Advances in Pure Mathematics, 2019, 9, 53-66
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Numerical Solution of Two-Dimensional Nonlinear Stochastic Itô-Volterra Integral Equations by Applying Block Pulse Functions

Guo Jiang, Xiaoyan Sang*, Jieheng Wu, Biwen Li

School of Mathematics and Statistics, Hubei Normal University, Huangshi, China
Email: gjiang@hbnu.edu.cn, *xysang@stu.hbnu.edu.cn, jiehengwu@stu.hbnu.edu.cn, lbw1818@163.com

How to cite this paper: Jiang, G., Sang, X.Y., Wu, J.H. and Li, B.W. (2019) Numerical Solution of Two-Dimensional Nonlinear Stochastic Itô-Volterra Integral Equations by Applying Block Pulse Functions. *Advances in Pure Mathematics*, **9**, 53-66.
https://doi.org/10.4236/apm.2019.92004

Received: January 15, 2019
Accepted: February 11, 2019
Published: February 14, 2019

Abstract

This paper investigates the numerical solution of two-dimensional nonlinear stochastic Itô-Volterra integral equations based on block pulse functions. The nonlinear stochastic integral equation is transformed into a set of algebraic equations by operational matrix of block pulse functions. Then, we give error analysis and prove that the rate of convergence of this method is efficient. Lastly, a numerical example is given to confirm the method.

Keywords

Block Pulse Functions, Integration Operational Matrix, Stochastic Itô-Volterra Integral Equations

1. Introduction

Two-dimensional stochastic Itô-Volterra integral equations arise from many phenomena in physics and engineering fields [1]. Some different orthogonal basis functions, polynomials and wavelets are used to approximate the solution of two-dimensional Volterra integral equations. For example, block pulse functions, triangular functions, modification of hat functions, Legender polynomials and Haar wavelet and the like (see [2] [3] [4] [5] [6]).

Especially, Fallahpour *et al.* [3] introduced the following two-dimensional linear stochastic Volterra integral equation by Haar wavelet

$$\begin{aligned} x(t_1,t_2) = f(t_1,t_2) &+ \int_0^{t_2}\int_0^{t_1} \tilde{k}(t_1,t_2,s_1,s_2)\,x(s_1,s_2)\,\mathrm{d}s_1\mathrm{d}s_2 \\ &+ \int_0^{t_2}\int_0^{t_1} \hat{k}(t_1,t_2,s_1,s_2)\,x(s_1,s_2)\,\mathrm{d}B(s_1)\,\mathrm{d}B(s_2), \end{aligned} \tag{1}$$

where $x(t_1,t_2)$ is unknown and called the solution of the Equation (1),

$\tilde{k}(t_1,t_2,s_1,s_2)$, $\hat{k}(t_1,t_2,s_1,s_2)$ and $f(t_1,t_2)$ are known functions $(t_1,t_2)\in[0,T_1)\times[0,T_2)$, $s_1\le t_1, s_2\le t_2$. $B(s_1)$ and $B(s_2)$ are two independent Brownian motions and $\int_0^{t_2}\int_0^{t_1}\hat{k}(t_1,t_2,s_1,s_2)x(s_1,s_2)\mathrm{d}B(s_1)\mathrm{d}B(s_2)$ is the double Itô integral. The authors transformed stochastic Volterra integral equations to algebra equations by Haar wavelet and gave the numerical solutions to the equations. Similarly, Fallahpour *et al.* [7] obtained a numerical method for two-dimensional linear stochastic Volterra integral equations by block pulse functions.

For nonlinear determinate Volterra integral equations, Maleknejad *et al.* [8] and Nemati *et al.* [6] used two-dimensional block pulse functions and Legendre polynomials to solve those respectively. Both Babolian *et al.* [2] and Maleknejad *et al.* [9] employed triangular functions to get the numerical solutions. Mirzaee *et al.* [5] [10] applied modified two-dimensional block pulse functions to approximate the following determinate equation

$$f(t_1,t_2)=\int_0^{t_1}\int_0^{t_2}\tilde{k}(t_1,t_2,s_1,s_2)\left[x(s_1,s_2)\right]^n\mathrm{d}s_2\mathrm{d}s_1,\ (t_1,t_2)\in[0,T_1)\times[0,T_2), \tag{2}$$

where nonlinear term $\left[x(s_1,s_2)\right]^n$ is power function and $x(s_1,s_2)$ is unknown, n is a positive integer. $\tilde{k}(t_1,t_2,s_1,s_2)$ is determinate kernel function $0\le s_1\le t_1\le T_1, 0\le s_2\le t_2\le T_2$. The authors revealed the accuracy and efficiency of the proposed method by some examples and gave the rate of convergence to the numerical solution.

However, as far as we known, there are hardly any papers about the numerical solution of two-dimensional nonlinear stochastic Itô-Volterra integral equations. Inspired by the above literatures, we introduce an efficient numerical method for the following nonlinear stochastic integral equation based on block pulse functions.

$$\begin{aligned}x(t_1,t_2)=x_0(t_1,t_2)&+\int_0^{t_2}\int_0^{t_1}\tilde{k}(t_1,t_2,s_1,s_2)\sigma\left(x(s_1,s_2)\right)\mathrm{d}s_1\mathrm{d}s_2\\&+\int_0^{t_2}\int_0^{t_1}\hat{k}(t_1,t_2,s_1,s_2)g\left(x(s_1,s_2)\right)\mathrm{d}B(s_1)\mathrm{d}B(s_2),\end{aligned} \tag{3}$$

where $x(t_1,t_2)$ is unknown function and is called the solution of the Equation (3) defined on district $D=[0,1)\times[0,1)$. $x_0(t_1,t_2)$ is known determinate function. $\tilde{k}(t_1,t_2,s_1,s_2)$ and $\hat{k}(t_1,t_2,s_1,s_2)$ are determinate kernel functions. $\int_0^{t_2}\int_0^{t_1}\hat{k}(t_1,t_2,s_1,s_2)g\left(x(s_1,s_2)\right)\mathrm{d}B(s_1)\mathrm{d}B(s_2)$ is the double Itô integral. $B(s_1)$ and $B(s_2)$ are two independent Brownian motions. σ and g are analytical functions.

In Section 2, we recall the definition and properties of block pulse function. In Section 3 and 4, we show the integration operational matrix about two-dimensional block pulse functions. In Section 5, an efficient numerical method to nonlinear stochastic Itô-Volterra integral equation is obtained. In Section 6, the error and the rate of convergence of this method are given. It's important to emphasize that the error is analyzed by Gronwall's inequality and the interchangeability of integral and expectation. However, the norm was used in the literature [11], it is a pity that the interchangeability of norm and integral wasn't proved. In Section

7, we give a numerical example to illustrate the validity of the method. In the final Section 8, we make some conclusions and look ahead to further work.

