Existence and Stability Property of Almost Periodic Solutions in Discrete Almost Periodic Systems

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Abstract

In this paper, we consider an almost periodic system which includes a system of the type $x(n+1) = \sum_{j-1}^{m} a_{ij}(n) x_j^k (1 \le i \le m)$, where k is a positive integer, a_{ij} are almost periodic in n and satisfy $a_{ij}(n) \ge 0$ for $i \ne j$, $\sum_{i=1}^{m} a_{ij}(n) = 0$ for $1 \le j \le m$. In the special case where $a_{ij}(n)$ are constant functions, above system is a mathematical model of gas dynamics and was treated by T. Carleman and R. D. Jenks for differential systems. In the main theorem, we show that if the $m \times m$ matrix $(a_{ij}(n))$ is irreducible, then there exists a positive almost periodic solution which is unique and has some stability. Moreover, we can see that this result gives R. D. Jenks' result for differential model in the case where $a_{ij}(n)$ are constant functions. In Section 3, we consider the linear system with variable cofficients $x(n+1) = A(n)x(n), x \in R^m$. Even in nonlinear problems, this linear system plays an important role, as their variational equations, and it is requested to determine the uniform asymptotically stability of the zero solution from the information about $A(n)$. In order to obtain the existence of almost periodic solutions of both linear and nonlinear almost periodic discrete systems: above linear system and $x_i(n+1) = \sum_{j=1}^{m} a_{ij}(n) g_j(x_j(n))$ for $1 \le i \le m$, respectively, we shall consider between certain stability properties, which are referred to as uniformly asymptotically stable, and the diagonal dominance matrix condition.

Keywords

Almost Periodic Solutions, Linear and Nonlinear Almost Periodic Discrete Systems, Uniformly Asymptotically Stable, Diagonal Dominance Matrix Condition

1. Introduction

System of almost periodic difference equations has been studied to describe phenomena of oscillations in the natural and social sciences. The investigation of almost periodic systems has been developed quite widely during the twentieth century, since relationships with the stability theory have been found. A main interest of the subject is the existence theorem for almost periodic solutions. Obviously an almost periodic solution is a bounded solution, but the existence of bounded solutions does not necessarily imply the existence of almost periodic solutions. Therefore, in order to prove the existence of almost periodic solutions, we need some additional conditions to the existence of bounded solutions. A main subject of the investigation has been to find such additional conditions, and up to now, many conditions have been considered (for example, in the linear system, J. Favard's separation condition [1]).

In the Section 4, we consider the nonlinear almost periodic system of

$$x_i(n+1) = \sum_{j=1}^{m} a_{ij}(n) x_j(n)^k, \quad 1 \le i \le m, \tag{1}$$

where k is a positive integer, $a_{ij}(n)$ are almost periodic in n and satisfy

i) $\sum_{i=1}^{m} a_{ij}(n) = 0, \ 1 \le j \le m,$

ii) $a_{ij}(n) \ge 0$ for $i \ne j$.

In the special case where $a_{ij}(n)$ are constant functions, system (1) is a mathematical model of gas dynamics and was treated by T. Carleman [2] and R. D. Jenks [3]. In the main theorem, we show that if the $m \times m$ matrix $\left(a_{ij}(n) \right)$ is irreducible, then there exists a positive almost periodic solution which is unique and some stability. Moreover, we can see that this result gives R. D. Jenks' result in the case where $a_{ij}(n)$ are constant functions. In the Section 5, we consider the linear almost periodic system with variable coefficients

$$x(n+1) = A(n)x(n), \ n \ge n_0 \ge 0, \tag{2}$$

where $x \in R^m$. Even in nonlinear problems, system (2) plays an important role, as their variational equations and moreover, it is requested to determine the uniformly asymptotic stability of the zero solution from the condition about $A(n)$. When $A(n)$ is a constant matrix, it is well known that the stability is equivalent to the following condition (cf. [4]);

"Absolute values of all eigenvalues of $A(n) \equiv A$ are less than one."

However, it is not true in the case of variable coefficients, and hence we need additional conditions to (2). In the main theorem, we show that one of the such conditions is the diagonal dominance matrix condition on $A(n)$ [5], that is, $A(n)$ satisfies

$$\left| a_{ii}(n) \right| \ge \sum_{k=1, k \ne i}^{m} \left| a_{ki}(n) \right|, \ 1 \le i \le m.$$

This result improves a stability criterion based on results of F. Nakajima [6] for differential equations.

2. Preliminaries

We denote by R^m the real Euclidean m-space. Let $R = (-\infty, \infty)$ and $R^+ = [0, \infty)$. $\mathbf{Z}$ is the set of integers, $\mathbf{Z}^+$ is the set of nonnegative integers. For $x \in R^m$, let $|x|$ be the Euclidean norm of x and x_i be the i-th component. Let

$$D = \left\{ x \in R^m \mid x_i \geq 0 \text{ for } 1 \leq i \leq m \right\},$$

$$\Omega = \left\{ x \in D \mid \sum_{i=1}^{m} x_i = 1 \right\}$$

and

$$\Pi = \left\{ x \in R^m \mid \sum_{i=1}^{m} x_i = 0 \right\}.$$

We introduce an almost periodic function $f(n,x) : \mathbf{Z} \times U \to R^m$, where U is an open set in R^m.

Definition 1. $f(n,x)$ is said to be almost periodic in n uniformly for $x \in U$, if for any $\epsilon > 0$ and any compact set K in U there exists a positive integer $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains an integer τ for which

$$\left| f(n + \tau, x) - f(n, x) \right| \leq \epsilon$$

for all $n \in \mathbf{Z}$ and all $x \in K$. Such a number τ in above inequality is called an ϵ-translation number of $f(n,x)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above definition, we discuss the concept of the normality of almost periodic functions. Namely, let $f(n,x)$ be almost periodic in n uniformly for $x \in U$. Then, for any sequence $\{h_k'\} \subset \mathbf{Z}$, there exist a subsequence $\{h_k\}$ of $\{h_k'\}$ and a function $g(n,x)$ such that

$$f(n + h_k, x) \to g(n,x) \tag{3}$$

uniformly on $\mathbf{Z} \times K$ as $k \to \infty$, where K is a compact set in U. There are many properties of the discrete almost periodic functions [7], which are corresponding properties of the continuous almost periodic functions $f(t,x) \in C(R \times U, R^m)$ [cf. [8] [9]]. We denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(n,x) = f(n + \tau, x), \quad \tau \in \mathbf{Z}. \tag{4}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (4). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and $f(n + n_k, x) \to g(n,x)$ uniformly on $\mathbf{Z} \times S$ for any compact subset S in R^m. Specially, for a function $f(n)$ on $\mathbf{Z}$ with values in R^m, $H(f)$ denotes the set of all function $g(n)$ such that for some sequence $\{n_k\}$,

$$f(n + n_k) \to g(n) \text{ in } \mathbf{Z} \text{ as } k \to \infty,$$

where the symbol "$\to$" stands for the uniformly convergence on any compact set

in **Z** (in short, "in **Z**"). Clearly, $f \in H(f)$.

By (3), if $f : \mathbf{Z} \times U \to R^m$ is almost periodic in n uniformly for $x \in U$, so is a function in $\Omega(f)$.

We define the irreducible matrix to need after.

Definition 2. An $m \times m$ matrix $A(n) = (a_{ij}(n))$ is said to be irreducible if for any two nonempty disjoint subsets I and J of the set of m integers $\{1, 2, \cdots, m\}$ with $I \bigcup J = \{1, 2, \cdots, m\}$, there exists an i in I and a j in J such that $a_{ij}(n) \not\equiv 0$. In the case where $A(n)$ is scalar, $A(n)$ is said to be irreducible if $A(n) \not\equiv 0$. Otherwise, $A(n)$ is said to be reducible, and we can assume that $A(n)$ takes the form of

$$A(n) = \left(\begin{array}{c|c} * & 0 \\ \hline *' & B(n) \end{array} \right),$$

where $*$ is $l \times l (1 \leq l \leq m)$ square matrix, $*'$ is $m - l \times l$ matrix, $B(n)$ is $m - l \times m - l$ zero or a square irreducible matrix.

