}_\nu$, although unknown here, it will satisfy in every particular silver interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$ a very useful inequality, which will be named fundamental inequality of the silver intervals.

It will also be proven true that for the probability of the relation (2.1):

$$\lim_{\nu\to\infty} \bar{P}_\nu = 0 \tag{2.9}$$

In the beautiful book "the secret life of numbers" professor of Mathematics Andrew Hodges mentions that one of the smartest tricks in the history of mathematics is the Euler transformation below:

$$\begin{gathered}\left(1+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)\left(1+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)\left(1+\frac{1}{5^1}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots\right)\\ \cdots\left(1+\frac{1}{q_\nu^1}+\frac{1}{q_\nu^2}+\frac{1}{q_\nu^3}+\frac{1}{q_\nu^4}+\cdots\right)\cdots = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\end{gathered} \tag{2.10}$$

The second part of 2.10 is the known harmonic sequence that as known is inexhaustible and corresponds to Riemann's function:

$$\zeta(z) = 1/1^z + 1/2^z + 1/3^z + \cdots$$

The proof of (2.10) results directly from the general form of writing the natural number:

$$a_\nu = q_{i_1}^{j_1} q_{i_2}^{j_2} q_{i_3}^{j_3} \cdots q_{i_k}^{j_k} \tag{2.11}$$

That was mentioned in the beginning of Section 2. Executing retrospectively the multiplication of the 1st part it will indeed lead to the 2nd part due to the appearance of all the combinations of (2.11) in the denominators, and so all the integers positive numbers etc.

The relation (2.10) is known from the time of Gauss, that leads directly to the conclusion found by Euclid thousands years ago, which is that the prime numbers are infinite. If they were not then the first part of (2.10) would be a product of finite multitude of derivatives, where each one of them would converge and therefore this product would not deviate from infinity. This however, is absurd, since the second part would also not deviate, which is indeed deviating, because it is the harmonic sequence that was previously mentioned. However, the author's shorter proof can be given here: "the relation (2.11) includes exponents that are natural integer numbers and therefore each one of them is developed again in the same way $j_\rho = q_{s_1}^{m_1} q_{s_2}^{m_2} q_{s_3}^{m_3} \cdots q_{s_r}^{m_r}$. However, the new exponents m_n, will be developed again in the same way and so on. It is therefore obvious, that if the prime numbers had finite multitude in *N*, then the combinations for the representation the natural numbers a_ν would be depleted, since these combinations would not obviously have the advantage of infinite different per-two mathematical (tree-like) representations. Hence, in that case the infinite natural numbers would not be represented by the relation (2.11) which is absurd".

One more not so well known relation of the bibliography (that is also mentioned in Section 6 of Andrew's Hodges book) for a random prime number q_ν, as symbolized here, is:

$$\left(1-\frac{1}{q_\nu}\right)^{-1}=1+\frac{1}{q_\nu^1}+\frac{1}{q_\nu^2}+\frac{1}{q_\nu^3}+\frac{1}{q_\nu^4}+\cdots \tag{2.12}$$

It is noted that (2.12) could be proven easily from the known Taylor formulation [3]:

$$f(x)=f(x_0)+\frac{(x-x_0)^1}{1!}f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\cdots$$

Plugging $x_0=0$ and $f(x)=(1-x)^{-1}$, and placing afterwards the differentiations of *x*, where *x* is $1/q_\nu$. So the same relation which can be used to convert the functions such as $\cos(x), \mathrm{e}^x$ etc. was used in a sequence of infinite terms.

Now, because the second part of (2.10) tends towards infinity when $\nu\to\infty$, as stated before, combining the relations (2.1), (2.10), (2.12) one immediately concludes to the proven (2.9)

The 1st part of (2.10) is equal to the function zeta

$$\zeta(1)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots \tag{2.13}$$

Because when executing the 1st part the multiplication in the denominators of the fractions, all the combinations of the products of all the prime derivatives, raised in all the powers, to infinity will appear. Hence, according to the relation $a_\nu=q_{i_1}^{j_1}q_{i_2}^{j_2}q_{i_3}^{j_3}\cdots q_{i_k}^{j_k}$ (which was reported in the beginning of this Section 2) the result will be all the natural numbers, therefore function $\zeta(1)$. Combining this fact with the one from (2.10) (2.12) and also with (2.1) the following known relation is concluded:

$$\zeta(1)=\lim_{\nu\to\infty}\frac{1}{\overline{P}_\nu} \tag{2.14}$$

3. The Tracker of Infinity (Eratosthenes Sieve)

One should think about the endless axis of positive natural numbers a_ν, that starts from 0 and contains, in equal distances, the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, … coloured light blue which are simultaneously stationary observers. Furthermore, the author suggests to visualise one more tracker that starts from position number 2 and walks towards the right side of the line of the natural numbers executing the following order "Every time you encounter a light blue natural number a_ν send on your right, to the abyssal infinity, a message to the light blue observers-natural numbers, which are integer multiples of a_ν, that says change your colour to black. The a_ν remains light blue and is registered in your log book". What will happen? Simply. In the route $2 \to 3$ all the even natural numbers will be black to infinity except of course for number 2. These will be called second-multiple (2-multiples) not including number 2. In the route $3 \to 4$ there will be black numbers except from the second-multiples and all the multiples of 3 to infinity except for the natural number 3. These multiples of 3 not including the initial number 3 will be called third-multiples (3-multiples) numbers. When the tracker reaches number 4, however, finds it black and does not send a message for colour changing to the observers-natural numbers. Number 5 is found light blue (unmarked) and a new message is sent, according to the order given, to mark all the integer multiples of the natural number 5 except for 5, with the colour black. The multiples of 5 except for 5, will be called fifth-multiples numbers and so on. Therefore, the tracker in this journey leaves behind as light blue only the prime numbers that have been registered in the log book. The integers, third-multiples, fifth-multiples, seventh-multiples etc. meaning all the natural numbers that are not prime numbers and have been marked black will be called prime-multiples numbers. According to what was shown regarding the silver interval during the route $2^2 \to 3^2$, that is, in the tracker's route inside the silver interval $\delta_1 = \left[2^2, 3^2\right)$, an encounter with all the now blackened multiples of 2 will take place. The numbers 5, 7 will remain light blue during this route and they are prime numbers. Similarly, during the new route in $\delta_2 = \left[3^2, 5^2\right)$, the tracker will encounter, marked in black, all the multiples of 2 and 3. That means that the multiples of the subsequence of the prime numbers in the silver interval that the tracker crosses each time, will be marked black.

The prime numbers that are integer multiples of the random prime number q_s, will be called q_s-multiples. Additionally, in a random silver interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$ the natural numbers will be called:

$$q_M^2, q_M^2 + 1, q_M^2 + 2, q_M^2 + 3, \cdots, q_{M+1}^2 - 1$$

respectively as 1st, 2nd, 3rd, … position in the interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$.

According to this last definition the question now is; which is the position of a first appearing q_s-multiple in $\delta_M = \left[q_M^2, q_{M+1}^2\right)$ and which is the position of the last q_s-level in this silver interval. This positions are called $\theta_s(1, M)$ and $\theta_s(\tau, M)$ respectively, ensuring that the symbols represent the information given accordingly. Here 1 represents the 1st and τ represents the last (from the Greek word "τελευταίος" that means last). The interval $b_s(M) = \left[\theta_s(1, M), \theta_s(\tau, M)\right]$ that includes only natural numbers will be called band of q_s-multiples (prime-multiples) of the silver interval δ_M. Furthermore, $d_M = q_{M+1}^2 - q_M^2$ symbolizes the "length" of a random silver interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$. Since the two successive prime numbers q_M, q_{M+1} will have a difference of greater or equal to 2 (2 occurs when they are twin prime numbers) there will be:

$$d_M = q_{M+1}^2 - q_M^2 \geq (q_M + 2)^2 - (q_M)^2 = 4(1 + q_M) > 4q_M \tag{3.1}$$

And since $q_M \geq q_s, \forall s = 1,2,3,4,\cdots,M$ one concludes that:

$$d_M = q_{M+1}^2 - q_M^2 > 4q_M \geq 4q_s, \forall s = 1,2,3,4,\cdots,M, \forall M > 1 \tag{3.2}$$

Also, due to the distance of the origin $\theta_s(1,M)$ of any band $b_s(M)$ from the origin of the silver interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$ being always less that the distance q_s of the two successive q_s-levels of its subsequence (that means $q_s \leq q_M$) there will be:

$$0 \leq \theta_s(1,M) - 1 < q_s,\ d_M - \theta_s(\tau,M) < q_s,\ \forall q_s \leq q_M \tag{3.3}$$

The "equal to 0" in the first relation (3.3) represents the case where $q_s = q_M$.