2. Two-Dimensional Block Pulse Functions

One dimensional block pulse functions (BPFs) have been widely studied and applied to solve different problems. For example, the article [12] and their relative references give a detailed description. A $m_1 m_2$-set of two-dimensional block pulse functions (2D-BPFs) $\phi_{a_1,a_2}(t_1,t_2)$ in the region of $D=[0,1)\times[0,1)$ are defined as:

$$\phi_{a_1,a_2}(t_1,t_2)=\begin{cases}1 & (a_1-1)h_1 \le t_1 < a_1 h_1, (a_2-1)h_2 \le t_2 < a_2 h_2 \\ 0 & \text{otherwise,}\end{cases}$$

where $a_i = 1,2,\cdots,m_i$, $h_i = \frac{1}{m_i}$, $m_i = 2^n$, m_i and n are arbitrary positive integers and $i=1,2$.

Similar to the one-dimensional case [12]. There are some elementary properties for 2D-BPFs as follows:

1) Disjointness:

$$\phi_{a_1,a_2}(t_1,t_2)\phi_{b_1,b_2}(t_1,t_2)=\begin{cases}\phi_{a_1,a_2}(t_1,t_2) & \text{if } a_1=b_1, a_2=b_2 \\ 0 & \text{otherwise,}\end{cases} \tag{4}$$

where $a_i, b_i = 1,2,\cdots,m_i$, $i=1,2$.

2) Orthogonality:

$$\int_0^1\int_0^1 \phi_{a_1,a_2}(t_1,t_2)\phi_{b_1,b_2}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2=\begin{cases}h_1 h_2 & \text{if } a_1=b_1, a_2=b_2 \\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

3) Completeness: for every $f\in\left(L^2(D)\right)$, when m_1 and m_2 approach to the infinity, Parseval's identity holds:

$$\int_0^1\int_0^1 f^2(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2=\sum_{a_1=1}^{\infty}\sum_{a_2=1}^{\infty} f_{a_1,a_2}^2\left\|\phi_{a_1,a_2}(t_1,t_2)\right\|^2, \tag{6}$$

where

$$f_{a_1,a_2}=\frac{1}{h_1 h_2}\int_0^1\int_0^1 f(t_1,t_2)\phi_{a_1,a_2}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2.$$

The set of 2D-BPFs may be written as a vector $\Phi(t_1,t_2)$ of dimension $(m_1 m_2)$:

$$\Phi_{m_1 m_2}(t_1,t_2)=\left(\phi_{1,1}(t_1,t_2),\cdots,\phi_{1,m_2}(t_1,t_2),\cdots,\phi_{m_1,1}(t_1,t_2),\cdots,\phi_{m_1,m_2}(t_1,t_2)\right)^{\mathrm{T}}, \tag{7}$$

where $(t_1,t_2)\in D$.

From the above representation and disjointness property, it follows that:

$$\Phi_{m_1 m_2}(t_1,t_2)\Phi_{m_1 m_2}^{\mathrm{T}}(t_1,t_2)=\begin{pmatrix}\phi_{1,1}(t_1,t_2) & 0 & \cdots & 0 \\ 0 & \phi_{1,2}(t_1,t_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{m_1,m_2}(t_1,t_2)\end{pmatrix}_{m_1 m_2\times m_1 m_2}, \tag{8}$$

$$\Phi_{m_1 m_2}^{\mathrm{T}}(t_1,t_2)\Phi_{m_1 m_2}(t_1,t_2)=1,$$

$$\Phi_{m_1m_2}(t_1,t_2)\Phi^{\mathrm{T}}_{m_1m_2}(t_1,t_2)G=\tilde{G}\Phi_{m_1m_2}(t_1,t_2), \quad (9)$$

where G is a (m_1m_2)-vector and the matrix $\tilde{\boldsymbol{G}}=diag(G)$. Moreover, it is easy to conclude that for every $(m_1m_2)\times(m_1m_2)$ matrix $\boldsymbol{A}$

$$\Phi^{\mathrm{T}}_{m_1m_2}(t_1,t_2)\boldsymbol{A}\Phi_{m_1m_2}(t_1,t_2)=\hat{A}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2), \quad (10)$$

where $\hat{A}$ is a (m_1m_2)-vector with elements equal to the diagonal entries of matrix $\boldsymbol{A}$.

Any function $x(t_1,t_2)$ which is square integrable in the interval D can be expanded in terms of BPFs as

$$x(t_1,t_2)\simeq x_{m_1m_2}(t_1,t_2)=\sum_{a_1=1}^{m_1}\sum_{a_2=1}^{m_2}x_{a_1,a_2}\phi_{a_1,a_2}(t_1,t_2)=X^{\mathrm{T}}_{m_1m_2}\Phi_{m_1m_2}(t), \quad (11)$$

where $x_{m_1m_2}(t_1,t_2)$ is m_1m_2 approximations of 2D-BPFs of $x(t_1,t_2)$, $x_{m_1m_2}(t_1,t_2)$ is a coefficient (m_1m_2)-vector, *i.e.*

$$X_{m_1m_2}=\left(x_{1,1},\cdots,x_{1,m_2},\cdots,x_{m_1,1},\cdots,x_{m_1,m_2}\right)^{\mathrm{T}}, \quad (12)$$

where the block pulse coefficients x_{a_1,a_2} are obtained as

$$x_{a_1,a_2}=\frac{1}{h_1h_2}\int_{(a_2-1)h_2}^{a_2h_2}\int_{(a_1-1)h_1}^{a_1h_1}x(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2.$$

Similarly, a function of four variables $k(t_1,t_2,s_1,s_2)$ on $L^2(D\times D)$ may be approximated with respect to 2D-BPFs such as

$$k(t_1,t_2,s_1,s_2)\simeq\Phi_{m_1m_2}(t_1,t_2)^{\mathrm{T}}\boldsymbol{K}\Phi_{m_1m_2}(s_1,s_2),$$

where $\Phi_{m_1m_2}(t_1,t_2)$ is a 2D-BPFs vector of dimension (m_1m_2), $\boldsymbol{K}$ is the $(m_1m_2)\times(m_1m_2)$ two-dimensional block pulse coefficient matrix in the following form

$$\boldsymbol{K}=\left(\boldsymbol{K}_{a_1b_1}\right)_{m_1\times m_1},\quad \boldsymbol{K}_{a_1b_1}=\left(k_{a_1a_2b_1b_2}\right)_{m_2\times m_2},$$

$a_i,b_i=1,\cdots,m_i,i=1,2$ and two-dimensional block pulse coefficients $k_{a_1a_2b_1b_2}$ are given by

$$k_{a_1a_2b_1b_2}=\frac{1}{h_1^2h_2^2}\left[\int_0^1\int_0^1\int_0^1\int_0^1 k(t_1,t_2,s_1,s_2)\phi_{b_1,b_2}(t_1,t_2)\phi_{a_1,a_2}(s_1,s_2)\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}s_1\mathrm{d}s_2\right]. \quad (13)$$

The more details can also reference to [7].