3. Linear Systems

We consider the system of linear difference equation

$$x(n+1) = A(n) x(n), \tag{5}$$

where $x \in R^m$ and the $m \times m$ matrix $A(n) = (a_{ij}(n))$ is bounded on **Z** and almost periodic function in n. We state discretization of Jenks and Nakajima' results for differential equations [3] [10].

Now we define stability properties with respect to the subset K in R^m. Here, we denote by $x(n, n_0, x_0)$ the solution of system (5) with initial condition (n_0, x_0).

Definition 3. The bounded solution $u(n)$ of system (5) defined on **Z** is said to be;

i) uniformly stable (in short, U.S.) in K on **Z⁺** if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $|u(n) - x(n, n_0, x_0)| < \epsilon$ for all $n \geq n_0$ whenever $x_0 \in K$ and $|u(n_0) - x_0| < \delta(\epsilon)$ at some n_0 in **Z⁺**.

ii) uniformly asymptotically stable (in short, U.A.S.) in K on **Z⁺** if it is U.S. in K on **Z⁺** and if there exists a $\delta_0 > 0$ and, if for any $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that $|u(n) - x(n, n_0, x_0)| < \epsilon$ for all $n \geq n_0 + T(\epsilon)$ whenever $x_0 \in K$ and $|u(n_0) - x_0| < \delta_0$ at some n_0 in **Z⁺**.

iii) uniformly asymptotically stable (in short, U.A.S.) in the whole K on **Z⁺** if it is U.S. in K on **Z⁺** and if for any $\epsilon > 0$ and $r > 0$ there exists a $T(\epsilon, r) > 0$ such that $|u(n) - x(n, n_0, x_0)| < \epsilon$ for all $n \geq n_0 + T(\epsilon, r)$ whenever $x_0 \in K$ and $|u(n_0) - x_0| < r$, at some n_0 in **Z⁺**.

When **Z⁺** in the definitions (i), (ii) and (iii) is replaced by **Z**, we say that $x(n)$ is U.S. in K on **Z**, U.A.S. in K on **Z** and U.A.S. in the whole K on **Z**, respectively. Clearly Definition 3 agrees with the definitions of the usual stability properties in the case where $K = R^m$.

Throughout this paper, we suppose the following conditions;

i) $\sum_{i=1}^{m} a_{ij}(n) = 0$ for $1 \le j \le m$,

ii) $a_{ij}(n) \ge 0$ for $i \ne j$

and

iii) each element in $H(A)$ is irreducible.

First of all, we prove the following lemmas.

Lemma 1. Consider the m-equations $x_i(n+1) = f_i(n, x(n))$, $1 \le i \le m$, where $f_i(n, x)$ is continuous on second variable x in R^m, and assume that the initial value problem has a unique solution.

a) If $\sum_{i=1}^{m} f_i(n, x) = 0$, then the set Π is invariant.

b) If $f_i(n, x) \ge 0$ for $x_i = 0$ and $x_j \ge 0$, then the set D is positively invariant, and in addition,

if $\sum_{i=1}^{m} f_i(n, x) = 0$, then the set Ω is positively invariant.

In the case of differential equations, the proof of the similar lemma is obvious (for instance, see [11]). We modify it to prove this lemma, but we omit it.

Lemma 2. If conditions (i) and (ii) are satisfied, then the trivial solution of system (5) is U.S. in Π on **Z** and also it is U.S. on **Z**.

By modifying theorem in [5], we can easily prove Lemma 2 at same technique.

Lemma 3. If each element in $H(A)$ is irreducible, then the each element in $H(A)$, we say $B \in H(A)$ and $B(n) = (b_{ij}(n))$, has the property that for any two nonempty disjoint subsets I and J of the set of m integers $\{1, 2, \cdots, m\}$ with $I \bigcup J = \{1, 2, \cdots, m\}$, there exists an $i \in I$ and $j \in J$ such that

$$\overline{\lim}_{n \to \infty} |b_{ij}(n)| \ne 0.$$

Proof. Suppose not, Then there exists a $B(n) = (b_{ij}(n))$ in $H(A)$ and two nonempty disjoint subsets I and J of $\{1, 2, \cdots, m\}$ with $I \bigcup J = \{1, 2, \cdots, m\}$ such that

$$\overline{\lim}_{n \to \infty} b_{ij}(n) = 0 \text{ for all } i \in I \text{ and } j \in J.$$

Since $B(n)$ is bounded on **Z**, there exists a subsequence $n_k, n_k \to \infty$ as $k \to \infty$, such that

$$B(n + n_k) \to C(n) \text{ in } \mathbf{Z} \text{ as } k \to \infty,$$

where $C(n) = (c_{ij}(n)) \in H(A)$. Clearly,

$$c_{ij}(n) = \lim_{k \to \infty} b_{ij}(n + n_k) = 0 \text{ for } n \in \mathbf{Z} \text{ and } i \in I, j \in J.$$

This show the reducibility of $C(n)$, which is a contradiction. This proves Lemma 3.

For system (5), we consider the system in $H(A)$ of

$$x(n+1) = B(n)x(n), \tag{6}$$

where $B \in H(A)$.

Lemma 4. Assume that conditions (ii) and (iii) are satisfied for system (5),

and let $x(n)$ be a nontrivial solution of system (6) such that $x(n) \in D$ on **Z**. Then there exists a constant $c > 0$ such that

$$x_i(n)/\|x(n)\| \geq c \text{ for } n \in \mathbf{Z} \text{ and } 1 \leq i \leq m.$$

Proof. Let $x(n) = (x_1(n), x_2(n), \cdots, x_m(n))$ be a solution of system (6) such that $x(n) \in D$ on **Z**. First of all, we show that if $x_i(n_0) = 0$ at some $n_0 \in \mathbf{Z}$, then

$$x_i(n) = 0 \text{ for all } n \geq n_0.$$

Since $x_i(n)$ satisfies the equation

$$x_i(n+1) = b_{ii}(n) x_i(n) + \sum_{j \neq i}^{m} b_{ij}(n) x_j(n),$$

where $(b_{ij}(n)) = B(n)$. Moreover, since $\sum_{j \neq i}^{m} b_{ij}(n) x_j(n) \geq 0$, we have

$$x_i(n+1) \geq b_{ii}(n) x_i(n), \tag{7}$$

which implies

$$x_i(n) \leq x_i(n_0) \prod_{s=n_0}^{n-1} b_{ii}(s) \text{ for } n \leq n_0.$$

Thus, we obtain

$$x_i(n) = 0 \text{ for } n \leq n_0.$$

Because $x_i(n_0) = 0$ and $x_i(n) \geq 0$ on **Z**. Now suppose that Lemma 4 is not true. Then for some B in $H(A)$, the corresponding system (6) has a nontrivial solution $x(n)$, $x(n) \in D$ on **Z**, such that for some sequence n_k,

$$x_1(n_k)/\|x(n_k)\| \to 0 \text{ as } k \to \infty. \tag{8}$$

Set $\phi_k(n) = x(n+n_k)/\|x(n_k)\|$. Then, $\phi_k(n)$ satisfies

$$x(n+1) = B(n+n_k) x(n)$$

and

$$\phi_k(n) \in D \text{ on } \mathbf{Z}, |\phi_k(0)| = 1.$$

Since the sequence $\{\phi_k(0)\}$ is bounded, $\{\phi_k(n)\}$ is uniformly bounded on any finite interval in **Z**, and hence there is a convergent subsequence of ϕ_k, which is again denoted by ϕ_k, such that

$\phi_k(n) \to y(n)$ in **Z** for some function $y(n)$ as $k \to \infty$.