For example:

$$\theta_M(1,M) = 1,\ \theta_1(1,M) = 2,\ \theta_1(\tau,M) = d_M, \forall M > 1$$

Because q_M^2, q_{M+1}^2 are odd natural numbers as squares of prime numbers (that are odd). Also, it should be reminded that $M = M_\nu = 1,2,3,4,5,6,\cdots$. Moreover, it will be symbolized as

$$l_s(M) = \theta_s(\tau,M) - \theta_s(1,M) + 1$$

the "length" of a random band $b_s(M)$. Therefore as an example one finds:

$l_1(M) = d_M - 2 + 1 = d_M - 1, \forall M > 1$, for example $l_1(2) = 16 - 1 = 15$

But $l_2(2) = 16$, $l_3(2) = 11$, $l_2(3) = 22$, $l_3(3) = 21$, $\cdots$.

As "lengths" for both the silver intervals and the bands were defined not the geometrical distances of the two ends but the multitude of the natural numbers that are included in the interval that corresponds each time to the silver interval (or band). In contrast, their normal length $l_s(M) - 1$ could be named geometrical length.

4. The Silver Intervals, the Fundamental Inequality and a First Solution

Summing up, the silver interval is defined as δ_M using the relation

$$\delta_M = \left[q_M^2, q_{M+1}^2\right) \tag{4.1}$$

with $M = M_\nu \in N$ and $q_1, q_2, q_3, \cdots, q_{M_\nu}$, being the respective subsequence of Y_M. Reminding that this subsequence of $q_1, q_2, q_3, \cdots, q_{M_\nu}$, of a silver interval consists of M successive prime natural numbers. q_s-multiples were named the multiples of the random q_s prime number and for the random silver interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$ the defined band of the specific q_s-multiple in the interval of natural numbers is:

$$b_s(M) = \left[\theta_s(1,M), \theta_s(\tau,M)\right] \tag{4.2}$$

Between the first (1) and the last (τ) position of the q_s-multiple in the silver interval δ_M. In general $\theta_s(\kappa,M)$ will be a position in the silver interval that corresponds to the κ^{th} in a row q_s-multiple. Furthermore, it was symbolised

$$d_M = q_{M+1}^2 - q_M^2 \tag{4.3}$$

the "length" of a random silver interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$. Also, the multitude of the natural numbers that contains a random band $b_s(M)$ as its "length" was defined and symbolized as

$$l_s(M) = \theta_s(\tau, M) - \theta_s(1, M) + 1 = nq_s + 1 \tag{4.4}$$

In this length $l_s(M)$ there are obviously $n + 1$ q_s-multiples. Also the distance nq_s of the $b_s(M)$ band will be called, as stated before its geometrical length. At last the relation (3.2) was proven:

$$d_M = q_{M+1}^2 - q_M^2 > 4q_M \geq 4q_s, \forall s = 1,2,3,4,\cdots,M \quad \text{and} \quad \forall M > 1 \tag{4.5}$$

It should be clarified that all the composites of δ_M are necessarily q_s-multiples of its subsequence, as it was proven in Section 2.

In the below **Figure 1** we see that the band $b_s(M) = [\gamma, \zeta]$ has a "length" $d_M = 3q_s$ and not $d_M > 4q_s$ as the (4.5) implies above, however that was done die to the simple supervision and obviously it will not interfere with the proving methodology that will be shown below. (A composite natural number is the one that is not prime). In general all the prime-multiples numbers of a random prime number q_s (meaning all its integer multiples, that were named q_s-multiple numbers) function as erasers in the list of prim number candidates, since they erase the possibility of being the natural numbers with which the prime numbers coincide. Using this interpretation they will be named q_s-multiples erasers. These are the ones that the tracker changes their color from light blue to black (erasure) as soon as the tracker meets the first q_s number on the unending travel of the axis $a_v = v$ of natural numbers.

In **Figure 1** are shown the two boarders $G = q_M^2$ and $L = q_{M+1}^2 - 1$ of the random silver interval $\delta_M = \left[q_M^2, q_{M+1}^2\right)$ where:

$$L = l - 1 = q_{M+1}^2 - 1 \tag{4.6}$$

Also in that same figure are seen the two bands $b_s(M) = [\gamma, \zeta]$ of q_s and the $b_\rho(M) = [\alpha, \eta]$ of q_ρ. Where $\lambda, G, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, L, l, \mu$ obviously natural numbers on the axis of natural numbers a_v. The band $b_s(M)$ with geometrical length nq_s and $n = 3$ for **Figure 1**, will include $n + 1$ multitude of

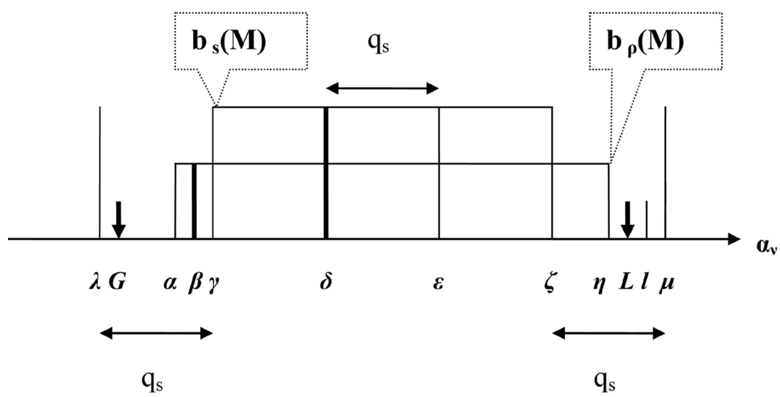

Figure 1. Silver interval.

q_s -multiples erasers that is **Figure 1** are the $\gamma,\delta,\varepsilon,\zeta$ with properties: $\gamma-\lambda=\delta-\gamma=\varepsilon-\delta=\zeta-\varepsilon=\mu-\zeta=q_s$. In addition, β is a random q_ρ -multiple number of another random band $b_\rho(M)$ that as shown in the figure happens to be overlapping with $b_s(M)$. The relation (2.1) shown in Section 2:

$$\bar{P}(M)=\bar{P}_v=\prod_{j=1}^{M_v}\bar{F}_j=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots\left(1-\frac{1}{q_{Mv}}\right) \tag{4.7}$$

is based on the assumption that the density of q_j -multiples erasers in the silver interval δ_M, where $j=1,2,3,\cdots,M$ is:

$$\bar{\rho}_j=1:q_j=\frac{1}{q_j} \tag{4.8}$$

Thus, the active multitude of q_j -multiples erasers in δ_M, with "length" d_M will be:

$$\bar{K}_j=\frac{d_M}{q_j} \tag{4.9}$$

Hence, the non-erased natural numbers in δ_M from the q_j -multiples will have as active multitude

$$\bar{F}_j=\left(1-\frac{1}{q_j}\right)d_M \tag{4.10}$$

However, owing to two random bands $b_s(M)$ and $b_\rho(M)$ of δ_M not having the same boarders, as shown in **Figure 1**, meaning that theirs boarders do not coincide (*i.e.* $\alpha\neq\gamma,\zeta\neq\eta$ in **Figure 1**) the active multitude of the erasers of the assumption in δ_M will not coincide with the true multitude of erasers.