3. Operational Matrix of Integration

Let $\boldsymbol{M}=(\xi_{ij})_{M_1\times M_2}$ and $\boldsymbol{N}=(\eta_{ij})_{N_1\times N_2}$ be matrices. M_l,N_l are positive integers, $l=1,2$. We have

$$\boldsymbol{M}\otimes\boldsymbol{N}=(\xi_{ij}\boldsymbol{N})=\begin{pmatrix}\xi_{11}\boldsymbol{N} & \xi_{12}\boldsymbol{N} & \cdots & \xi_{1M_2}\boldsymbol{N}\\ \xi_{21}\boldsymbol{N} & \xi_{22}\boldsymbol{N} & \cdots & \xi_{2M_2}\boldsymbol{N}\\ \vdots & \vdots & \ddots & \vdots\\ \xi_{M_11}\boldsymbol{N} & \xi_{M_12}\boldsymbol{N} & \cdots & \xi_{M_1M_2}\boldsymbol{N}\end{pmatrix}_{M_1N_1\times M_2N_2},$$

where $\otimes$ denotes the Kronecker product defined as [13]. Each $\xi_{ij}\boldsymbol{N}$ is a block of size $N_1 \times N_2$, $\boldsymbol{M} \otimes \boldsymbol{N}$ is of size $M_1N_1 \times M_2N_2$.

Then the vector $\Phi_{m_1m_2}(t_1,t_2)$ can be showed as following

$$\begin{aligned}
&\Phi_{m_1m_2}(t_1,t_2) \\
&= \Phi_{m_1}(t_1) \otimes \Phi_{m_2}(t_2) \\
&= \left(\phi_1(t_1),\phi_2(t_1),\cdots,\phi_{m_1}(t_1)\right)^{\mathrm{T}} \otimes \left(\phi_1(t_2),\phi_2(t_2),\cdots,\phi_{m_2}(t_2)\right)^{\mathrm{T}} \\
&= \left(\phi_1(t_1)\phi_1(t_2),\cdots,\phi_1(t_1)\phi_{m_2}(t_2),\cdots,\phi_{m_1}(t_1)\phi_1(t_2),\cdots,\phi_{m_1}(t_1)\phi_{m_2}(t_2)\right)^{\mathrm{T}}
\end{aligned}$$

where $\phi_{a_i}(t_i)$ are one dimensional BPFs, $\Phi_{m_i}(t_i)$ are vectors of one dimensional BPFs, $a_i = 1,2,\cdots,m_i, i = 1,2$.

The integration of the vector $\Phi_{m_1m_2}(t_1,t_2)$ defined in (7) can be approximately obtained as following

$$\begin{aligned}
\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 &= \int_0^{t_2}\int_0^{t_1}\Phi_{m_1}(s_1)\otimes\Phi_{m_2}(s_2)\mathrm{d}s_1\mathrm{d}s_2 \\
&= \int_0^{t_1}\Phi_{m_1}(s_1)\mathrm{d}s_1 \otimes \int_0^{t_2}\Phi_{m_2}(s_2)\mathrm{d}s_2 \\
&\simeq \boldsymbol{P}_1\Phi_{m_1}(t_1)\otimes\boldsymbol{P}_2\Phi_{m_2}(t_2) \\
&= (\boldsymbol{P}_1\otimes\boldsymbol{P})_2\,\Phi_{m_1m_2}(t_1,t_2) \\
&= \boldsymbol{P}\Phi_{m_1m_2}(t_1,t_2),
\end{aligned} \tag{14}$$

where $t_1 \in [0,1), t_2 \in [0,1)$, $\boldsymbol{P}$ is the $(m_1m_2)\times(m_1m_2)$ operational matrix of integration for 2D-BPFs and $\boldsymbol{P}_i$, $(i=1,2)$ are the operational matrix of one-dimensional BPFs [12] defined over $[0,1)$ as following.

$$\boldsymbol{P}_i = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(m_i\times m_i)}.$$

For details, see [7], so

$$\begin{aligned}
\int_0^{t_2}\int_0^{t_1}x(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 &\simeq \int_0^{t_2}\int_0^{t_1}X_{m_1m_2}^T\Phi_{m_1m_2}(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 \\
&= X_{m_1m_2}^T\boldsymbol{P}\Phi_{m_1m_2}(t_1,t_2).
\end{aligned} \tag{15}$$

4. Stochastic Integration Operational Matrix

Similarly, we obtain the stochastic integration of the vector $\Phi_{m_1m_2}(t_1,t_2)$ defined in (7) as following

$$\begin{aligned}
&\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}(s_1,s_2)\mathrm{d}B(s_1)\mathrm{d}B(s_2) \\
&= \int_0^{t_2}\int_0^{t_1}\Phi_{m_1}(s_1)\otimes\Phi_{m_2}(s_2)\mathrm{d}B(s_1)\mathrm{d}B(s_2) \\
&= \int_0^{t_1}\Phi_{m_1}(s_1)\mathrm{d}B(s_1)\otimes\int_0^{t_2}\Phi_{m_2}(s_2)\mathrm{d}B(s_2) \\
&\simeq \boldsymbol{P}_{s_1}\Phi_{m_1}(t_1)\otimes\boldsymbol{P}_{s_2}\Phi_{m_2}(t_2) \\
&= \left(\boldsymbol{P}_{s_1}\otimes\boldsymbol{P}_{s_2}\right)\Phi_{m_1m_2}(t_1,t_2) = \boldsymbol{P}_s\Phi_{m_1m_2}(t_1,t_2),
\end{aligned} \tag{16}$$

where $t_1 \in [0,1), t_2 \in [0,1)$, $\boldsymbol{P}_s$ is the $(m_1 m_2)\times(m_1 m_2)$ stochastic operational matrix of integration for 2D-BPFs and $\boldsymbol{P}_{s_i}$, $(i=1,2)$ are the stochastic operational matrix of one-dimensional BPFs [12] defined over $[0,1)$ as following.

$$\boldsymbol{P}_{s_i} = \begin{pmatrix} B\left(\frac{h_i}{2}\right) & B(h_i) & B(h_i) & \cdots & B(h_i) \\ 0 & B\left(\frac{3h_i}{2}\right)-B(h_i) & B(2h_i)-B(h_i) & \cdots & B(2h_i)-B(h_i) \\ 0 & 0 & B\left(\frac{5h_i}{2}\right)-B(5h_i) & \cdots & B(3h_i)-B(2h_i) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B\left(\frac{(2m_i-1)h_i}{2}\right)-B((m_i-1)h_i) \end{pmatrix}_{m_i\times m_i}. \quad (17)$$

For details, see [7]. Therefore,

$$\begin{aligned} \int_0^{t_2}\int_0^{t_1} x(s_1,s_2)\,\mathrm{d}B(s_1)\,\mathrm{d}B(s_2) &\simeq \int_0^{t_2}\int_0^{t_1} X_{m_1m_2}^{\mathrm{T}}\Phi_{m_1m_2}(s_1,s_2)\,\mathrm{d}B(s_1)\,\mathrm{d}B(s_2) \\ &= X_{m_1m_2}^{\mathrm{T}}\boldsymbol{P}_s\Phi_{m_1m_2}(t_1,t_2). \end{aligned} \quad (18)$$

5. Numerical Method

In this section, we first provide a useful result for solving two-dimensional nonlinear stochastic Itô-Volterra integral Equation (3).