We can also assume that

$$B(n+n_k) \to C(n) \text{ in } \mathbf{Z} \text{ as } k \to \infty,$$

where $C \in H(A)$ and $C(n) = (c_{ij}(n))$. Therefore, $y(n)$ is the solution of the system

$$y(n+1) = C(n) y(n), \tag{9}$$

$y(n) \in D$ on **Z** and $|y(0)| = 1$. Moreover (8) implies that $y_1(0) = 0$. Thus, as was proved above, we have

$$y_1(n) = 0 \text{ for } n \leq 0,$$

For this $y(n)$, we define two subsets I and J of $\{1, 2, \cdots, m\}$ by $I = \{1 \leq i \leq m \mid y_i(n) \equiv 0 \text{ for } n \leq N_i$, where N_i depends on $y_i(n)\}$ and $J = \{1 \leq i \leq m \mid y_i(n) > 0 \text{ on } \mathbf{Z}\}$. Then $I \bigcup J = \{1, 2, \cdots, m\}$, $\{1\} \in I$ and $J \neq \phi$ since $y(n) \neq 0$. By Lemma 3,

$$\overline{\lim}_{n \to \infty} \left| c_{i_0 j_0}(n) \right| \neq 0 \text{ for some } i_0 \in I \text{ and some } j_0 \in J. \tag{10}$$

Now the i_0-th equation of system (9) takes the form of

$$y_{i_0}(n+1) = \sum_{k \in I} c_{i_0 k}(n) y_k(n) + \sum_{k \in J} c_{i_0 k}(n) y_k(n),$$

and hence

$$\sum_{k \in J} c_{i_0 k}(n) y_k(n) = 0 \text{ for } n+1 \leq \min_{i \in I} N_i, \tag{11}$$

because of the definition of the set I. Since each term in the left hand side of (11) is nonnegative, all of them are equal to zero. Therefore

$$c_{i_0 j_0}(n) y_{j_0}(n) = 0 \text{ for } n+1 \leq \min_{i \in I} N_i,$$

which implies, by (10),

$$y_{j_0}(n_0) = 0 \text{ at some } n_0.$$

This contradicts the definition of the set of J. The proof is completed.

The following proposition is an immediate result of Lemma 4.

Proposition 1. Under conditions (ii) and (iii), system (6) has no nontrivial solution $x(n)$ such that

$$x(n) \in \partial D \text{ on } \mathbf{Z},$$

where $\partial D = \{x \in D \mid x_i = 0 \text{ for some } i, 1 \leq i \leq m\}$.

We next consider a non-homogeneous system corresponding to system (5)

$$x(n+1) = A(n)x(n) + f(n) \tag{12}$$

and assume that $A(n)$ satisfies conditions (i), (ii) and (iii).

Lemma 5. If $f(n)$ is bounded on $\mathbf{Z}$ with values in R^m and $\sum_{i=1}^{m} f_i(n-1)$ is bounded on $\mathbf{Z}^+$, then all solutions of system (12) are bounded on $\mathbf{Z}^+$.

Proof. It is sufficient to show that (12) has at least one bounded solution on $\mathbf{Z}^+$, because the trivial solution of (5) is U.S. by Lemma 2. We consider the system with real parameter ϵ

$$x(n+1) = A(n)x(n) + \epsilon f(n) \tag{13}$$

and show that for a sufficiently small ϵ, system (13) has a bounded solution on $\mathbf{Z}^+$, which implies the existence of a bounded solution on $\mathbf{Z}^+$ for system (12) by replacing x in (13) with x/ϵ. For a $0 < \delta < 1/\sqrt{m}$ and for the m-vector e each of whose components is 1, let D' be a convex cone defined by

$$D' = \left\{ x \in R^m \mid \langle e, x \rangle \geq |e| \cdot |x| \cdot \delta \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product and $|x|^2 = \langle x, x \rangle$. Clearly, $D \subset D'$.

Every solution $x(n)$ of (13) satisfies

$$\sum_{i=1}^{m} x_i(n+1) = \epsilon \sum_{i=1}^{m} f_i(n)$$

because of condition (i). By replacing n with n-1 of the above both sides,

$$\sum_{i=1}^{m} x_i(n) = \epsilon \sum_{i=1}^{m} f_i(n-1) \leq \tilde{M} + \epsilon M \quad \text{for } x(0) \in \Omega,$$

where

$\tilde{M} > 0$ is sufficient small number and $M = \sup_{n>0} \sum_{i=1}^{m} f_i(n-1)$.

When $x(n) \in D'$, we have

$$\sum_{i=1}^{m} x_i(n) = \langle e, x(n) \rangle \geq |x(n)| \cdot |e| \cdot \delta,$$

and hence,

$$\left(\tilde{M} - \epsilon M\right) / |e| \leq |x(n)| \leq \left(\tilde{M} + \epsilon M\right) / \left(|e| \cdot \delta\right), \tag{14}$$

Therefore, in order to show the boundedness of $x(n)$ with $x(0)$ in Ω, it is sufficient to prove that $x(n) \in D'$ on $\mathbf{Z}^+$ if ϵ is sufficiently small. Now suppose that for each solution $x_\epsilon(n)$ of (13) with $x_\epsilon(0)$ in Ω, there exists an $l_\epsilon > 0$ such that

$$x_\epsilon(l_\epsilon) \in \partial D'.$$

We can assume that

$$x_\epsilon(n_\epsilon) \in \partial D \quad \text{at some } n_\epsilon, 0 \leq n_\epsilon < l_\epsilon$$

and

$$x_\epsilon(n) \in \overline{D' - D} \quad \text{for } n_\epsilon \leq n \leq l_\epsilon,$$

where ∂K and $\overline{K}$ denote the boundary and the closure of the set K, respectively. If we set $y_\epsilon(n) = x_\epsilon(n + n_\epsilon)$, $y_\epsilon(n)$ is a solution of the system

$$y(n+1) = A(n + n_\epsilon) y(n) + \epsilon f(n + n_\epsilon)$$

such that $y_\epsilon(0) \in \partial D, y_\epsilon(\tau_\epsilon) \in \partial D'$ at $\tau_\epsilon = l_\epsilon - n_\epsilon > 0$ and $y_\epsilon(n) \in \overline{D' - D}$ for $0 \leq n \leq \tau_\epsilon$. Thus, by (14),

$$\left(\tilde{M} - \epsilon M\right) / |e| \leq |y_\epsilon(n)| \leq \left(\tilde{M} + \epsilon M\right) / \left(|e| \cdot \delta\right) \quad \text{for } 0 \leq n \leq \tau_\epsilon,$$

The same argument in the proof of Lemma 4 enables us to assume that
$y_\epsilon(n) \to w(n)$ in $\mathbf{Z}$ for some function $w(n)$ as $\epsilon \to 0$
and
$A(n + n_\epsilon) \to B(n)$ in $\mathbf{Z}$ for some $B \in H(A)$ as $\epsilon \to 0$
Therefore, $w(n)$ satisfies $w(n+1) = B(n)w(n)$ and clearly, for $\tau = \liminf_{\epsilon \to 0} \tau_\epsilon$,

$$w(n) \in \overline{D' - D} \quad \text{for } 0 \leq n < \tau \tag{15}$$

and

$$\tilde{M}/|e| \le |w(n)| \le \tilde{M}/(|e| \cdot \delta) \quad \text{for } 0 \le n \le \tau. \tag{16}$$

Moreover we have $w(0) \in \partial D$, which implies by Lemma 1 that

$$w(n) \in D \text{ on } \mathbf{Z}^+.$$

From this and (15) it follows that

$$w(n) \in D \cap \overline{D' - D} = \partial D \quad \text{for } 0 \le n < \tau. \tag{17}$$

Now we show that $\tau = \infty$. In fact, if $\tau < \infty$, we have

$$y_\epsilon(\tau_\epsilon) \to w(\tau) \text{ as } \epsilon \to 0.$$

Thus $w(\tau) \in \partial D'$ because $y_\epsilon(\tau_\epsilon) \in \partial D'$, and hence

$$w(\tau) \in \partial D' \cap \partial D = \{0\},$$

which contradicts (16). Therefore (16) and (17) hold for $\tau = \infty$. Moreover this enables us to assume that

$$\tilde{M}/|e| \le |w(n)| \le \tilde{M}/(|e| \cdot \delta) \quad \text{for all } n \in \mathbf{Z}$$

and

$$w(n) \in \partial D \quad \text{for all } n \in \mathbf{Z}.$$

Because $H(B)$ is compact in the sense of the convergence. This contradicts the conclusion in Proposition 1. This proves that $x(n) \in D'$ on $\mathbf{Z}^+$ if ϵ is sufficiently small. The proof is completed.