Let K_s the unknown multitude of q_s -multiples of these erasers of δ_M. It was shown before that all these numbers belong to $b_s(M)$ of δ_M and are of multitude $n+1$ (with $n=3$ in **Figure 1**). Hence, in general it will be true:

$$K_s=n+1 \tag{4.11}$$

However due to the relation (4.8), (4.9) and based on the help of **Figure 1** in the general form it would be

$$\bar{K}_s=\bar{\rho}_j d_M=\frac{d_M}{q_s}=\frac{l-G}{q_s}=\frac{\gamma-G}{q_s}+\frac{\zeta-\gamma}{q_s}+\frac{l-\zeta}{q_s}=\frac{(\gamma-G)+(l-\zeta)}{q_s}+n$$

The remainders between the silver interval δ_M and the band $b_s(M)$ are the intervals $\gamma-G$ and $l-\zeta$, that obviously each one of these is less or equal to q_s not however both equal to q_s, because then $q_s=q_M=q_{M+1}$, which is absurd, hence it is true that

$$0<\frac{(\gamma-G)+(l-\zeta)}{q_s}<2$$

Combining the last relation with the expression of $\bar{K}_s$ from above, it is given:

$$\bar{K}_s<2+n \tag{4.12}$$

And based on (4.11) it will be true

$$K_s > \bar{K}_s - 1 \quad (4.13)$$

This relation (4.13) shows how the expected multitude of prime numbers in the δ_M interval will have to be as a whole greater than the true multitude, since the active multitude of the erasers is generally smaller that their true multitude, fact that was confirmed using computer based calculations and will be further explained in the following analysis. In the table below is shown the changes between the true and expected multitude of prim numbers in the first 12 silver intervals. The expected or active multitude of prime numbers is calculated using the relation $\bar{Q}_M = Int\left[\bar{P}(M)\cdot d_M\right] =$ Integral part of $\bar{P}(M)\cdot d_M$.

A random eraser, prime-multiple, of band $b_s(M)$, let that be δ, would have non-zero probability coinciding with another eraser q_ρ-multiple, let that be β, of another band $b_\rho(M)$ (that is overlapping with band $b_s(M)$ and β is not in the overlapping region) if the left boarders γ, α of $b_s(M)$ and $b_\rho(M)$ were coinciding. In this case [which is the implied acceptance of (2.1) or (4.7)] there is a possibility of the creation of an additional prime number, than it would in reality, because this way the capability of erasing of the eraser δ is cancelled, since it is degenerating into the erasing that is already fulfilled by the other prime-multiple eraser β. To reverse the possible redundancy by 1 of multitude of active erasers in comparison to the multitude of real, due to the boarders γ, α, not overlapping and instead to increase it, it is enough to increase $\bar{K}_s$ by 1. However, how would one explain the choice to increase "by 1"? The proof, for its necessity, is that if the left boarder γ, of a random band $b_s(M)$, was overlapping with the left boarder G of the silver interval of δ_M, then $b_s(M)$ would include at most one more eraser x. It should be highlighted here that only this eraser introduced error (and thus causes the difference in active and real values), because this is the only one missing from the area $G\gamma < q_s$, resulting in the statistically expected cancelling of the erasing ability of another eraser of another band being included in the calculations of the relation (2.1)—only from this x—, due to the statistically expected overlapping of x, β in $G\gamma$. With this virtual cancelling will reasonably be born in the statistical calculations of (2.1) fewer prospective (active) erasers. This way, however, there will be, from (2.1), more prime numbers than the real ones, because x does not exist (*i.e.* virtual), hence it did not accomplish statistical erasing, like β did. So to reverse the result x needs to be added to become from virtual to real. The same will be true for the right boarder ζ of $b_s(M)$.

Similarly, because the right boarders e.g. ζ, η of $b_s(M)$, $b_\rho(M)$ are possibly, in general, not overlapping, there needs to be anew an increase in the active multitude $\bar{K}_s$, for the same reason as before, by one more unit. In total, by increasing $\bar{K}_s$ by 2, the new active multitude of erasers for $b_s(M)$ will be for every case greater than the real multitude of K_s, owing to the fact that it is like placing in each one of the two intervals $[G, \gamma)$ and $(\zeta, L]$ one q_j-multiple eraser, which is over-covering, because the λ, μ do not, as a rule, overlap with G, L respectively. Thus one concludes in the relation

$$K_{s,NEW} = \bar{K}_s + 2 \geq K_s \quad (4.14)$$

More specifically, one can show (4.14) in a different way, shown below. If all the bands had, in the ideal case, their left and right boarders same as the borders G and L respectively of δ_M (**Figure 1**) then the easers would increase by multitude $2M-2$. The -2 is present because the $b_M(M)$ band of q_M has already the same left boarder (limit) with G, but also another one, the $b_1(M)$ of $q_1=2$ has the same right boarder with L. Splitting the multitude of the new erasers (that transform, the distribution of prime numbers q_s in the δ_M interval into an ideal one, and thus over-covering the error), in the band M of δ_M. There will be an increment for every band of this ideal case [of (4.7)] by $(2M-2)/M = 2-(2/M) < 2$. So with 2 further erasers, for every band, the over-covering mentioned before is certain. Because in this way a new ideal distribution is born "that it is certain to realise all the missing erasing and even more". That leads to an inequality.

A more analytical proof of the above is given here. Let μ_{mk_m} be the new increased (by 2) multitude of erasers of the band $b_{k_m}(m)$, which is the band k_m of δ_m silver interval. Let $I = 1/Q_M$ be the definition of a new unit with $Q_M = \prod_{m=1}^{m=M} \prod_{k_m=1}^{T_m} \mu_{mk_m}$. The new unit I (instead of the previous one which is 1) subdivides now, to more parts the interval, that is defined by the subsequence of M of sequential silver intervals on the axis of natural numbers, with multitude $T_m \in N$ of bands, the catholic selected among them δ_m. It is clear that every unit of natural numbers, which is the distance 1 of two sequential natural numbers on their axis, to the last silver interval δ_m, will be subdivided in multitudes Q_M equal parts. Afterwards, the erasers of every band are equally distributed in every single silver interval. It is obvious that there will be created M multitude new virtual silver intervals that will have all their erasers q_{k_m}-multiples placed on the new marks that define the smaller according to the factor Q_M new subdivisions, where each subdivision has length I. So after the increment by 2 erasers of every band $b_k(m)$ and after equally distribution of erasers in all bands and in all these, M multitude, virtual silver intervals, we conclude that the erasers of every band $b_k(m)$ become more frequent and thus finally the previous ability of distribution:

$$\bar{\rho}_{k_m} = \frac{1\cdot Q_M}{Q_M q_{k_m}} = \frac{1}{q_{k_m}}$$

transforms in a new greater distribution density:

$$\rho_{k_m} = \frac{1\cdot Q_M + Q_M \xi_{k_m}}{Q_M q_{k_m}} = \frac{1+\xi_{k_m}}{q_{k_m}}$$

where according to the previous ones and because the increment for every band by 2 erasers; the following will be true:

$$\xi_{k_m} = \frac{2}{u_{k_m}} > 0,\ \forall k_m, \forall m$$

where $u_{k_m} \in N$ represents the old (before the increment by 2) multitude of erasers of the band $b_{k_m}(m)$. It becomes clear, on the one hand, that in every single virtual silver interval δ_m, due to the equal distribution of the erasers of every band in it, it would now be true the exact corresponding relation of the previous (4.7) so as to estimate the prime numbers in every silver interval δ_m. That means that now the ideal—for the precise calculations of the multitude of prime numbers—new virtual density ρ_{k_m} (that will be used and directly below) is used instead of the real and non-ideal $\bar{\rho}_{k_m}$ that was used before in (4.7). On the other hand, it is realised that in this way there will be less virtual prim numbers (in every δ_m), since there are more erasers for every band of every silver interval δ_m. The index k_m obviously corresponds to the index s of δ_M for the random now intermediate or non silver interval δ_m. Leaving the above boarder M of the arbitrary elected sequence, of the successive silver intervals δ_m, to tend towards infinity; all the different cases are covered. It is very important to emphasize that this method of creating the virtual silver interval functions as follows: "In order the true multitude of erasers to coincides with their active multitude, all erasers should belong to bands which would had their boundaries in common with the boundaries of their silver space δ_m, so that in this case, on the one side the random band $b_j(M)$ would had the predicted density $1/q_j$ of erasers, and on the other side this band would had equal distribution of all its erasers in its total δ_m. So in this case would be made the coincidences of erasers predicted by the statistics of relationship (4.7), so that finally the relationship (4.7) would function correctly. But that does not happen. Therefore, with the virtual silver interval we achieve the equal distribution of the erasers of random band in the total δ_m, that is we achieve the realization of the statistically predicted coincidences from the relationship (4.7), and on the other hand simultaneously we succeed that these erasers to have greater density than the real density $1/q_j$. So finally in the virtual silver interval we will surely have more write-offs than the real ones. In others words we will have something that required from the asking inequality". This proof, combined with the definition of the virtual silver interval of precise calculations of prime numbers (due to the equal-distribution of the erasers of the bands as mentioned), will clarify the analysis below, of the inequalities mentioned and justified before, with a different additional way.