Lemma 1. *Let* $\sigma(t)=\sum a_j t^j$, $g(t)=\sum b_j t^j$ *be the analytic functions for positive integer* $j\in(0,\infty)$, *then*

$$\sigma\left(x_{m_1m_2}(t_1,t_2)\right)=\sigma^{\mathrm{T}}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(t_1,t_2),$$

$$g\left(x_{m_1m_2}(t_1,t_2)\right)=g^{\mathrm{T}}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(t_1,t_2),$$

where $\Phi_{m_1m_2}(t_1,t_2)$ and $X_{m_1m_2}$ are derived in (7) and (12),

$$\sigma^{\mathrm{T}}\left(X_{m_1m_2}\right)=\left(\sigma(x_{1,1}),\cdots,\sigma(x_{1,m_2}),\cdots,\sigma(x_{m_1,1}),\cdots,\sigma(x_{m_1,m_2})\right),$$

$$g^{\mathrm{T}}\left(X_{m_1m_2}\right)=\left(g(x_{1,1}),\cdots,g(x_{1,m_2}),\cdots,g(x_{m_1,1}),\cdots,g(x_{m_1,m_2})\right).$$

Proof. By virtue of the known conditions and the disjointness properties of 2D-BPFs defined in (4), we can get

$$\begin{aligned} \sigma\left(x_{m_1m_2}(t_1,t_2)\right) &= \sum a_j\left(x_{m_1m_2}(t_1,t_2)\right)^j = \sum a_j\left[\sum_{a_1=1}^{m_1}\sum_{a_2=1}^{m_2} x_{a_1,a_2}\phi_{a_1,a_2}(t_1,t_2)\right]^j \\ &= \sum a_j\left[x_{1,1}\phi_{1,1}(t_1,t_2)+\cdots+x_{1,m_2}\phi_{1,m_2}(t_1,t_2)+\cdots+x_{m_1,1}\phi_{m_1,1}(t_1,t_2)\right. \\ &\quad \left.+\cdots+x_{m_1,m_2}\phi_{m_1,m_2}(t_1,t_2)\right]^j \\ &= \sum a_j\left(x_{1,1}^j,\cdots,x_{1,m_2}^j,\cdots,x_{m_1,1}^j,\cdots,x_{m_1,m_2}^j\right)\Phi_{m_1m_2}(t_1,t_2) \\ &= \sigma^{\mathrm{T}}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(t_1,t_2), \end{aligned}$$

thus,

$$\sigma\left(x_{m_1m_2}(t_1,t_2)\right)=\sigma^{\mathrm{T}}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(t_1,t_2)=\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\sigma\left(X_{m_1m_2}\right), \quad (19)$$

$$g\left(x_{m_1m_2}\left(t_1,t_2\right)\right)=g^{\mathrm{T}}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}\left(t_1,t_2\right)=\Phi_{m_1m_2}^{\mathrm{T}}\left(t_1,t_2\right)g\left(X_{m_1m_2}\right). \tag{20}$$

The proof is completed. □

Now we suppose $x(t_1,t_2)$, $x_0(t_1,t_2)$, $\sigma(x(t_1,t_2))$, $g(x(t_1,t_2))$, $\tilde{k}(t_1,t_2,s_1,s_2)$ and $\hat{k}(t_1,t_2,s_1,s_2)$ can be approximated in terms of 2D-BPFs.

$$x(t_1,t_2)\simeq x_{m_1m_2}(t_1,t_2)=X_{m_1m_2}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)=\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)X_{m_1m_2}, \tag{21}$$

$$x_0(t_1,t_2)\simeq x_{0_{m_1m_2}}(t_1,t_2)=X_{0_{m_1m_2}}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)=\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)X_{0_{m_1m_2}}, \tag{22}$$

$$\sigma\left(x(t_1,t_2)\right)\simeq\sigma\left(x_{m_1m_2}(t_1,t_2)\right)=\sigma^{\mathrm{T}}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(t_1,t_2)=\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\sigma\left(X_{m_1m_2}\right), \tag{23}$$

$$g\left(x(t_1,t_2)\right)\simeq g\left(x_{m_1m_2}(t_1,t_2)\right)=g^{\mathrm{T}}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(t_1,t_2)=\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)g\left(X_{m_1m_2}\right), \tag{24}$$

$$\tilde{k}(t_1,t_2,s_1,s_2)\simeq\tilde{k}_{m_1m_2}(t_1,t_2,s_1,s_2)=\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_1\Phi_{m_1m_2}(s_1,s_2), \tag{25}$$

$$\hat{k}(t_1,t_2,s_1,s_2)\simeq\hat{k}_{m_1m_2}(t_1,t_2,s_1,s_2)=\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_2\Phi_{m_1m_2}(s_1,s_2), \tag{26}$$

where $X_{m_1m_2}$, $X_{0_{m_1m_2}}$, $\sigma\left(X_{m_1m_2}\right)$ and $g\left(X_{m_1m_2}\right)$ are two-dimensional block pulse coefficient vectors. $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are two-dimensional block pulse coefficient matrices.

Now, by (21)-(26), we approximate the Equation (3)

$$\begin{aligned}
&X_{m_1m_2}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)\\
&=X_{0_{m_1m_2}}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)\\
&\quad+\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_1\Phi_{m_1m_2}(s_1,s_2)\Phi_{m_1m_2}^{\mathrm{T}}(s_1,s_2)\sigma\left(X_{m_1m_2}\right)\mathrm{d}s_1\mathrm{d}s_2\\
&\quad+\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_2\Phi_{m_1m_2}(s_1,s_2)\Phi_{m_1m_2}^{\mathrm{T}}(s_1,s_2)g\left(X_{m_1m_2}\right)\mathrm{d}B(s_1)\mathrm{d}B(s_2)\\
&=X_{0_{m_1m_2}}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)\\
&\quad+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_1\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}(s_1,s_2)\Phi_{m_1m_2}^{\mathrm{T}}(s_1,s_2)\sigma\left(X_{m_1m_2}\right)\mathrm{d}s_1\mathrm{d}s_2\\
&\quad+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_2\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}(s_1,s_2)\Phi_{m_1m_2}^{\mathrm{T}}(s_1,s_2)g\left(X_{m_1m_2}\right)\mathrm{d}B(s_1)\mathrm{d}B(s_2)\\
&=X_{0_{m_1m_2}}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)\\
&\quad+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_1\int_0^{t_2}\int_0^{t_1}\tilde{\sigma}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2\\
&\quad+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_2\int_0^{t_2}\int_0^{t_1}\tilde{g}\left(X_{m_1m_2}\right)\Phi_{m_1m_2}(s_1,s_2)\mathrm{d}B(s_1)\mathrm{d}B(s_2)\\
&=X_{0_{m_1m_2}}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)\\
&\quad+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_1\tilde{\sigma}\left(X_{m_1m_2}\right)\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}(t_1,t_2)\mathrm{d}s_1\mathrm{d}s_2\\
&\quad+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_2\tilde{g}\left(X_{m_1m_2}\right)\int_0^{t_2}\int_0^{t_1}\Phi_{m_1m_2}(t_1,t_2)\mathrm{d}B(s_1)\mathrm{d}B(s_2),
\end{aligned}$$