Lemma 6. Under the assumptions (i) and (ii), if for each B in $H(A)$, the trivial solution of the system

$$x(n+1) = B(n)x(n)$$

is U.S. on $\mathbf{Z}$ and U.A.S. on $\mathbf{Z}^+$, then the trivial solution of system (5) is U.A.S. on $\mathbf{Z}$.

Proof. Let $x(n, n_0, x_0)$ be the solution of (5). Since the trivial solution of (5) is U.S. on $\mathbf{Z}$ by Lemma 2, as is seen from (ii) in Definition 3, it is sufficient to show that for any $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that

$$|x(n, n_0, x_0)| < \delta(\epsilon) \quad \text{for some } n, n_0 \le n \le n_0 + T(\epsilon),$$

whenever $n_0 \in \mathbf{Z}$ and $|x_0| < \delta_0 = \delta(1)$, where $\delta(\cdot)$ is the number in (i) of Definition 3.

Now suppose that there exists an $\epsilon > 0$ and sequences $\{n_k\}$ in $\mathbf{Z}$ and $\{x_k\}$ in R^m such that $|x_k| \le \delta_0$ and

$$|x(n, n_k, x_k)| \ge \delta(\epsilon) \quad \text{for all } n_k, n_k \le n \le n_k + k.$$

Since $|x_k| < \delta_0 = \delta(1)$,

$$\delta(\epsilon) \le |x(n, n_k, x_k)| \le 1 \quad \text{for } n_k, n_k \le n \le n_k + k.$$

Set $\phi_k(n) = x(n + n_k, n_k, x_k)$. Then, $\phi_k(n)$ satisfies

$$x(n+1) = A(n + n_k)x(n)$$

and

$$\delta(\epsilon) \le |\phi_k(n)| \le 1 \text{ for } 0 \le n \le k.$$

We can assume that

$$\phi_k(n) \to y(n) \text{ in } \mathbf{Z}^+ \text{ for some function } y(n) \text{ as } k \to \infty$$

and

$$A(n + n_k) \to B(n) \text{ in } \mathbf{Z} \text{ for some } B \in H(A) \text{ as } k \to \infty.$$

Therefore $y(n)$ is a solution of the system

$$y(n+1) = B(n) y(n) \tag{18}$$

and

$$\delta(\epsilon) \le |y(n)| \le 1 \text{ on } \mathbf{Z}^+.$$

On the other hand, we have

$$y(n) \to 0 \text{ as } n \to \infty,$$

because the trivial solution of (18) is U.A.S. on $\mathbf{Z}^+$. Therefore there arises a contradiction. Thus the proof is completed.

We show the following theorem, before we will mention a definition of the exponential dichotomy of a linear system;

System (5) is said to possess an exponential dichotomy if there exists a projection matrix P and positive constants $K_1, K_2, \rho_1, \rho_2, \sigma_1$ and σ_2 such that

$$\left\| \Phi(n) P \Phi^{-1}(l) \right\| \le K_1 \rho_1^{n-l} \left(0 < \rho_1 = e^{-\sigma_1} \right), \text{ for } n \ge l,$$

$$\left\| \Phi(n)(I - P) \Phi^{-1}(l) \right\| \le K_2 \rho_2^{n-l} \left(0 < \rho_2 = e^{-\sigma_2} \right), \text{ for } n \le l,$$

where, I is a identical matrix and Φ is a fundamental matrix solution of system (5) (cf. [4] [5] [8]).

Theorem 1. Assume that system (5) satisfies conditions (i), (ii) and (iii), Then the trivial solution of system (5) is U.A.S. in Π on $\mathbf{Z}$.

Proof. On the set Π which is invariant for system (5), the system is written as the $(m-1)$-system

$$\tilde{x}(n+1) = \tilde{A}(n) \tilde{x}(n) \tag{19}$$

where $\tilde{x} = (x_1, x_2, \cdots, x_{m-1}) \in R^{m-1}$ and $\tilde{A}(n)$ is an $(m-1) \times (m-1)$ matrix whose (i, j) element is given by $a_{ij}(n) - a_{im}(n)$ for $1 \le i, j \le m-1$. First of all, we can show that for each $\tilde{B}$ in $H(\tilde{A})$, the system

$$\tilde{x}(n+1) = \tilde{B}(n) \tilde{x}(n) \tag{20}$$

has an exponential dichotomy on $\mathbf{Z}^+$ since (20) has at least one bounded solution, and as is well known (cf. [12]), it is equivalent to show the system

$$\tilde{x}(n+1) = \tilde{B}(n) \tilde{x}(n) + \tilde{f}(n) \tag{21}$$

possesses at least one bounded solution on $\mathbf{Z}^+$ for any bounded function $\tilde{f}(n)$ on $\mathbf{Z}^+$. For each $\tilde{B}(n)$ in $H(\tilde{A})$ there corresponds some $B(n) = (b_{ij}(n))$ in $H(A)$ such that the (i, j) element of $\tilde{B}(n)$ is equal to $b_{ij}(n) - b_{im}(n)$ for $1 \le i, j \le m-1$. For $\tilde{f}(n) = (f_1(n), f_2(n), \cdots, f_{m-1}(n))$, let $g(n)$ be defined by

$$g_i(n) = f_i(n) \text{ for } 1 \le i \le m-1,$$

$$g_m(n) = -\sum_{i=1}^{m-1} f_i(n).$$

Obviously $g(n)$ and $\sum_{i=1}^{m} g_i(n-1)(=0)$ are bounded on $\mathbf{Z}^+$. Applying Lemma 5 to the m-system

$$x(n+1) = B(n)x(n) + g(n),$$

we obtain the bounded solution $x(n)$ on $\mathbf{Z}^+$ with $x(0) = 0$, and $\sum_{i=1}^{m} g_i(n-1) = 0$ which yields

$$\sum_{i=1}^{m} x_i(n) = 0.$$

Hence we can verify that $\tilde{x}(n) = (x_1(n), x_2(n), \cdots, x_{m-1}(n))$ is a bounded solution on $\mathbf{Z}^+$ of system (21). The exponential dichotomy of (20) implies further that the trivial solution is U.A.S. on $\mathbf{Z}^+$, because the trivial solution is U.S. on $\mathbf{Z}$ by Lemma 2. Therefore it follows from Lemma 6 that the trivial solution of (19) is U.A.S. on $\mathbf{Z}$, *i.e.*, the trivial solution of (5) is U.A.S. in Π on $\mathbf{Z}$. The proof is completed.

4. Nonlinear Systems

We consider the nonlinear almost periodic system of

$$x_i(n+1) = \sum_{j=1}^{m} a_{ij}(n) g_j(x_j(n)) \text{ for } 1 \le i \le m, \tag{22}$$

where $A(n) = (a_{ij}(n))$ is almost periodic function of n with conditions

(iv) $\sum_{i=1}^{m} a_{ij}(n) = 0, \ 1 \le j \le m$

and

(v) $a_{ij}(n) \ge 0$ for $i \ne j$.

In addition, assume that $g_j(u)$ are continuously differentiable for $u \ge 0$, $g_j(0) = 0$ and $\dot{g}_j(u) > 0$ for real number $u > 0$, where $\dot{g}_j(u)$ is the derivative of $g_j(u)$ at u.

We first consider the linear system

$$x(n+1) = L(n)x(n) \tag{23}$$

and its perturbed system

$$x(n+1) = L(n)x(n) + f(n, x(n)), \tag{24}$$

where $L(n)$ is an $m \times m$ matrix function, almost periodic function in n, $f(n, x)$ is continuous with respect to its second argument and $f(n, x) = o(|x|)$ uniformly for $n \in \mathbf{Z}$. Assume that the set Π is invariant for both system (23) and (24).