It is observed that in the relation (4.5) it was proved that $d_M > 4q_s$. So now, the two additional erasers of relation (4.14) must be distributed in more than 4 q_s-multiples of $b_s(M)$. [The band $b_s(M)$ of **Figure 1** has geometrical length $d_M = 3q_s < 4q_s$, instead of $d_M > 4q_s$, owing to the relation (4.5), however, this is not inadequate, because as stated it was done only due to the simple impediments and it clearly does not influence the probative methodology]. So, with this distribution that creates the new active multitude of its erasers (4.14) based on the old one, on every one of the old erasers there will be added at most 2/4 = 1/2 erasers. In more details from the relation (4.14) one gets: $\frac{\bar{K}_s + 2}{d_M} \geq \frac{K_s}{d_M}$ and due to the $d_M > 4q_s$ results in $\frac{\bar{K}_s}{d_M} + \frac{2}{4q_s} > \frac{K_s}{d_M}$ so $\bar{\rho}_s + \frac{1/2}{q_s} > \rho_s$, and due to $\bar{\rho}_s = \frac{1}{q_s}$ of (4.8) one results:

$$\left\{\rho_s < \frac{2}{q_s}, \forall s > 1, \text{consequently } \rho_1 < \frac{1.75}{q_1} = \frac{1.75}{2} = \frac{7}{8}\right\} \tag{4.15}$$

So for the absolute validity of the inequalities (4.15) the fact that the active density $\bar{\rho}_s = 1/q_s$ of every band $b_s(M)$ [that is used in the relation (2.1) or (4.7)] was considered to probably be a bit greater, because every band (that is enclosed on its whole in the silver interval δ_M) does not include exactly n multitude of q_s-multiples erasers, as implied by the expression $\bar{\rho}_s = 1/q_s$, but $n+1$, which is explained using **Figure 1**. [For example $\bar{\rho}_s = 1/q_s$ means that $b_s(M)$ in **Figure 1** will include 1 q_s-multiple eraser in each one of these three intervals [γ, δ), [δ, ε), [ε, ζ], that means 3 and not 4 of q_s-multiples, that it indeed contains]. Therefore, in order to be led in the two relations (4.15) one is obligated to use the expression $\bar{\rho}_s = (n+1)/(nq_s) = [1/q_s] + [1/(nq_s)]$, instead of the relation $\bar{\rho}_s = 1/q_s$. Additionally, because it was said $d_M > 4q_s$, $n > 4$ will be true [owing to the band $b_s(M)$ necessarily includes all the q_s-multiples of d_M], so in every case there is $\bar{\rho}_s < 5/(4q_s)$. Thus, based on all the previous $\rho_s < 1.75/q_s < 2/q_s$ will be true, which led to the two relations (4.15) that will be used below, because it was shown that they are generic and true for every case. On the grounds that, even if the observation $\bar{\rho}_s = (n+1)/(nq_s)$ is not taken into consideration and by simply accepting that $\bar{\rho}_s = 1/q_s$, then $\rho_s < 1.5/q_s < 2/q_s$ will arise, which drive to the same conclusion that the above relation (4.15) is true.

As a result the true probability will be defined respectively by (4.7) using the true density ρ_s which satisfies the relation (4.15), meaning:

$$p(\nu) = P_\nu = \prod_{j=1}^{M_\nu} F_j = (1-\rho_1)(1-\rho_2)(1-\rho_3)\cdots(1-\rho_M) \tag{4.16}$$

And now obviously due to the (4.15) it is true that:

$$p(\nu) = P_\nu > \frac{1}{8}\prod_{j=2}^{M}\left(1-\frac{2}{q_j}\right), \ \forall M \in N \tag{4.17}$$

However, in this way an interesting scenario occurs, a sequence of inequalities:

$$\begin{aligned} p(\nu) = P_\nu &> \frac{1}{8}\prod_{j=2}^{M_\nu} F_j = \frac{1}{8}\left(1-\frac{2}{3}\right)\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{11}\right)\cdots\left(1-\frac{2}{q_M}\right) \\ &= \frac{1}{8}\left(\frac{1}{3}\frac{3}{5}\frac{5}{7}\frac{9}{11}\frac{11}{13}\frac{15}{17}\cdots\frac{q_{M-1}-2}{q_{M-1}}\frac{q_M-2}{q_M}\right) \\ &> \frac{1}{8}\left(\frac{1}{3}\frac{3}{5}\frac{5}{7}\left(\frac{7}{8}\frac{8}{9}\right)\frac{9}{11}\frac{11}{13}\left(\frac{13}{14}\frac{14}{15}\right)\frac{15}{17}\cdots\frac{q_{M-1}-2}{q_{M-1}}\left(\frac{q_{M-1}}{q_{M-1}+1}\cdots\frac{q_M-3}{q_M-2}\right)\frac{q_M-2}{q_M}\right) \\ &= \frac{1}{8}\frac{1}{q_M}, \ \forall \nu \in \delta_M, \ \forall M \in N \end{aligned}$$

Particularly, all the successive fractions of the type $\nu/(\nu+1)$ were inserted in brackets (…) exactly where they were missing, which creates a more enhanced inequality.

The last arose after the erasing of the equal numerators and denominators. Consequently for the true function $p(\nu)$, that defines the exact number of prime numbers in the random silver interval δ_M, the result will be:

$$\bar{P}_\nu \geq p(\nu) = P_\nu > \frac{1}{8q_M} \geq \frac{1}{8\sqrt{\nu}}, \quad \forall \nu \in \delta_M \quad \text{and} \quad \forall M \in N \tag{4.18}$$

The last inequality from the tree inequalities of the relation (4.18) derives obviously from the relation $q_M^2 \leq \nu = \alpha_\nu < q_{M+1}^2$ that defines the natural numbers ν of the silver interval δ_M, whilst the first inequality (4.18) derives from everything that was mentioned before for the consequences of the non-overlapping of the limits of $b_s(M)$ and $b_\rho(M)$, however this will not be used in this proof. The inequality $p(\nu) > 1/(8q_M)$ of the relation (4.18) will be named fundamental inequality of the silver intervals. The inequalities (4.18) are these that as mentioned will be proved in regards to the relation (2.1) when mentioned in Section 3 that (2.1) along with everything that will be shown related to it below, includes all the available catholic information.

Previously, the probability $p(\nu) = P(M) = P_\nu$ was characterized as exact function meaning that it calculates the exact number of prime numbers in δ_M. The function $p(\nu)$, even unknown is said here to be exact in the sense that the inequality (4.18) can be used for it, exactly like an inequality can be used for number π in the sense that in theory this number exists in any desirable precision. The proof of (2.1) is completed based in the independent divisibility of the prime numbers, like it was determined in Section 2. It is noticed that the fundamental inequality is the "function" of each silver interval, and this concept is a form of correcting (2.1) specifically for every silver interval, because the equal-distributions of prime numbers are disturbed in the limits of δ_M. Hence now, the wanted tally of the events of twins (and maybe of other formed prime numbers) to infinity can be assessed whether it has a finite result or not.

Observing **Table 1** the reason can be understood. The generator of prime-multiples numbers is, as explained in Section 3, an ideal mathematical generator that would define precisely the multitude of prime numbers in every single one of the silver intervals δ_M if the bands had the same limits, which is shown by the procedure of calculation of the active probability $\bar{p}(\nu) = \bar{P}(M) = \bar{P}_\nu$ in the proof of (2.1) [or (4.7)] that means one precise calculation based on the absolutely know tally of the prime-multiples natural numbers that consist a perfect repetitive procedure in every silver interval δ_M. If now (2.1) is combined with the exact constraint of (4.17), that was shown it is understood that (4.17) is not refuted in any δ_M to infinity. An assessment, without essential meaning, by a computer in a depth of hundreds of millions of natural numbers verified relation (4.17) even with $g < 1.5$, instead of $g = 2$ that all the factors of the inequality (4.17) have. Of course it was already proven that (4.17) is true for $g = 1.75$;

Table 1. Prime multitude in silver intervals.

Silver intervals	True multitude of prime number	Expected multitude of prime numbers
[4, 8]	2	2
[9, 24]	5	5
[25, 48]	6	6
[49, 120]	15	16
[121, 168]	9	9
[169, 288]	22	23
[289, 360]	11	12
[361, 528]	27	28
[529, 840]	47	51
[841, 960]	16	18
[961, 1368]	57	62
[1369, 1680]	44	46

however $g = 2$ was chosen because that was the one that allowed the erasing of sequential fractions, which at the end led to the proof of (4.18) that in turn proved to be sufficient for the calculation of the multitude of twin prime numbers, as it will be shown.