by (15) and (18), we have

$$\begin{aligned}
&X_{m_1m_2}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)\\
&=X_{0_{m_1m_2}}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_1\tilde{\sigma}\left(X_{m_1m_2}\right)\boldsymbol{P}\Phi_{m_1m_2}(t_1,t_2)\\
&\quad+\Phi_{m_1m_2}^{\mathrm{T}}(t_1,t_2)\boldsymbol{K}_2\tilde{g}\left(X_{m_1m_2}\right)\boldsymbol{P}_s\Phi_{m_1m_2}(t_1,t_2),
\end{aligned}$$

let $\boldsymbol{Q}=\boldsymbol{K}_1\tilde{\sigma}\left(X_{m_1m_2}\right)\boldsymbol{P}$ and $\boldsymbol{Q}_s=\boldsymbol{K}_2\tilde{g}\left(X_{m_1m_2}\right)\boldsymbol{P}_s$, they both are $(m_1m_2)\times(m_1m_2)$

matrices. By (10), we have

$$X_{m_1m_2}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)=X_{0_{m_1m_2}}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)+\hat{Q}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)+\hat{Q}_s^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2),$$

where $\hat{Q}$ and $\hat{Q}_s$ are (m_1m_2)-vectors with elements equal to the diagonal entries of matrices $\boldsymbol{Q}$ and $\boldsymbol{Q}_s$. Then

$$X_{m_1m_2}^{\mathrm{T}}=X_{0_{m_1m_2}}^{\mathrm{T}}+\hat{Q}^{\mathrm{T}}+\hat{Q}_s^{\mathrm{T}}. \tag{27}$$

There are various methods to solve the nonlinear system of Equation (27) of $X_{m_1m_2}$. In this paper, we will use the int () function provided by Matlab 2015b [14] to solve it. According to the coefficient vector $X_{m_1m_2}$, we obtain that the approximation solution of Equation (3) $x_{m_1m_2}(t_1,t_2)=X_{m_1m_2}^{\mathrm{T}}\Phi_{m_1m_2}(t_1,t_2)$.

6. Error Analysis

In this section, for convenience, we assume $m_1=m_2=m$ and prove that the approximation solution is convergent of order $O(h), h=\frac{1}{m}$.

Lemma 2. *Let $v(s_1,s_2)$ be an arbitrary bounded function on $D=[0,1)\times[0,1)$ and $\tilde{e}_{mm}(s_1,s_2)=v(s_1,s_2)-v_{mm}(s_1,s_2)$, which $v_{mm}(s_1,s_2)$ is m^2 approximations of 2D-BPFs of $v(s_1,s_2)$, then*

$$\|\tilde{e}\|_{L^2(D)}^2=\int_0^1\int_0^1\tilde{e}_{mm}^2(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2\le O(h^2). \tag{28}$$

Proof. Similar to [15] [16]. □

Lemma 3. *Let $v(t_1,t_2,s_1,s_2)$ be an arbitrary bounded function on $D\times D$ and $\hat{e}_{mm}(t_1,t_2,s_1,s_2)=v(t_1,t_2,s_1,s_2)-v_{mm}(t_1,t_2,s_1,s_2)$, which $v_{mm}(t_1,t_2,s_1,s_2)$ is m^2 approximations of 2D-BPFs of $v(t_1,t_2,s_1,s_2)$, then*

$$\|\hat{e}\|_{L^2(D\times D)}^2=\int_0^1\int_0^1\int_0^1\int_0^1\hat{e}_{mm}^2(t_1,t_2,s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2\le O(h^2). \tag{29}$$

Proof. Similar to [15] [16]. □

Next, let

$$\begin{aligned}
e_{mm}(t_1,t_2)&=x(t_1,t_2)-x_{mm}(t_1,t_2)\\
&=x_0(t_1,t_2)-x_{0_{mm}}(t_1,t_2)+\int_0^{t_2}\int_0^{t_1}\Big[\tilde{k}(t_1,t_2,s_1,s_2)\sigma\big(x(s_1,s_2)\big)\\
&\quad-\tilde{k}_{mm}(t_1,t_2,s_1,s_2)\sigma\big(x_{mm}(s_1,s_2)\big)\Big]\mathrm{d}s_1\mathrm{d}s_2\\
&\quad+\int_0^{t_2}\int_0^{t_1}\Big[\hat{k}(t_1,t_2,s_1,s_2)g\big(x(s_1,s_2)\big)\\
&\quad-\hat{k}_{mm}(t_1,t_2,s_1,s_2)g\big(x_{mm}(s_1,s_2)\big)\Big]\mathrm{d}B(s_1)\mathrm{d}B(s_2).
\end{aligned} \tag{30}$$

where $x_{mm}(t_1,t_2)$ is the approximation solution of $x(t_1,t_2)$ defined in (3), $x_{0_{mm}}(t_1,t_2)$, $\tilde{k}_{mm}(t_1,t_2,s_1,s_2)$ and $\hat{k}_{mm}(t_1,t_2,s_1,s_2)$ are m^2 approximations of 2D-BPFs of $x_0(t_1,t_2),\tilde{k}(t_1,t_2,s_1,s_2)$ and $\hat{k}(t_1,t_2,s_1,s_2)$, respectively.

Theorem 1. *For analytic functions σ and g, there are constant numbers satisfy the following conditions:*

1) $|\sigma(x)-\sigma(y)|\le l_1|x-y|$, $|g(x)-g(y)|\le l_3|x-y|$,
2) $|\sigma(x)|\le l_2,|g(y)|\le l_4$,

where $x, y \in R$ and let $\left|\tilde{k}(t_1,t_2,s_1,s_2)\right| \le l_5, \left|\hat{k}(t_1,t_2,s_1,s_2)\right| \le l_6$ be determinate bounded kernel functions, where l_i, $i = 1,2,\cdots,6$ are constant numbers. Then,

$$\int_0^T \int_0^T \mathbb{E}\left(\left|e_{mm}(t_1,t_2)\right|^2\right) \mathrm{d}t_1 \mathrm{d}t_2$$
$$= \int_0^T \int_0^T \mathbb{E}\left(\left|x(t_1,t_2) - x_{mm}(t_1,t_2)\right|^2\right) \mathrm{d}t_1 \mathrm{d}t_2 \le O\left(h^2\right), \quad T \in [0,1).$$

Proof. For (30), we have

$$\mathbb{E}\left(\left|e_{mm}(t_1,t_2)\right|^2\right) \le 3\Bigg[\mathbb{E}\left(\left|x_0(t_1,t_2) - x_{0_{mm}}(t_1,t_2)\right|^2\right)$$
$$+ \mathbb{E}\Bigg(\Bigg|\int_0^{t_2}\int_0^{t_1}\Big[\tilde{k}(t_1,t_2,s_1,s_2)\sigma(x(s_1,s_2))$$
$$- \tilde{k}_{mm}(t_1,t_2,s_1,s_2)\sigma(x_{mm}(s_1,s_2))\Big]\mathrm{d}s_1\mathrm{d}s_2\Bigg|^2\Bigg)$$
$$+ \mathbb{E}\Bigg(\Bigg|\int_0^{t_2}\int_0^{t_1}\Big[\hat{k}(t_1,t_2,s_1,s_2)g(x(s_1,s_2))$$
$$- \hat{k}_{mm}(t_1,t_2,s_1,s_2)g(x_{mm}(s_1,s_2))\Big]\mathrm{d}B(s_1)\mathrm{d}B(s_2)\Bigg|^2\Bigg)\Bigg].$$