First of all, we can prove the following lemmas.

Lemma 7. If the trivial solution of system (23) is U.A.S. in Π on $\mathbf{Z}$, then the trivial solution of system (24) has also the same stability property.

Proof. Let $\tilde{x} = (x_1, x_2, \cdots, x_{m-1})$ for $x = (x_1, x_2, \cdots, x_m) \in R^m$. Then there are

positive constants c_1 and c_2 such that

$$c_1|\tilde{x}| \leq |x| \leq c_2|\tilde{x}| \text{ for } x \in \Pi, \tag{25}$$

because $x_m = -(x_1 + x_2 + \cdots + x_{m-1})$. On the set Π, systems (23) and (24) are written as

$$\tilde{x}(n+1) = \tilde{L}(n)\tilde{x}(n) \tag{26}$$

and

$$\tilde{x}(n+1) = \tilde{L}(n)\tilde{x}(n) + g(n, \tilde{x}(n)) \tag{27}$$

respectively, where the (i, j) element of $\tilde{L}(n), 1 \leq i, j \leq m-1$, is given by $l_{ij}(n) - l_{im}(n)$ for $L(n) = (l_{ij}(n))$ and $g(n, \tilde{x}) = o(|\tilde{x}|)$ uniformly for $n \in \mathbf{Z}$. Inequality (25) shows that the trivial solution of (23) is U.A.S. in Π if and only if the trivial solution of (26) is U.A.S., and we have also the same equivalence between (24) and (27). As is well known, if the trivial solution of (26) is U.A.S., then the trivial solution of (27) has also the same stability property. Thus our assertion is proved.

The following lemma is obtained by the slight modification of the difference equation to Seifert's result [13]. Then, we will omit the proof (cf. [9]).

We consider the almost periodic nonlinear system

$$x(n+1) = f(n, x(n)), n \in \mathbf{Z}, x \in R^m, \tag{28}$$

where $f(n, x)$ is almost periodic in n uniformly for $x \in R^m$ and for a constant $L^* > 0$, $|f(n, x) - f(n, y)| \leq L^*|x - y|$ for $n \in \mathbf{Z}$ and $x, y \in \Omega$.

Lemma 8. Assume that the set Ω is positively invariant for system (28) and all solutions in Ω on $\mathbf{Z}$ are U.A.S. in Ω on $\mathbf{Z}$. Then the set of such solutions is finite and consists of only almost periodic solutions $\phi_1, \phi_2, \cdots, \phi_m$ which satisfy $|\phi_i(n) - \phi_j(n)| \geq \beta$ on $\mathbf{Z}$ for $i \neq j$ and some constant $\beta > 0$.

Now we can show the following theorem. Since the last statements of the following theorem are alternative, under each assumption of these statements we can prove the existence of almost periodic solutions in Ω and the module containment.

Theorem 2. Under the assumptions (iv) and (v), system (22) has a nontrivial almost periodic solution in Ω whose module is contained in the module of $A(n)$. In addition to the above assumptions, if $A(n)$ is irreducible, then the almost periodic solution of (22) is unique in Ω, which remains in Ω^0 on $\mathbf{Z}$, and it is U.A.S. in the whole Ω on $\mathbf{Z}$, where $\Omega^0 = \{x \in \Omega \mid x_i > 0 \text{ for all } i, 1 \leq i \leq m\}$. Moreover, if $A(n)$ is reducible, then at least one of the above almost periodic solutions $p(n)$ satisfies that $p(n) \in \partial\Omega$ on $\mathbf{Z}$, where $\partial\Omega = \{x \in \Omega \mid x_i = 0 \text{ for some } i, 1 \leq i \leq m\}$.

Proof. First of all, we consider the case where $A(n)$ is irreducible. Since system (22) satisfies the conditions of Lemma 1, the set Ω is positively invariant, namely, $y(n) \in \Omega$ on $\mathbf{Z}^+$ for a solution $y(n)$ of (22) with $y(0) \in \Omega$, and furthermore we can assume that

$$y(n) \in \Omega \text{ on } \mathbf{Z}$$

because of the almost periodicity of $A(n)$. We can show that this $y(n)$ is U.A.S. in Ω on $\mathbf{Z}$. If we set $x = y(n) + w$ in system (22), then $w \in \Pi$ for x in Ω and

$$w_i(n+1) = \sum_{j=1}^{m} a_{ij}(n) \dot{g}_j(y_j(n)) w_j + o(|w|) \text{ for } 1 \leq i \leq m, \tag{29}$$

And Π is invariant for the above system. Considering the first approximation of system (29)

$$w(n+1) = L(n) w(n), \tag{30}$$

where $L(n) = (l_{ij}(n))$ is defined by $l_{ij}(n) = a_{ij}(n) \dot{g}_j(y_j(n))$, condition (iv) implies that Π is also invariant for (30). Then, by Lemma 6, if the trivial solution of (30) is shown to be U.A.S. in Π on $\mathbf{Z}$, then the trivial solution of (29) has the same stability, and consequently $y(n)$ is U.A.S. in Ω on $\mathbf{Z}$. Therefore it is sufficient to show that the trivial solution of (30) is U.A.S. in Π on $\mathbf{Z}$. Clearly $L(n)$ is bounded and we have

$$\sum_{i=1}^{m} l_{ij}(n) = \sum_{i=1}^{m} a_{ij}(n) \dot{g}_j(y_j(n)) = 0 \text{ for } 1 \leq j \leq m$$

and

$$l_{ij}(n) = a_{ij}(n) \dot{g}_j(y_j(n)) \geq 0 \text{ for } i \neq j \tag{31}$$

because of conditions (iv) and (v), respectively. Thus $L(n)$ satisfies conditions (i) and (ii). Condition (iii) will be verified in the following way. Applying the same argument as in the proof of Lemma 4 to system (22), we can see that there exists a constant $c > 0$ such that

$$1 \geq y_i(n) \geq c \text{ for } n \in \mathbf{Z} \text{ and } 1 \leq i \leq m, \tag{32}$$

and hence there is a constant $\tilde{c} > 0$ such that

$$\dot{g}_j(y_j(n)) \geq \tilde{c} \text{ for } n \in \mathbf{Z} \text{ and } 1 \leq j \leq m.$$

Therefore, (31) implies

$$l_{ij}(n) \geq \tilde{c} a_{ij}(n) \text{ for } i \neq j,$$

which guarantees that each element of $H(L)$ is irreducible, because $A(n)$ is irreducible and almost periodic. Thus it follows from Theorem 1 that the trivial solution of (30) is U.A.S. in Π on $\mathbf{Z}$, *i.e.*, all solutions of system (22) in Ω on $\mathbf{Z}$ are U.A.S. in Ω on $\mathbf{Z}$. Therefore Lemma 8 concludes that system (22) possesses an almost periodic solution in Ω which remains in Ω^0 by (32), and the set of solutions in Ω on $\mathbf{Z}$ is finite and consists of only almost periodic solutions $\phi_1, \phi_2, \cdots, \phi_m$ which satisfy

$$|\phi_i(n) - \phi_j(n)| \geq \beta \text{ on } \mathbf{Z} \text{ for } i \neq j \text{ and some constant } \beta > 0.$$

Next we can show that there exists a $T > 0$ such that each solution $x(n, n_0, x_0)$ of (22) with $x_0 \in \Omega$ satisfies that for some ϕ_j and the constant δ_0 of (ii) in Definition 3,

$$\left| x\left(n, n_0, x_0\right) - \phi_j\left(n\right) \right| < \delta_0 \quad \text{at some } n, n_0 \le n \le n_0 + T,$$

which implies

$$\left| x\left(n, n_0, x_0\right) - \phi_j\left(n\right) \right| \to 0 \quad \text{as } n \to \infty, \tag{33}$$

because ϕ_j is U.A.S. in Ω. Suppose that this is not true. Then there exists a small constant $\alpha > 0$ less than β and sequences $\{n_k\}$ in $\mathbf{Z}$ and $\{x_k\}$ in Ω such that

$$\left| x\left(n, n_k, x_k\right) - \phi_j\left(n\right) \right| \ge \alpha \quad \text{for all } n \in \left[n_k, n_k + k\right] \text{ and all } j, 1 \le j \le m.$$