It was proven before, that an increment of ideal erasers (active) of every band for a mean multitude $(2M-2)/M = 2-(2/M)$, will cause expected correction. Indeed, by calculating the mean increment-correction of each erasers, [one in every interval $q_s \Rightarrow 1/q_s$ in (4.7) or (2.1)] for every band $b_s(M)$, approximately equal to:

$$\Delta n_s \cong \frac{2-(2/M)}{1+n} \cong \frac{2-(2/M)}{1+\left[\left(q_{M+1}^2 - q_M^2\right)/q_s\right]}$$

Once can set in (4.16) $\rho_s = (1+\Delta n_s)/q_s$ instead of $\bar{\rho}_s = 1/q_s$ of (4.7) or (2.1). Meaning

$$\rho_s = \frac{1+\Delta n_s}{q_s}, \quad s = 1, 2, \cdots, M \tag{4.18a}$$

Based on this correction (4.16) forms the previous **Table 1** as follows.

It is observed from **Table 2** that there is indeed an important correction from an initial active multitude of 278 expected prime numbers, to 257.3 now. Meaning an error of approximately 1.4% from 6.5% that was before in **Table 1**.

On the subject of the twin pair, which was studied after Introduction, in Section 1, it was seen by relation (1.1) that a specific event with a probability p with regards to its appearance in a single repetition, in a multitude of N independent repetitions, will have a mean multitude of twin pairs, (which will be precise in infinity repetitions) equal to:

Table 2. Correction of prime multitude in silver intervals.

Silver intervals	True multitude of prime number	Expected multitude of prime numbers
[4, 8]	2	2.5
[9, 24]	5	4.36
[25, 48]	6	5
[49, 120]	15	14.6
[121, 168]	9	8
[169, 288]	22	20.65
[289, 360]	11	10.6
[361, 528]	27	26
[529, 840]	47	47.9
[841, 960]	16	16
[961, 1368]	57	58.8
[1369, 1680]	44	42.8
Sums:	264	257.3

$$R = (N-1)p^2 \tag{4.19}$$

The precise relation (4.19) derives from the independency of repetitions in a multitude of $N-1$ boarders among these.

The known theorem of prime numbers that dictates a logarithmic distribution [1] [3] [6] is essentially a statistical theorem. To be exact, in this paper's Statistics, based on the relations (2.1), (4.18) that were proven and will be utilized below, it will additionally be validated that "the catholic (random) selection of a prime number q_a (that was named in the beginning of this paper) in an also catholic (randomly) selected silver interval δ_M, does not give the catholic information (that was also named in the beginning of this paper) that the probability of appearance of the next prime number q_b is changing in the very same silver interval according to the distance of its position from q_a". The useful meaning of this catholic capacity is that the prime numbers are distributed in the catholic silver interval in such way that it does not statistically favor, after all in their infinity multitude, neither them getting closer nor away from each other. One of the initial reasons for this is that the creation of the prime numbers from the tracker does not produce any logical suggestion that will neither dictate them getting closer to each other in the same silver interval, which statistically would favor the twin paring among them in their infinity multitude, nor them getting away from each other in the same silver interval which statistically would complicate the twin pairing among them in their infinity multitude. In other words, all the catholic information for the distribution of prime numbers in provided by the relations (2.1) and (4.18) along with the improvement of (4.18a) etc. If the position of a prime number was affecting the position of a neighboring prime number in the same silver interval based on the catholic information (with a proposal of generic validity in *N*), then this catholic event would prevent the probative method of creation of (2.1) to ignore it, and thus it would be disclosed in that way. Consequently, since in mathematics nothing happens without a reason, the distribution of the prime numbers in a random (or otherwise catholic) selected silver interval will be the one that dictates the catholic relation (2.1). The catholic relations (4.18), (4.18a), but also any of their improvements, do not change the above conclusion because they are objective inequalities. Meaning the reason is that this inequalities of catholic validity are exclusively due to the disorder of calculations that creates the border between two sequential catholic selected silver intervals, which however does not change the way the erasers that create the relation (2.1) work; using the equal-distribution of the probabilities of the prime numbers in this catholic selected silver interval, but this border prevents their precise calculations due to the non-overlapping of the limits of the erasers bands with the limits of the silver intervals, as it was already mentioned.

Proof

It is known, (and easy to be shown), that the minimum distance L_{ij} between two multiples of the random (meaning; catholic selected) prime numbers q_i, q_j is changing cyclically and therefore all the values of L_{ij} are equiprobable in a catholic (random) selection of L_{ij}. For example, without damaging the generality if one selects $q_i = 5$, $q_j = 3$, then this minimum distance will change cyclically from 0 to 2. Hence, the action of the catholic information (relations which are true in all *N*) that a "catholically selected natural number $a_\nu \in \delta_M$ is not divided by a prime number $q_i \in Y_M$" will again leave equal the probabilities to be not divided by the same $q_i \in Y_M$, and also by any other prime number $q_j \in Y_M$, one of its following natural numbers $a_\kappa \in \delta_M$. The obvious reason for this is that the catholic sentence "catholic selected natural number $a_\nu \in \delta_M$ is not divided by a prime number $q_i \in Y_M$", based on everything that was said for L_{ij}, not produce any catholic suggestion (meaning there is not any catholic information) to produce differentiation on the (catholic or general expressed) probability for divisibility of the following natural numbers $a_\kappa \in \delta_M$, (which follow after $a_\nu \in \delta_M$) by one of prime numbers (separately) of the subsequence Y_M of this catholic selection silver interval. This conclusion directly entails that the catholic information that "$a_\nu \in \delta_M$ is a prime number" does not changes the probability (in function with the distance of its position) "to have a following natural prime number $a_\kappa \in \delta_M$ at any distance in this same catholic selected silver interval". Also the same will happen if for this catholic selected number is valid the proposition $a_\kappa \in \delta_{M+1}$ instead the before one, because the change of the probability, due to the existence of an additional factor in the relation (2.1), will be independent of the previous proof. We have said in introduction that without catholic information nothing happening in Mathematics because Mathematics is absolutely based on causality.

Summing up, based on the previous paragraph, but also based on those mentioned at the end of the Introduction (for the Proposition of Catholic Information and PDI), we proved that the prime numbers in their infinite multitude statistically neither approach each other nor the inverse thus they will have a statistically unbiased distribution regarding their probability of appearance in a random (catholic) chosen silver interval. Specifically, the multiples of any prime numbers (e.g. the prime numbers 5, 7 that were mentioned in the beginning of the Introduction) appear as independent events in N (as results of an ideal roulette) of the classical theory of probability. Therefore, the prime numbers per two will have their multiples as independent events of classical theory of probabilities in N. So, whether a random natural number $\nu = \alpha_\nu$ is prime or not depends on the interaction of independent events of the probability theory that is the interaction of multiples of the preceding prime numbers in the subsequence Y_M of this randomly selected silver interval δ_M. These events are distributed like the result of an ideal imaginary roulette, as stated. It is the most representative distribution of an ideal roulette. So the prime numbers will be independent events of the probability theory. A question that is raised is if the boundaries of silver intervals affect on the distribution of prime numbers in them. The answer is no. The proof for this is based on the fact that the prime numbers have multiples independent per two and that means that the same will happen with the square of the prime numbers which define the boundaries of the silver intervals. Thus these boundaries do not catholically affect on the distribution of prime numbers in the catholic silver interval. So during the selection of a random (representative catholic selected) silver interval it will occur inductively for its boundaries to also be independent events regarding the prime number multiples of the subsequence of this catholic selected interval. Thus these boundaries will be catholically independent regarding the position that the prime numbers will appear in random silver interval.