According to Itô isometry, Cauchy-Schwartz inequality and Lipschitz conditions, we can write

$$\mathbb{E}\left(\left|e_{mm}(t_1,t_2)\right|^2\right)$$
$$\le 3\Bigg[\mathbb{E}\left(\left|x_0(t_1,t_2) - x_{0_{mm}}(t_1,t_2)\right|^2\right)$$
$$+ \mathbb{E}\left(\int_0^{t_2}\int_0^{t_1}\left|\tilde{k}(t_1,t_2,s_1,s_2)\sigma(x(s_1,s_2)) - \tilde{k}_{mm}(t_1,t_2,s_1,s_2)\sigma(x_{mm}(s_1,s_2))\right|^2 \mathrm{d}s_1\mathrm{d}s_2\right)$$
$$+ \mathbb{E}\left(\int_0^{t_2}\int_0^{t_1}\left|\hat{k}(t_1,t_2,s_1,s_2)g(x(s_1,s_2)) - \hat{k}_{mm}(t_1,t_2,s_1,s_2)g(x_{mm}(s_1,s_2))\right|^2 \mathrm{d}s_1\mathrm{d}s_2\right)\Bigg]$$
$$= 3\Bigg[\mathbb{E}\left(\left|x_0(t_1,t_2) - x_{0_{mm}}(t_1,t_2)\right|^2\right)$$
$$+ \int_0^{t_2}\int_0^{t_1}\mathbb{E}\Big(\Big|\tilde{k}(t_1,t_2,s_1,s_2)\big[\sigma(x(s_1,s_2)) - \sigma(x_{mm}(s_1,s_2))\big]$$
$$+ \sigma(x_{mm}(s_1,s_2))\big[\tilde{k}(t_1,t_2,s_1,s_2) - \tilde{k}_{mm}(t_1,t_2,s_1,s_2)\big]\Big|^2\Big)\mathrm{d}s_1\mathrm{d}s_2$$
$$+ \int_0^{t_2}\int_0^{t_1}\mathbb{E}\Big(\Big|\hat{k}(t_1,t_2,s_1,s_2)\big[g(x(s_1,s_2)) - g(x_{mm}(s_1,s_2))\big]$$
$$+ g(x_{mm}(s_1,s_2))\big[\hat{k}(t_1,t_2,s_1,s_2) - \hat{k}_{mm}(t_1,t_2,s_1,s_2)\big]\Big|^2\Big)\mathrm{d}s_1\mathrm{d}s_2\Bigg]$$
$$\le 3\Bigg[\left|x_0(t_1,t_2) - x_{0_{mm}}(t_1,t_2)\right|^2$$
$$+ 2l_1^2 l_5^2 \int_0^{t_2}\int_0^{t_1}\mathbb{E}\left(\left|e_{mm}(s_1,s_2)\right|^2\right)\mathrm{d}s_1\mathrm{d}s_2$$
$$+ 2l_2^2 \int_0^{t_2}\int_0^{t_1}\left|\tilde{k}(t_1,t_2,s_1,s_2) - \tilde{k}_{mm}(t_1,t_2,s_1,s_2)\right|^2 \mathrm{d}s_1\mathrm{d}s_2$$
$$+ 2l_3^2 l_6^2 \int_0^{t_2}\int_0^{t_1}\mathbb{E}\left(\left|e_{mm}(s_1,s_2)\right|^2\right)\mathrm{d}s_1\mathrm{d}s_2$$
$$+ 2l_4^2 \int_0^{t_2}\int_0^{t_1}\left|\hat{k}(t_1,t_2,s_1,s_2) - \hat{k}_{mm}(t_1,t_2,s_1,s_2)\right|^2 \mathrm{d}s_1\mathrm{d}s_2\Bigg].$$

Then, we can get

$$\mathbb{E}\left(\left|e_{mm}\left(t_1,t_2\right)\right|^2\right) \le \beta\left(t_1,t_2\right) + \alpha\int_0^{t_2}\int_0^{t_1}\mathbb{E}\left(\left|e_{mm}\left(s_1,s_2\right)\right|^2\right)\mathrm{d}s_1\mathrm{d}s_2,$$

where,

$$\alpha = 6\left(l_1^2 l_5^2 + l_3^2 l_6^2\right).$$

$$\beta\left(t_1,t_2\right) = 3\Big[\left|x_0\left(t_1,t_2\right) - x_{0_{mm}}\left(t_1,t_2\right)\right|^2 + 2l_2^2\int_0^{t_2}\int_0^{t_1}\left|\tilde{k}\left(t_1,t_2,s_1,s_2\right) - \tilde{k}_{mm}\left(t_1,t_2,s_1,s_2\right)\right|^2 \mathrm{d}s_1\mathrm{d}s_2 + 2l_4^2\int_0^{t_2}\int_0^{t_1}\left|\hat{k}\left(t_1,t_2,s_1,s_2\right) - \hat{k}_{mm}\left(t_1,t_2,s_1,s_2\right)\right|^2 \mathrm{d}s_1\mathrm{d}s_2\Big].$$

Let $f\left(t_1,t_2\right) = \mathbb{E}\left(\left|e_{mm}\left(t_1,t_2\right)\right|^2\right)$, we get

$$f\left(t_1,t_2\right) \le \beta\left(t_1,t_2\right) + \alpha\int_0^{t_2}\int_0^{t_1} f\left(\tau_1,\tau_2\right)\mathrm{d}\tau_1\mathrm{d}\tau_2, \quad \tau_1 \in \left[0,t_1\right), \tau_2 \in \left[0,t_2\right).$$

By Gronwall's inequality, we have

$$f\left(t_1,t_2\right) \le \beta\left(t_1,t_2\right) + \alpha\int_0^{t_2}\int_0^{t_1} \mathrm{e}^{\int_0^{\tau_2}\int_0^{\tau_1}\alpha\mathrm{d}s_1\mathrm{d}s_2}\beta\left(\tau_1,\tau_2\right)\mathrm{d}\tau_1\mathrm{d}\tau_2, \quad t_1,t_2 \in \left[0,1\right).$$