Since $f\left(n, x\right)$ is almost periodic in n uniformly for $x \in \Omega$, we can choose a sequence $\{\tau_k\}, n_k + \left[k/2\right] < \tau_k < n_k + k$, such that

$$f\left(n + \tau_k, x\right) \to f\left(n, x\right) \quad \text{in } \mathbf{Z} \times \Omega \text{ as } k \to \infty.$$

If we set $\psi_j^{(k)}\left(n\right) = \phi_j\left(n + \tau_k\right)$ for $1 \le j \le m$ and $\psi_{m+1}^{(k)}\left(n\right) = x\left(n + \tau_k, n_k, x_k\right)$, these $m+1$ functions $\psi_j^{(k)}\left(n\right)$ satisfy

$$x\left(n+1\right) = f\left(n + \tau_k, x\left(n\right)\right)$$

and

$$\psi_j^{(k)}\left(n\right) \in \Omega \quad \text{on } \mathbf{Z} \text{ for } 1 \le j \le m,$$

$$\psi_{m+1}^{(k)}\left(n\right) \in \Omega \quad \text{for } n \ge -\left[k/2\right] \ge n_k - \tau_k,$$

because $\psi_{m+1}^{(k)}\left(n_k - \tau_k\right) = x_k \in \Omega$. Moreover,

$$\left| \psi_j^{(k)}\left(0\right) - \psi_i^{(k)}\left(0\right) \right| \ge \alpha \quad \text{for } i \ne j, 1 \le i, j \le m+1.$$

We can assume that $\psi_j^{(k)}\left(n\right) \to \psi_j\left(n\right)$ in $\mathbf{Z}$ for some function $\psi_j, 1 \le j \le m+1$, as $k \to \infty$. Therefore $\psi_j\left(j = 1, 2, \cdots, m\right)$ are solutions of system (22), because $f\left(n + \tau_k, x\right) \to f\left(n, x\right)$ in $\mathbf{Z} \times \Omega$ as $k \to \infty$, and

$$\psi_j\left(n\right) \in \Omega \quad \text{on } \mathbf{Z} \text{ for } 1 \le i \le m+1$$

and

$$\left| \psi_j\left(0\right) - \psi_i\left(0\right) \right| > \alpha \quad \text{for } i \ne j, 1 \le i, j \le m+1,$$

which shows that system (22) has $m+1$ distinct solutions in Ω on $\mathbf{Z}$. This is contradiction. Therefore, $\psi_j\left(n\right)$ is U.A.S. in the whole Ω on $\mathbf{Z}$, if the uniqueness of ϕ_j is shown.

Now we will prove the uniqueness of ϕ_j. Suppose $\phi_1 \ne \phi_j$ for $j \ge 2$ and set

$$S_1 = \left\{ x_0 \in \Omega \middle| \left| x\left(n, 0, x_0\right) - \phi_1\left(n\right) \right| \to 0 \text{ as } n \to \infty \right\}$$

and

$$S_2 = \left\{ x_0 \in \Omega \middle| \left| x\left(n, 0, x_0\right) - \phi_j\left(n\right) \right| \to 0 \text{ for some } \phi_j, j \ge 2, \text{ as } n \to \infty \right\}.$$

Then S_1 and S_2 are open sets in Ω, and moreover these sets are nonempty and disjoint, because $\left| \phi_1\left(n\right) - \phi_j\left(n\right) \right| \ge \alpha$ on $\mathbf{Z}$ for $j \ge 2$. On the other hand, (33) shows that $\Omega = S_1 \bigcup S_2$, which contradicts the connectedness of Ω. Thus

the uniqueness of an almost periodic solution is proved, and moreover, as is seen from [14], this uniqueness guarantees the module containment of the almost periodic solution.

Now consider the case where $A(n)$ is reducible. We can assume $A(n)$ takes the form of

$$A(n) = \left(\begin{array}{c|c} * & 0 \\ \hline * & B(n) \end{array} \right),$$

where $B(n)$ is zero or a square irreducible matrix of order $l, 2 \leq l \leq m-1$. If $B(n)$ is zero, system (22) obviously has the constant solution $p(n)$ in $\partial\Omega$ such that $p_i(n) = 0$ for $1 \leq i \leq m-1$ and $p_m(n) = 1$. In the latter case, if we set in system (22)

$$x_k = 0 \text{ for } 1 \leq k \leq l - m \text{ and } y_i = x_{m-l+i} \text{ for } 1 \leq i \leq l,$$

then system (22) is reduced to the lower dimensional system

$$y_i(n+1) = \sum_{j=1}^{l} b_{ij}(n) g_j(y_j(n)) \text{ for } 1 \leq i \leq l, \tag{34}$$

where $B(n) = (b_{ij}(n))$. Since $B(n)$ is irreducible, the above system (34) has an almost periodic solution $y(n)$ such that

$$y_i(n) > 0 \text{ for } 1 \leq i \leq l \text{ and } \sum_{i=1}^{l} y_i(n) = 1 \tag{35}$$

and furthermore the module of $y(n)$ is contained in the module of $B(n)$, *i.e.*, of the module of $A(n)$. Thus, system (22) has an almost periodic solution $p(n)$ in $\partial\Omega$ on **Z** such that $p_i(n) = 0$ for $1 \leq i \leq m - l$ and $p_i(n) = y_{i-m+l}(n)$ for $m - l + 1 \leq i \leq m$. The proof is completed.

Remark 1. As will be seen from the module containment, the above almost periodic solution is a critical point in the case where $A(n)$ is a constant. Hence Theorem 2 is a discretization of Nakajimas' result (Theorem 2 in [10]).

5. A Stability Criteria of Linear Systems

We consider a stability criterion for solutions of a linear system with coefficient matrix of diagonal dominance type.

We again consider a linear system (5).
Let $A(n) = (a_{ij}(n)), i, j = 1, 2, \cdots, m$ be an $m \times m$ matrix of functions for $n \in \mathbf{Z}$. We assume the following conditions;

$$a_{jj}(n) \leq 0 \text{ for } 1 \leq j \leq m \text{ and } n \in \mathbf{Z}, \tag{36}$$

$$\det A(n) \neq 0, \tag{37}$$

where, $\det A(n)$ denotes the determinant of matrix $A(n)$ and

$$\sum_{i=1, i \neq j}^{m} |a_{ij}(n)| \leq |a_{jj}(n)| \text{ for } 1 \leq j \leq m \text{ and } n \in \mathbf{Z}. \tag{38}$$

At first, we need the following lemmas for main theorem.

Lemma 9. If a square matrix A is irreducible and satisfies (38) and if for at least one j,

$$\sum_{i=1, i \neq j}^{m} |a_{ij}| < |a_{jj}|,$$

then A is nonsingular.

For the proof, see [15].