We can finally prove that not only the probability $\overline{P}_\nu$ is constant within a randomly selected silver interval δ_M (or in others words $\overline{P}_\nu$ independent of the position of a random candidate prime number $a_\nu \in \delta_M$) but and the variation $\Delta\overline{P}_\nu < 0$ of $\overline{P}_\nu$, which is being introduced by the inequality (4.18), is also a constant in the same random δ_M. Indeed, we observe that $\Delta\overline{P}_\nu$ depends solely on the positions and range of overlap of the bands and so the $\Delta\overline{P}_\nu$ ultimately depends on the bounds of the bands. Let be B_1, B_2 two such bands of q_{s1}-multiples and q_{s2}-multiples that means q_{s1}, q_{s2} are the primes which their multiples define the bounds of B_1, B_2 respectively. Then the distances a_1, a_2 of two boundaries of B_1 from random (CS) position a_ν will be (in the general case of choosing random δ_M) independent of the two other distances b_1, b_2 of the boundaries of B_2 from the same position a_ν. The reason is the catholic property of multiples of prime numbers q_{s1}, q_{s2}:

$$p\left(\text{multiple of } q_{s1} / \text{multiple of } q_{s2}\right) = p\left(\text{multiple of } q_{s1}\right)$$

where p represents the probability of event in its parenthesis. That is what we prove before in introduction. As we prove there this last relation is valid for the same position μ and thus therefore it will be valid for different positions μ_1, μ_2. Therefore, the reducing the probability of the deletions, that we also mentioned, as well its result $\Delta\bar{P}_\nu$, eventually will be both independent of the position of a_ν in general choice of δ_M. That means that the correct probability $p(\nu)=\bar{P}_\nu+\Delta\bar{P}_\nu$ will be independent of position $a_\nu\in\delta_M$. In others words as $\bar{P}_\nu$ as well $\Delta\bar{P}_\nu$ are independent of position $\nu=a_\nu\in\delta_M$ because both these probabilities in catholic case are defined of the prime multiples which relative to that position ν have appearance frequencies independent each other. Therefore according to the Proposition of Catholic Information which we refer before in Introduction, if would exist general relationships between a random position ν (in random δ_M) and range of overlap of some catholic (random) band in δ_M then we would conclude that must exist also catholic relationships between the appearance frequencies of these primes multiples in random δ_M. But on the basis of PDI that we prove in Introduction this last conclusion is wrong. In other words according to the Proposition of Catholic Information we can't write general formulas between frequencies of appearance of two prime multiples in random δ_M using in this formula only the general symbols q_μ, q_s for them. According to the Shannon definition of information we know that the information is the other side of probability: $I=\log_2(1/p)$, and thus in our solution the catholic expressions (written in general form) are the only ones which include and transfer information of total space that is information of set N relative to some general defined form, for example to some interval in its random choice. And as we have said in introduction, in Mathematics nothing happens in the total space of some set without the existence of catholic information which according its definition is flowing from corresponding general form that must concerns the total space of this set. So in the same random δ_M the statistical tendency of primes will be neither their approach nor their depart, but simply they will follow the statistic dictated by the relations (2.1) and (4.18). In other words, we have prove that the general relations (2.1) and (4.18) that constitute catholic information (because they are written in general form and they are referenced on random δ_M) have generally been the ones that define the distribution of prime numbers. This process is valid until to the infinite, a fact ensures correctness of our statistical calculations bellow.

For the clarification of the definition inductively, that was used previously, it should be explained that the creation of prime numbers, by the procedure that was used to prove the relation (2.1) (which drives to creation of prime numbers in silver intervals and based on the independent multiples of prime numbers of their subsequence) starts without any overruling of the above proof form the beginning of the set of the silver intervals. Thus it is necessary at this point to investigate the inaugural structure at the proof above. Indeed, the first silver interval $\delta_0=(1^2,2^2)$ has, as we said shortly after the relation (2.7), as its subsequence the empty set. Therefore, from δ_0, necessarily, all of its elements will be selected the natural number 2, 3 as new prime numbers according to the selection rule which in this case states only that "the unreal multiples of unreal primes in the subsequence Y_0 not divide the natural numbers 2, 3". This due to the fact that the subsequence Y_0 of δ_0 does not includes any element and so therefore the sentence "the numbers 2, 3 are not multiples of a non-existent prime number" is true. Afterwards, it is noticed that the multiples of these initial prime numbers 2 and 3 are independent from each other in the set N and thus the prime numbers that will be selected, based on the same selection rule, in the next silver intervals δ_1, δ_2 will be always, according to the previous proving procedure, independent events of the probability theory and so on inductively. Additionally, the boarders of the following silver interval that is created from the squares of the prime numbers 2, 3 do not contribute any logical catholic suggestion that will affect the divisibility both in this silver interval and in a catholic (randomly) selected one below. In this way we understand that the squares of prime numbers which define the silver intervals (because they define their boundaries) will be catholically independent events of the established probability theory.

Hence, according to the Proposition of catholic Information, that was mentioned in the Introduction of this article, there is no catholic Information that will force the infinity multitude of prime numbers to being thicken or dilute in a statistically random (CS) silver interval, and thus (based on the obvious axiomatic logical Proposition of catholic Information, in the introduction) one can execute (for infinity multitude of prime numbers) accurate statistical calculation (exactly because this multitude is infinite) using the established probability theory. And this is something that we will make below. In other words the total appearance of prime numbers in N is same with one of the infinite results-games (with infinite rotations for each game) of an ideal roulette that in every game has on its rotating disc (in every rotation) the multiples of those prime numbers which inductively produced using the previous prime numbers, which were found in the same way as before, and so on. Saying "random" way we mean "without additional catholic information except the one dictated by the relation (2.1)". The reason is that the action of inequalities (4.18) and (4.18a) etc are only preventing the accurate calculation of the multitude of the prime numbers in a silver interval and do not affect the catholic power of (2.1) for the reason stated in previous paragraphs.

Returning to the problem of the twin prime numbers, according to everything that was stated, the system of the aforementioned relations (2.1) and (4.18) will dictate a catholic distribution of prime numbers, where the frequency of their appearance will remain constant in the same silver interval. Thus the positions of prime numbers are per two independent from each other and is reduced in a rate dictated by the two relations (2.1) and (4.18) from the one silver interval to the next. Thus the frequency of appearance of prime numbers is changing only between silver intervals and not in the same silver interval, and all these are determined statistically by the catholic relations (2.1) and (4.18), etc.

Thereupon, for the natural number $a_\nu = \nu$ of a silver interval δ_M, the relation (4.18) in the respective (cumulative) here application of the relation (4.19) for the tallying of the twin prime numbers, that form the independent events of the appearance of prime numbers, meaning the independent prime numbers, gives:

$$R_M = \sum_{\nu=q_M^2}^{q_{M+1}^2-1} P_\nu^2 > \frac{1}{64} \sum_{\nu=q_M^2}^{q_{M+1}^2-1} \frac{1}{\nu} \tag{4.20}$$

Hence, the infinity multitude of the silver intervals one has:

$$R = \sum_{M=2}^{\infty} R_M > \frac{1}{64} \sum_{M=2}^{\infty} \left(\sum_{\nu=q_M^2}^{q_{M+1}^2-1} \frac{1}{\nu} \right) = \frac{1}{64} \sum_{\nu=9}^{\nu=\infty} \frac{1}{\nu} = \frac{1}{64} \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \right) \tag{4.21}$$

However, the second part of (4.21) becomes infinity because the brackets are the result of the subtraction of a finite multitude of terms from the known harmonic series, that as proven in the introduction of unit 1, it becomes infinity:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{\nu} + \cdots = \infty \tag{4.22}$$

Therefore, it was proven that the wanted multitude of the twin prime numbers in the infinite multitude of natural numbers will also be infinite.

Without damaging the generality, assuming the twin pairs from the left, that is assuming the twin pairs of every new prime natural number candidate in the random silver interval δ_M (with probability P_ν) with the one smaller (on its left) also new candidate prime (and hence its twin) natural number (with probability $P_{\nu-1}$) it is found:

$$R_{M(left)} = P_{q_M^2-1} P_{q_M^2} + \sum_{\nu=q_M^2+1}^{q_{M+1}^2-2} P_\nu^2 > R'_M = \sum_{\nu=q_M^2}^{q_{M+1}^2-1} P_\nu^2$$

Taking into consideration, in every δ_M for $M > 0$, that the terms are of even multitude. The obvious reason for the clarification is that the disorder, that is caused by the relation above (in the tally R of twin numbers) also refers to the boarders of the sequential silver intervals, due to the fact that for these the probability P_ν changes. So for the relations (4.21) there is:

$$R_{left} = \sum_{M=2}^{\infty} R_{M(left)} > R > \frac{1}{64} \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \right) = \infty$$

which is a condition able to make the multitude of twin prime pairs in the set N of natural numbers infinity and thus concludes this proof, since it is true that $R_{left} = \infty$. The question here is why it was not taken into consideration the causal datum that "the twin prime pairs of two successive natural numbers is excluded". The answer is that this calculation ignores this kind of causal data, because it is based only on the possibility of twin pairs. If, however, on the base of a seemingly stricter proving procedure one considers this datum (information), then the above calculation is repeated using the set of the odd natural integers. In such a case, because $3/(2\nu+1) > 1/(2\nu+2), \forall \nu > 0, \nu \in N$, it is easily seen that:

$$R_{left,\pi\varepsilon\rho\iota\tau\tau\omega\nu} > \frac{1}{4} R_{left} > \frac{1}{4} R > \frac{1}{256}\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\cdots\right) = \infty$$

Hence, the conclusion is the same.