Then, for $T \in \left[0,1\right)$

$$\begin{aligned}
&\int_0^T\int_0^T f\left(t_1,t_2\right)\mathrm{d}t_1\mathrm{d}t_2 \\
&= \int_0^T\int_0^T \mathbb{E}\left(\left|e_{mm}\left(t_1,t_2\right)\right|^2\right)\mathrm{d}t_1\mathrm{d}t_2 \\
&\le \int_0^T\int_0^T\left(\beta\left(t_1,t_2\right) + \alpha\int_0^{t_2}\int_0^{t_1}\mathrm{e}^{\int_0^{\tau_2}\int_0^{\tau_1}\alpha\mathrm{d}s_1\mathrm{d}s_2}\beta\left(\tau_1,\tau_2\right)\mathrm{d}\tau_1\mathrm{d}\tau_2\right)\mathrm{d}t_1\mathrm{d}t_2 \\
&= \int_0^T\int_0^T\beta\left(t_1,t_2\right)\mathrm{d}t_1\mathrm{d}t_2 + \alpha\int_0^T\int_0^T\int_0^{t_2}\int_0^{t_1}\mathrm{e}^{\int_0^{\tau_2}\int_0^{\tau_1}\alpha\mathrm{d}s_1\mathrm{d}s_2}\beta\left(\tau_1,\tau_2\right)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}t_1\mathrm{d}t_2 \\
&\le \int_0^T\int_0^T\beta\left(t_1,t_2\right)\mathrm{d}t_1\mathrm{d}t_2 + \alpha\mathrm{e}^{\alpha T^2}\int_0^T\int_0^T\int_0^{t_2}\int_0^{t_1}\beta\left(\tau_1,\tau_2\right)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}t_1\mathrm{d}t_2 \\
&= 3\int_0^T\int_0^T\left|x_0\left(t_1,t_2\right) - x_{0_{mm}}\left(t_1,t_2\right)\right|^2\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + 6l_2^2\int_0^T\int_0^T\int_0^{t_2}\int_0^{t_1}\left|\tilde{k}\left(t_1,t_2,s_1,s_2\right) - \tilde{k}_{mm}\left(t_1,t_2,s_1,s_2\right)\right|^2\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + 6l_4^2\int_0^T\int_0^T\int_0^{t_2}\int_0^{t_1}\left|\hat{k}\left(t_1,t_2,s_1,s_2\right) - \hat{k}_{mm}\left(t_1,t_2,s_1,s_2\right)\right|^2\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + \alpha\mathrm{e}^{\alpha T^2}\Big[3\int_0^T\int_0^T\int_0^{t_2}\int_0^{t_1}\left|x_0\left(\tau_1,\tau_2\right) - x_{0_{mm}}\left(\tau_1,\tau_2\right)\right|^2\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + 6l_2^2\int_0^T\int_0^T\int_0^{t_2}\int_0^{t_1}\int_0^{\tau_2}\int_0^{\tau_1}\left|\tilde{k}\left(\tau_1,\tau_2,s_1,s_2\right) - \tilde{k}_{mm}\left(\tau_1,\tau_2,s_1,s_2\right)\right|^2\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + 6l_4^2\int_0^T\int_0^T\int_0^{t_2}\int_0^{t_1}\int_0^{\tau_2}\int_0^{\tau_1}\left|\hat{k}\left(\tau_1,\tau_2,s_1,s_2\right) - \hat{k}_{mm}\left(\tau_1,\tau_2,s_1,s_2\right)\right|^2\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}t_1\mathrm{d}t_2\Big] \\
&= 3I_1 + 6l_2^2 I_2 + 6l_4^2 I_3 + \alpha\mathrm{e}^{\alpha T^2}\left[3I_4 + 6l_2^2 I_5 + 6l_4^2 I_6\right],
\end{aligned}$$

by using (28) (29), the integrals

$$I_i \le c_i h^2, \ i = 1,2,\cdots,6,$$

the last equation can be converted into

$$\begin{aligned}
&\int_0^T\int_0^T\mathbb{E}\left|e_{mm}\left(t_1,t_2\right)\right|^2\mathrm{d}t_1\mathrm{d}t_2 \\
&\le \left[\left(3c_1 + 6l_2^2 c_2 + 6l_4^2 c_3\right) + \alpha\mathrm{e}^{\alpha T^2}\left(3c_4 + 6l_2^2 c_5 + 6l_4^2 c_6\right)\right]h^2 \le O\left(h^2\right).
\end{aligned}$$

where $c_i, i = 1,2,\cdots,6$ are independent nonnegative constants.

The proof is completed. □

7. Numerical Examples

In the last section, we give a numerical example which illustrates the feasibility of the above method. The approximation solutions and mean solutions of the equations are shown in Figures 1-4.

Example 1. Consider the following two-dimensional nonlinear stochastic Itô-Volterra integral equation (one-dimensional case can reference to Example 1 in [17]).

$$x(t_1,t_2)=\frac{1}{10}-\left(\frac{1}{30}\right)^2\int_0^{t_2}\int_0^{t_1}x(s_1,s_2)\left(1-x^2(s_1,s_2)\right)\mathrm{d}s_1\mathrm{d}s_2 \\ +\frac{1}{30}\int_0^{t_2}\int_0^{t_1}\left(1-x^2(s_1,s_2)\right)\mathrm{d}B(s_1)\mathrm{d}B(s_2).$$

The front view and the top view of the approximation solutions of the Example 1 for $m = 8$ are given in Figure 1.

The front view and the top view of the mean solutions of the Example 1 for $m = 8$ are given in Figure 2.

The front view and the top view of the approximation solutions of the Example 1 for $m = 16$ are given in Figure 3.

The front view and the top view of the mean solutions of the Example 1 for $m = 16$ are given in Figure 4.

From these figures, we find the general trends of the solutions are similar for different m, and the absolute error of mean solution is very small. This method is efficient and the accuracy is credible.

8. Conclusion

For some stochastic Volterra integral equations, exact solutions cannot be expressed. But, the numerical solution can be conveniently obtained based on different stochastic numerical methods. As the complexity of the system, we use

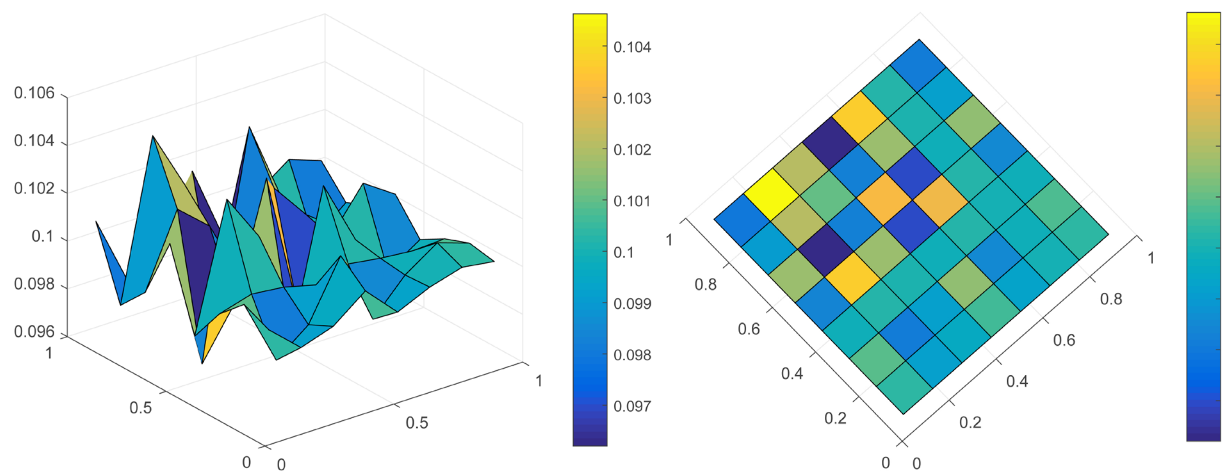

Figure 1. The front view and top view of the approximation solutions for $m = 8$.