Lemma 10. If a nonsingular $m \times m$ matrix $A = (a_{ij})$ satisfies (38), then all principal minors of A are nonsingular, namely,

$$\det \begin{pmatrix} a_{j_1 j_1} & a_{j_1 j_2} & \cdots & a_{j_1 j_l} \\ a_{j_2 j_1} & a_{j_2 j_2} & \cdots & a_{j_2 j_l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j_l j_1} & a_{j_l j_2} & \cdots & a_{j_l j_l} \end{pmatrix} \neq 0 \quad \text{for } 1 \leq j_1 < j_2 < \cdots < j_l \leq m.$$

Proof. Let A_1 be an $l \times l (l < m)$ principal minor of A. Then, for a permutation matrix Q,

$$QAQ^{\mathrm{T}} = \begin{pmatrix} A_1 & * \\ A_2 & * \end{pmatrix},$$

where A_2 has $(m-l)$ rows and l columns and Q^{T} denotes the transposed matrix of Q. Moreover, from the definition of irreducibility, we can choose an $l \times l$ permutation matrix Q_l such that

$$Q_l A_1 Q_l^{\mathrm{T}} = \begin{pmatrix} B_1 & C_2 & & \\ & B_2 & \ddots & \\ & & \ddots & C_p \\ O & & & B_p \end{pmatrix} \tag{38_1}$$

where B_i is an $r_i \times r_i$ irreducible matrix, $\sum_{i=1}^{p} r_i = l$, and C_i has $(r_1 + r_2 + \cdots + r_{i-1})$ row and r_i columns for $2 \leq i \leq p$. In particular, in the case where A_1 is irreducible, B_1 must be A_1 itself, and the matrices $B_2, B_3, \cdots, B_p, C_2, C_3, \cdots, C_p$ are not present. Setting $P = (Q_l \oplus I)Q$ for $(m-l) \times (m-l)$ unit matrix I, where $Q_l \oplus I$ is the direct sum of Q_l and I, we have

$$B = PAP^{\mathrm{T}} = \begin{pmatrix} Q_l A_1 Q_l^{\mathrm{T}} & * \\ A_2 Q_l^{\mathrm{T}} & * \end{pmatrix} = \left(\begin{array}{ccccc|c} B_1 & C_2 & & & C_p & \\ & B_2 & C_3 & & C_p & * \\ & & \ddots & \ddots & & \\ & & & B_{p-1} & C_p & \\ & O & & & B_p & \\ \hline D_1 & D_2 & & & D_p & * \end{array} \right)$$

where $A_2 Q_l^{\mathrm{T}} = (D_1, D_2, \cdots, D_p)$ and D_i has $m-l$ rows and r_i columns. Since the diagonal dominance condition (38) is invariant under the permutation of indexes, B also satisfies (38). Hence, letting

$$B_i = (b_{jk}), C_i = (c_{jk}) \text{ and } D_i = (d_{jk})$$

for a fixed $i, 1 \le i \le p$, we have

$$\sum_{j=1, j \ne k}^{r_i} |b_{jk}| + \sum_{j=1, j \ne k}^{r_i} |c_{jk}| + \sum_{j=1, j \ne k}^{r_i} |d_{jk}| \le |b_{kk}| \quad (1 \le k \le r_i), \tag{39}$$

where the summations on j are taken along columns and $C_1 = O$ for convenience. If $C_i \ne O$ or $D_i \ne O$, then

$$\sum_{j=1, j \ne k}^{r_i} |c_{jk}| + \sum_{j=1, j \ne k}^{r_i} |d_{jk}| > 0 \quad \text{for some } k, 1 \le k \le r_i,$$

and hence for this k,

$$\sum_{j=1, j \ne k}^{r_i} |b_{jk}| < |b_{kk}|$$

by (39). Therefore it follows from Lemma 9 that

$$\det B_i \ne 0, \tag{40}$$

since B_i is irreducible. If $C_i = O$ and $D_i = O$, then we have the form of

$$\det B = \det B_i \times (\cdots)$$

which also implies (40), because $\det B \ne 0$. In any case, we have $\det B_i \ne 0$. Since these are true for all $i, 1 \le i \le p$, it follows from (38$_1$) that

$$\det A_1 \ne 0.$$

this proves Lemma 10.

Lemma 11. If system (5) satisfies conditions (36) and (38), then the norm of solution $x(n)$ such that $|x(n)| = \sum_{i=1}^{m} |x_i(n)|$, is non-increasing, and consequently the zero solution is U.S..

For the proof, we can see (cf. [5]).

In the following theorem, we can prove that the zero solution is U.A.S., if $A(n)$ is bounded on Z and if condition (36), (37) and (38) are satisfied.

Theorem 3. In system (5), let $A(n)$ be bounded on **Z**. Assume that conditions (36) and (38) are satisfied for all $n \in \mathbf{Z}$ and that there is a constant $\alpha > 0$ such that

$$|\det A(n)| \ge \alpha, \text{ on } \mathbf{Z}.$$

Then the zero solution is U.A.S..

Proof. As is stated in Lemma 11, the zero solution is U.S., and hence it is sufficient to show that for any $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that

$$|x(n, n_0, x_0)| < \epsilon \text{ at some } n, n_0 \le n \le n_0 + T(\epsilon),$$

whenever $|x_0| \le 1$. Suppose that this is not true. Then there exists a constant $\epsilon > 0$, a sequence of solution $\{x^{(k)}(n)\}$ of (5) and a sequence n_k such that

$$|x^{(k)}(n)| \ge \epsilon \text{ on } n_k \le n \le n_k + k^2$$

and

$$|x^{(k)}(n_k)| \le 1.$$

Since $\left| x^{(k)}(n) \right|$ is non-increasing, we have

$$\epsilon \le \left| x^{(k)}(n) \right| \le 1, \text{ on } n_k \le n \le n_k + k^2$$

and there exists a subinterval $\left[s_k, s_k + k \right]$ of $\left[n_k, n_k + k^2 \right]$ such that

$$\left\| \left| x^{(k)}(n) \right| - \left| x^{(k)}(n') \right| \right\| < \frac{1}{k}, \text{ for } s_k \le n, n' \le s_k + k.$$

Set $\varphi^{(k)}(n) = x^{(k)}(n + s_k)$. Then, we obtain

$$\varphi^{(k)}(n+1) = A(n + s_k)\varphi^{(k)}(n), \tag{41}$$

$$\epsilon \le \left| \varphi^{(k)}(n) \right| \le 1, \text{ on } 0 \le n \le k \tag{42}$$

and

$$\left\| \left| \varphi^{(k)}(n) \right| - \left| \varphi^{(k)}(n') \right| \right\| < \frac{1}{k}, \text{ for } 0 \le n, n' \le k. \tag{43}$$

Since $A(n)$ is bounded, it follows from (41) and (42) that $\left\{ \varphi^{(k)}(n) \right\}$ is uniformly bounded on any finite interval of **Z**, and thus, taking a subsequence, $\left\{ \varphi^{(k)}(n) \right\}$ can be assumed to converge uniformly on any finite interval of **Z**. Defining $y(n)$ by

$$y(n) = \lim_{k \to \infty} \varphi^{(k)}(n),$$

it follows from (42) and (43) that there is a constant $\beta > 0, \epsilon \le \beta \le 1$, such that

$$\left| y(n) \right| = \beta, \text{ for } n \ge 0.$$

Since $y(n) = \left(y_1(n), y_2(n), \cdots, y_m(n) \right)$ is defined on **Z**, we can choose an interval $I = \left[\sigma_1, \sigma_2 \right]$ (for some $\sigma_i \in \mathbf{Z}(i = 1, 2), 0 < \sigma_1 < \sigma_2$) such that

$$y_{j_1}(n), y_{j_2}(n), \cdots, y_{j_h}(n) > 0 \text{ on } I,$$

$$y_{j_{h+1}}(n), y_{j_{h+2}}(n), \cdots, y_{j_l}(n) < 0 \text{ on } I$$

and

$$y_{j_{l+1}}(n) \equiv y_{j_{l+2}}(n) \equiv \cdots \equiv y_{j_m}(n) \equiv 0 \text{ on } I.$$

Here we note that $\left\{ j_1, j_2, \cdots, j_l \right\} \ne \phi$ because $y(n) \not\equiv 0$. Then

$$\left| y(n) \right| = \sum_{p=1}^{h} y_{j_p}(n) - \sum_{p=h+1}^{l} y_{j_p}(n) = \beta \text{ on } I.$$

Let

$$f^{(k)}(n) = \sum_{p=1}^{h} \varphi^{(k)}_{j_p}(n) - \sum_{p=h+1}^{l} \varphi^{(k)}_{j_p}(n). \tag{44}$$