The reader can easily note that the steps from the relation (4.20) to the relation (4.22) can once more verify that the multitude of the prime numbers in the set of the natural N is infinite, if P_ν^2 is replaced by P_ν in (4.20) calculating now the multitude of prime natural numbers.

Furthermore, it will be shown that the probability of appearance of a prime number in a silver interval tends towards zero, meaning:

$$\lim_{\nu\to\infty} p(\nu) = 0 \tag{4.23}$$

From the inequality (4.18) that was shown here and the relation (2.9) that was shown in Section 2; the relation (4.23) is immediately proven:

$$\lim_{\nu\to\infty} p(\nu) \le \lim_{\nu\to\infty} \overline{P}_\nu = \frac{1}{\zeta(1)} = 0 \tag{4.24}$$

This relation is very important, because it states according to the definition given by Shannon that the information that the prime numbers enclose is infinite. Considering the set of natural numbers N quantified, in the sense that both its definition and all of its properties can be supported on the numbering, without breaking the quantum "1" that defines on the basis of the numbering all of the properties of set N, one concludes that the information will also be quantified in N. Indeed, since there is no prime number factorised, then according to the Shannon definition every new prime number in N defines the information $I_\nu = -\log_2\left[p(\nu)\right]$ which has the property to not be broken in a sum of two or more information terms smaller than the prime number. Consequently, in accordance with (4.24) every new prime number creates a new form of information, because it will not be analysed in a previous prime numbers information, additionally this information tends towards infinity. In the author's book "The twins of infinity and the Riemann hypothesis", Ziti ("Ζήτη") publications, it is defined as quantum q the least quantity of a quality Q. And as quality Q is defined a notifier set of properties, that are altered only during the break of quantum q. For example, the quantum of the quality "water" will be the "molecule of water". And therefore the q will be the physical unit of set Q with which one counts the repetitions of Q in a phenomenon in physics (or in mathematics). For instance, according to these the quantum of the Euclidean space-time of the Special Theory of Relativity will be four-dimensional and elementary hypercube of the space-time with an edge length equal to the length of Planck (10^{-35} m). Furthermore, based on the above definition, every well-defined pure physical size, will owing to be quantified and that will be the definition of the pure physical size. And it is truly very charming the question, whether on the infinite information of prime numbers is paradoxically mirrored an infinite and unchanged hyper-verse of events.

5. The Dark Paths of Infinity and a Second Solution

This is at last the final step of this research. It will define one more kind of intervals on an axis of natural number, because their inconceivable length will be named dark or gloomy intervals. The first of these will be the interval of 25 natural numbers:

$$Z(1) = [1+2.3, 1+2.3.5] = [7, 31]$$

The second will also be an interval of natural numbers as follows:

$$Z(2) = [2+2\times3\times5, 1+2\times3\times5\times7\times11\times13\times17\times19\times23\times29\times31]$$
$$= [32, 1+200560490130]$$

The third will be the interval of natural numbers:

$$Z(3) = \left[2+200560490130, 1+2\times3\times5\times q_4 q_5 \cdots q_{L(2)}\right]$$
$$= \left[200560490132, 1+\Omega(3)\right]$$

where $q_{L(2)}$ is the last prime number, meaning the greatest prime number that will be included in the immediately previous gloomy interval, meaning it will be the one that will be included in the second which is $Z(2)$. In general for $i > 1$ there will be:

$$Z(i) = \left[2+\Omega(i-1), 1+q_1 q_2 q_3 \cdots q_{L(i-1)}\right] = \left[2+\Omega(i-1), 1+\Omega(i)\right] \quad (5.1)$$

And obviously:

$$\Omega(i) = \prod_{j=1}^{L(i-1)} q_j = q_1 q_2 q_3 \cdots q_{L(i-1)} \quad (5.2)$$

The natural number $L(i)$ is obviously representing the multitude of the prime numbers that will be included up to the gloomy interval $Z(i)$. It will be again defined as "length" of $Z(i)$ the multitude of natural numbers that include:

$$l_Z(i) = \left[1+\Omega(i)\right] - \left[2+\Omega(i-1)\right] + 1 = \Omega(i) - \Omega(i-1) \quad \text{meaning}$$
$$l_Z(i) = \Omega(i) - \Omega(i-1) \quad (5.3)$$

Also, the multitude of prime numbers that are included in the interval:

$$G(i) = Z(i) - \{\Omega(i)-1, \Omega(i)+1\} \quad (5.4)$$

will be symbolized as $\omega(i)$.

The gloomy intervals that are defined in this way are infinite, however their length is increased in an outrageous rate! They have however a very important ability: "if $\omega(i) = 0$, meaning if the interval $G(i)$ that was defined does not include any prime number, then the $\Omega(i)-1, \Omega(i)+1$ will constitute a twin pair". This will be named twin pair proposition of the gloomy intervals. The proof is very simple and is derived from its definition (5.2):

If the hypothesis of this proposition is true, then all the previous prime numbers of the $\Omega(i)-1, \Omega(i)+1$, will be $q_1, q_2, q_3, \cdots, q_{L(i-1)}$, because the common first term $\Omega(i)$, of the natural numbers $\Omega(i)-1, \Omega(i)+1$, is obviously divided by all the $q_1, q_2, q_3, \cdots, q_{L(i-1)}$, however their second term by none. Hence, if the hypothesis of the proposition is true, then the $\Omega(i)-1, \Omega(i)+1$ will both be prime numbers and since their difference is 2, these will constitute the twin pair. This proof is similar with that of Euclid.

The question, whether there are gloomy intervals that do not contain any prime number, has a negative answer. The same question for the silver intervals that were defined in the previous sections is not known if it has the same answer, but this fact does not affect in any case the proof given. In the below Figure 2 is shown a gloomy interval $Z(i)$ with its limits, as well as the prime number q_j, if there is even one prime number in it not equal to the natural numbers $\Omega(i)-1, \Omega(i)+1$, where $W_i = 2\lambda + \Omega(i-1)$.

It is proven that if $G(i)$ of the relation (5.4) does not contain prime numbers then the gloomy interval $Z(i)$ will contain a twin because then the pair $\{\Omega(i)-1, \Omega(i)+1\}$ will be twin. What will happen though if there is only one prime number in $G(i)$? This will create four possible events, independent (due to the prime number independency) but not incompatible:

A = "q_j has a right sequential prime number, that is $q_j + 2$"

B = "q_j has a left sequential prime number, that is $q_j - 2$"

X = "q_j divides the natural number $\alpha = \Omega(i) - 1$"

Y = "q_j divides the natural number $\beta = \Omega(i) + 1$"

It should be clarified here that the above events of divisibility X, Y have catholic probability (frequency of their appearance) equal to $1/q_j$, exactly as it is required to apply with absolute accuracy the probability theory below. The reason is that the "random" q_j is an event (during its appearance) independent regarding the appearance of every prime factor of the "random" $\Omega(i)$, hence it will also be an event independent from the natural number α, β, according always to the definitions of the independency of events in the probability theory that was mentioned in the beginning. It is about the catholic or random selection exactly as required by an accurate calculation in an infinity multitude of events.

The probabilities of the two prime events A and B will satisfy the relations (4.18) shown before in Section 4. That means that the following relations will be true:

$$p(A) = p(q_j + 2) > \frac{1}{8}\frac{1}{\sqrt{q_j + 2}}, \quad p(B) = p(q_j - 2) > \frac{1}{8}\frac{1}{\sqrt{q_j - 2}} \tag{5.5}$$

The events X, Y will have equal probabilities:

$$p(X) = p(Y) = \frac{1}{q_j} \tag{5.6}$$

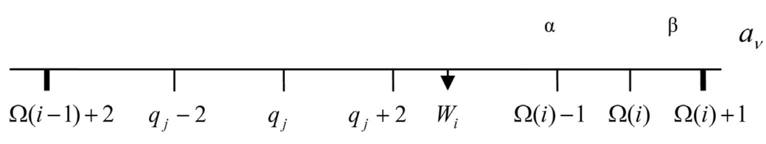

Figure 2. Dark intervals.