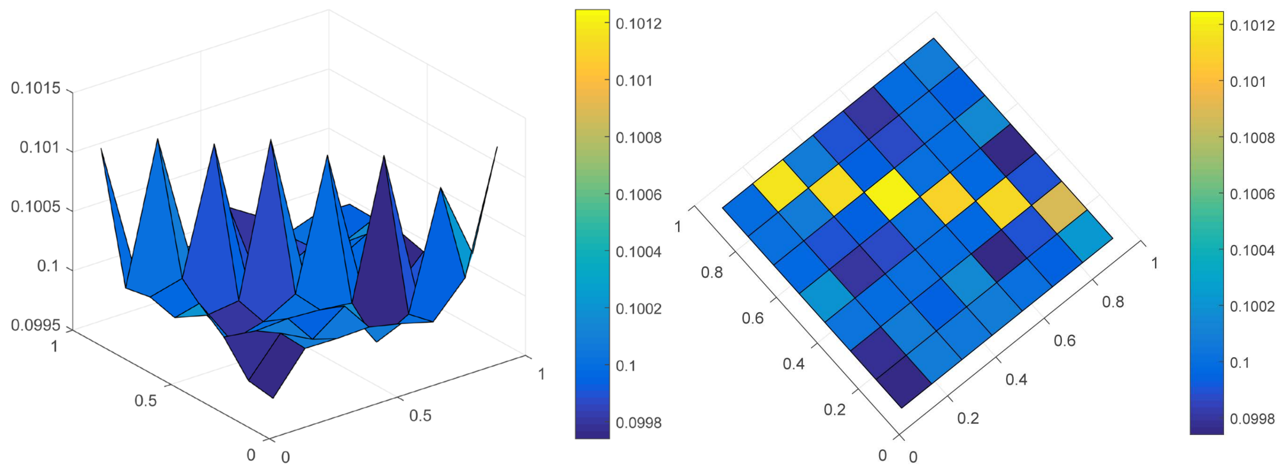

Figure 2. The front view and top view of the mean solutions for $m = 8$.

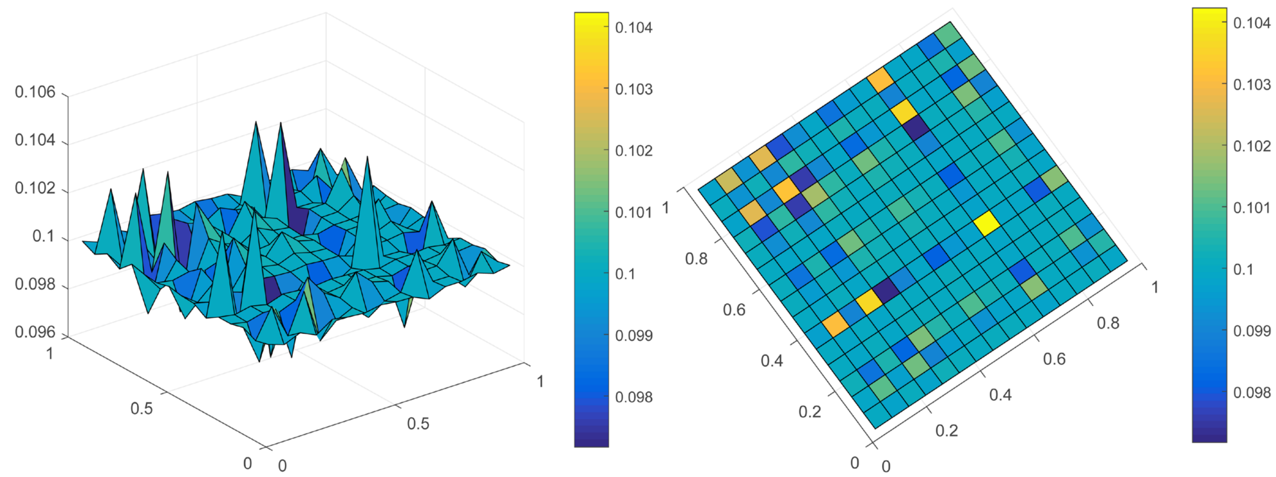

Figure 3. The front view and top view of the approximation solutions for $m = 16$.

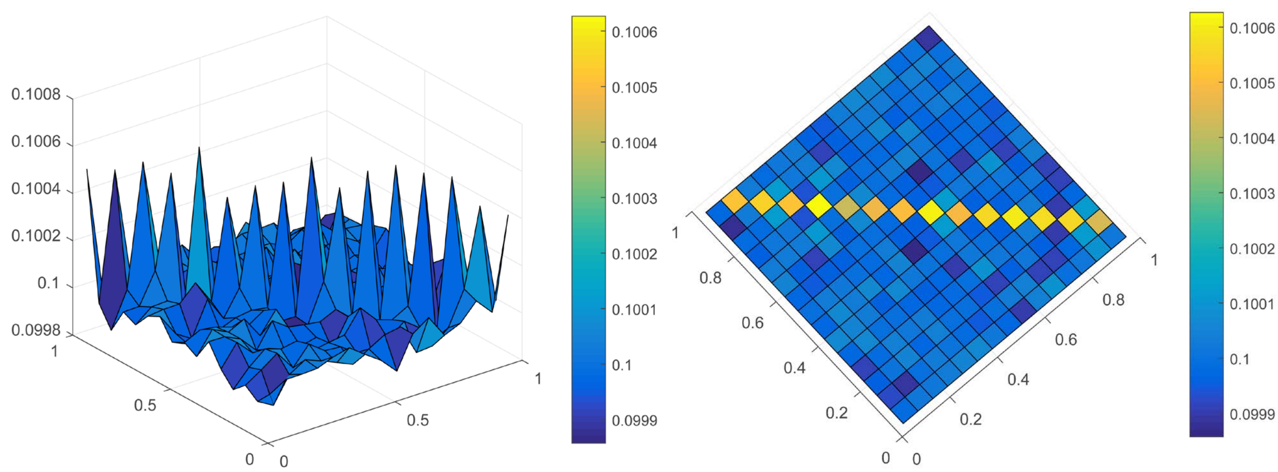

Figure 4. The front view and top view of the mean solutions for $m = 16$.

BPFs as the basis function to solve the two-dimensional nonlinear stochastic Volterra integral equation. This numerical method is simple and effective. In the future, we will try to extend it to n-dimensional space and solve more problems.

Acknowledgements

We thank the Editors and the Reviewers for their helps and comments. This article is funded by NSF Grants 11471105 of China, NSF Grants 2016CFB526 of Hubei Province, Innovation Team of the Educational Department of Hubei Province T201412, and Innovation Items of Hubei Normal University 2018032 and 2018105. These supports are greatly appreciated.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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