The relations (5.6) have an obvious cause. For example if $q_j = 3$ then the probability of a random natural number to be divided by 3 is 1/3, since one of the three natural numbers is third-multiple, meaning the integer multiple of the prime number 3. Similarly the probability to be divided as an integer a random natural number (as above the unspecified *α*, *β*) by q_j is $1/q_j$, since one in the q_j-multitude natural numbers is multiple of q_j. Based on (5.5) one gets:

$$p(B) \geq p(A) > \frac{1}{8}\frac{1}{\sqrt{q_j+2}} > \frac{2}{q_j}, \ \forall q_j > 258 \tag{5.7}$$

The last inequality of (5.7) is equivalent to the $q_j^2 - 256q_j - 512 > 0$, where obviously $\forall q_j > 258$ will be true, based on the properties of the triphony. On the other hand based on (5.6) there will be:

$$\frac{2}{q_j} = \frac{1}{q_j} + \frac{1}{q_j} > p(X) + p(Y) - p(X)p(Y \mid X) = p(X \cup Y) \tag{5.8}$$

The probability $P_{\Delta i}(q_j) = p(X \cup Y)$ in (5.8) is obviously the probability of "destroying" the twin $\Delta_i = \{\Omega(i)-1, \Omega(i)+1\}$ of $Z(i)$ from the appearance of the prime number q_j. Combining now the two relations (5.7) and (5.8) it is found:

$$p(A) > P_{\Delta i}(q_j) = p(X \cup Y), \ \forall q_j > 258$$

or

$$\delta_i(q_j) = p(A) - P_{\Delta i}(q_j) > 0, \ \forall q_j > 258 \tag{5.9}$$

The relation (5.9) states something very important "the probability to create at least one twin, the pair $\{q_j, q_j+2\}$ in the gloomy interval $Z(i)$ from the appearance of the prime number q_j is greater (significantly greater) than the probability for the potential twin $\Delta_i = \{\Omega(i)-1, \Omega(i)+1\}$ to be destroyed by the appearance of q_j, that from now on will be called the potential gloomy twin of $Z(i)$ interval". Provided that $q_j > 258$, which is not of interest, because for this proof the infinite last gloomy intervals $Z(i)$ are enough *i.e.* with $i > 2$.

What will happen now if a second prime number $q_{j\pm\lambda}$ is created? The relation (5.9) now with $q_{j\pm\lambda}$ in the place of q_j it will once more be true, with the main difference now being that except from the change in the probability of (5.9) there is another additional probability for the creation of a twin, because there will be the probability of the new of the new $q_{j\pm\lambda}$ to become neighbors with the already existing prime number q_j in the gloomy interval $Z(i)$ and so on. In conclusion there is a more powerful inequality for the relation of tallying the mean multitude of the twin pairs in $Z(i)$ where $i > 2$.

$$N_i > 1 + \sum_{j=j_1}^{j=W_i} \delta_i(q_j) > 1, \ \forall i > 2 \tag{5.10}$$

Number 1 on the left part of the second inequality (5.10), exists due to the potential gloomy twin $\Delta_i = \{\Omega(i)-1, \Omega(i)+1\}$, which is accounted by this unit, however this is increased with the ability of appearance of new prime number in all possible positions in $Z(i)$. Thereupon, the total multitude of twins in the infinite multitude of the gloomy intervals due to (5.10) will:

$$R=\sum_{i=3}^{\infty}N_i>\sum_{i=3}^{\infty}\left(1+\sum_{j=j_1}^{j=W_i}\delta_i\left(q_j\right)\right)=\infty \quad (5.11)$$

In that way it is verified with an additional way the result found at the end of Section 4, which is that the multitude of prime twins is infinite, because from (5.11); $R=\infty$ is derived. The first inequality of (5.10) is due to not considering the probability of creating a twin with the appearance of the prime number q_j on the left during the tallying, because the event B that was mentioned before was ignored.

There last statements can also be presented by a more mathematically concise way. The phrase "at least one" as shown and in Section 2, in the proof of relation (2.1), means the connection of all the respective probable events. The probability of the appearance of a right twin due to the appearance of at one prime number q_j, (**Figure 2**) in the positions:

$$a_k \text{ with } a_k=k=2j+\Omega(i-1),\ j=1,2,3,4,\cdots,\lambda,\ i>2$$

and with $W_i=2\lambda+\Omega(i-1)\le\beta/2$ it will be

$$\Delta p_{i+}=1-\prod_{j=2}^{\lambda+1}\left[1-p\left[2j+\Omega(i-1)\right]\right] \quad (5.12)$$

The last inequality $W_i\le\beta/2$ is set because there is obviously a probability that the random number q_j divides one of the two parts of the gloomy twin, if q_j appears in a position further away from the middle of $Z(i)$.

The probabilities $p\left[2j+\Omega(i-1)\right], j=2,3,\cdots,\lambda+1$ refer to the positions of $2j+\Omega(i-1), j=2,3,\cdots,\lambda+1$ in the neighborhood of whom, on the left as well, there might be the number q_j, therefore there is a possibility of one more prime number to appear, (so as a twin to arise), with the respective probabilities. The proof is completely the same with the proof of the relation (2.1).

$$\text{The product: } \prod_{j=2}^{\lambda+1}\left[1-p\left[2j+\Omega(i-1)\right]\right]$$

In (5.12) above represents the probability of not creating any neighboring prime number of q_j, from the right, and from the "section" of the independent events there is no appearance of prime numbers in the positions:

$$2j+\Omega(i-1),\ j=2,3,\cdots,\lambda+1$$

In the same exactly way there is the probability of Δp_{i-} to be divided by one of the parts of the potential gloomy twin of the interval $Z(i)$ from the appearance of at least on prime number q_j in any appropriate position in $Z(i)$, meaning till the natural number $W_i\le\beta/2$. The probability p_{j-} of the natural number $a_k=2j+\Omega(i-1), j\in\{1,2,3,\cdots,\lambda\}$ of the interval $Z(i)$ to divide one at least of the two parts of the potential gloomy twin of $Z(i)$, will satisfy the relationships:

$$p_{j-}<\frac{2}{2j+\Omega(i-1)}<p\left[2j+2+\Omega(i-1)\right],\ j\in\{1,2,3,\cdots,\lambda\},\ \forall i>2$$

That arise in a completely similar way with the relation (5.9), from the comparison of (5.7) and (5.8).

For the correlation of the changes Δp_{i-} and Δp_{i+}, taking into consideration the previous inequalities and since completely similar with the proof of (5.12), one gets:

$$\Delta p_{i-} = 1 - \prod_{j=1}^{\lambda} \left(1 - p_{j-}\right)$$

and afterwards is led:

$$\Delta p_{i-} < 1 - \prod_{j=1}^{\lambda} \left[1 - \frac{2}{2j + \Omega(i-1)}\right] < 1 - \prod_{j=2}^{\lambda+1} \left[1 - p\left[2j + \Omega(i-1)\right]\right] = \Delta p_{i+}$$

these last relations reveal that:

$$\Delta p_{i+} > \Delta p_{i-}, \ \forall i > 2 \tag{5.13}$$

The relation (5.13), that was shown above, states that the change of the probability to create at least one twin from the complex event of the appearance of at least one prime number (meaning one, two or three prime numbers etc.), in $Z(i)$ will be "much" greater than the change in the probability of this complex event to be destroyed by the potential existing gloomy twin of $Z(i)$. Then again, like before, by applying the last relation (5.13), the total multitude of twins in $Z(i)$ will be:

$$N_i > 1 + \Delta p_{i+} - \Delta p_{i-} > 1 \tag{5.14}$$

And following that, based on relation (5.14), for the whole of interval N of the natural numbers it will be true that:

$$R = \sum_{i=3}^{\infty} N_i > \sum_{i=3}^{\infty} \left(1 + \Delta p_{i+} - \Delta p_{i-}\right) = \infty$$

Therefore, finally the multitude of the twins in the interval N of natural numbers will be:

$$R = \infty$$

This is the last signal the tracker send from infinity [3] [7] [8] from this world of intangible beings of the pure information that is mirrored in the mental world of the Pythagoras' numbers, where paradoxically the tracker seems to have the advantage of the instantaneous transmission of the infinite information, like the goodness of reputation in the mysterious place of Ovid:

In the middle of the world
There is a place
Among the countries
In the waves of the sea
And the heavenly slopes
The boarder of the triple world…

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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