Then, we have

$$\lim_{k \to \infty} f^{(k)}(n) = \left| y(n) \right| = \beta \text{ on } I,$$

and there is a sequence $\left\{ \theta_k \right\} \subset I$ such that

$$\lim_{k \to \infty} \Delta f^{(k)}(\theta_k) = 0. \tag{45}$$

Moreover, since I is compact and $A(n)$ is bounded on **Z**, we can assume

that

$$\lim_{k \to \infty} \theta_k = \theta, \quad \text{for some } \theta \in I$$

and

$$\lim_{k \to \infty} A(s_k + \theta_k) = B, \quad \text{for some } m \times m \text{ matrix } B.$$

Clearly B satisfies (36), (38) and $|\det B| \geq \alpha$. Taking a difference of both sides of (44) at $n = \theta_k$ and using relation (41), we find

$$\Delta f^{(k)}(\theta_k) = \sum_{p=1}^{h} \left(\sum_{q=1}^{m} a_{j_p j_q}(s_k + \theta_k) \varphi_{j_q}^{(k)}(\theta_k, k) \right) - \sum_{p=h+1}^{l} \left(\sum_{q=1}^{m} a_{j_p j_q}(s_k + \theta_k) \varphi_{j_q}^{(k)}(\theta_k) \right),$$

and

$$\lim_{k \to \infty} \Delta f^{(k)}(\theta_k) = \sum_{p=1}^{h} \left(\sum_{q=1}^{l} b_{j_p j_q} y_{j_q}(\theta) \right) - \sum_{p=h+1}^{l} \left(\sum_{q=1}^{l} b_{j_p j_q} y_{j_q}(\theta) \right),$$

where $B = (b_{ij})$, since we have

$$\lim_{k \to \infty} \varphi_j^{(k)}(\theta_k) = y_j(\theta), \quad \text{for } 1 \leq j \leq m$$

and

$$y_{j_q}(\theta) = 0 \quad \text{for } l+1 \leq q \leq m.$$

By (45), we have

$$0 = \sum_{q=1}^{l} \left(\sum_{p=1}^{h} b_{j_p j_q} - \sum_{p=h+1}^{l} b_{j_p j_q} \right) y_{j_q}(\theta). \tag{46}$$

Since B satisfies (36) and (38),

$$\sum_{p=1}^{h} b_{j_p j_q} - \sum_{p=h+1}^{l} b_{j_p j_q} \begin{cases} \leq 0, & \text{for } 1 \leq q \leq h, \\ \geq 0, & \text{for } h+1 \leq q \leq l. \end{cases}$$

Therefore each term of right hand side of (46) is non-positive, because we have

$$y_{j_q} \begin{cases} > 0, & \text{for } 1 \leq q \leq h, \\ < 0, & \text{for } h+1 \leq q \leq l. \end{cases}$$

Then it follows from (46) that

$$\left(\sum_{p=1}^{h} b_{j_p j_q} - \sum_{p=h+1}^{l} b_{j_p j_q} \right) y_{j_q}(\theta) = 0, \quad \text{for } 1 \leq q \leq l,$$

which implies

$$\sum_{p=1}^{h} b_{j_p j_q} - \sum_{p=h+1}^{l} b_{j_p j_q} = 0, \quad \text{for } 1 \leq q \leq l,$$

since $\left| y_{j_q}(\theta) \right| \neq 0$ for $1 \leq q \leq l$. Thus we have

$$\det \begin{pmatrix} b_{j_1 j_1} & b_{j_1 j_2} & \cdots & b_{j_1 j_l} \\ b_{j_2 j_1} & b_{j_2 j_2} & \cdots & b_{j_2 j_l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j_l j_1} & b_{j_l j_2} & \cdots & b_{j_l j_l} \end{pmatrix} = 0. \tag{46_1}$$

On the other hand, B satisfies (38) and $|\det B| \ge \alpha > 0$, and thus it follows from Lemma 10 that all principal minors of B are nonsingular, which contradicts (46_1). This proves that the zero solution of system (5) is U.A.S..

Corollary 1. If system (5) is defined only for $n \ge 0$ and all assumptions of Theorem 3 are satisfied for $n \ge 0$, then the zero solution is U.A.S. for $n \ge 0$.

Proof. We construct the system defined on **Z** by

$$x(n+1) = A_0(n)x(n), \quad x \in R^m, \tag{47}$$

where

$$A_0(n) = \begin{cases} A(n), & \text{for } n \ge 0, \\ A(0), & \text{for } n < 0. \end{cases}$$

Since system (47) satisfies all assumptions of Theorem 3 on **Z**, the zero solution is U.A.S. on **Z**, and furthermore, since system (5) coincides with system (47) for $n \ge 0$, this prove our conclusion.

6. Application

Before Example 1, we state the following lemma is a special case of Theorem 3 in [13].

In the nonlinear system

$$x(n+1) = F(n, x(n)), \tag{48}$$

let $F(n,x)$ be almost periodic in n uniformly for $x \in R^m$ and for any $r > 0$, let there exists a constant $L_* = L_*(r) > 0$ such that

$$|F(n,x) - F(n,y)| \le L_*|x-y| \text{ for } |x|, |y| \le r \text{ and } n \in \mathbf{Z}.$$

Lemma 12. If $x(n)$ is a bounded solution of (48) on **Z** and if for any solution $y(n)$ of (48), $|x(n) - y(n)|$ is monotone decreasing to zero as $n \to \infty$, then $x(n)$ is a unique almost periodic solution and its module is contained in the module of $F(n,x)$.

Example 1. Consider the variational linear difference equation

$$w(n+1) = A(n)w(n), \tag{49}$$

where

$$A(n) = \begin{pmatrix} -p(n) & \dfrac{\alpha}{2} \\ p(n) - \dfrac{2}{\alpha}q(n) & -\dfrac{\alpha}{2} \end{pmatrix}.$$

We now assume that $(u_0(n), v_0(n))$ is at least one bounded solution of (49) on **Z** and $(u(n), v(n))$ is any solution of (49) on **Z**, and $p(n)$, $q(n)$ are some bounded functions on **Z** such that

$$p(n) \ge \alpha/2 \text{ and } \beta_1 \le q(n) \le \beta_2,$$

for some positive constants α, β_1 and β_2 such that $\alpha^2 \ge 2\beta_2$. We can verify that $A(n)$ satisfies all assumptions in Corollary 1. First of all, $A(n)$ is

bounded in the future for $n \in \mathbf{Z}$, because $p(n)$ and $q(n)$ are bounded function on $\mathbf{Z}$. It is clear that the diagonal elements of $A(n)$ are negative and

$$\det A(n) = q(n) \geq \beta_1 \quad \text{on } \mathbf{Z}.$$

The diagonal dominance condition (38) for $A(n)$ requires that

$$-p(n) + \left| p(n) - \frac{2}{\alpha} q(n) \right| \leq 0, \quad \text{and} \quad -\frac{\alpha}{2} + \left| -\frac{\alpha}{2} \right| \leq 0,$$

which is equivalent to

$$q(n) \leq \alpha p(n)$$

and this is satisfied by

$$q(n) \leq \beta_2 \leq \alpha^2 / 2 \leq \alpha p(n).$$

Therefore, by Theorem 3, the zero solution of (49) is U.A.S. and

$$\left| u_0(n) - u(n) \right| + \left| v_0(n) - v(n) \right| \to 0 \quad \text{as } n \to \infty,$$

where the convergence is monotone decreasing by Lemma 11. Thus, applying Lemma 12 to system (49), we find that there exists a unique almost periodic solution with the module contained in the module of $A(n)$.

7. Conclusion

In this paper, we obtain the existence and stability property of almost periodic solutions in discrete almost periodic systems. First, in Section 1, the research background is introduced. In Section 2, the fundamental concepts of the almost periodic solutions in discrete almost periodic systems is given. In Section 3, we are introduced to the several lemmas and have uniformly asymptotically stability theory of the linear system, and moreover, in Section 4, we consider the generalized gas almost periodic system, and if linear part is irreducible matrix, then we obtain the existence of almost periodic solutions of this system. Finally, in Sections 5 and 6, we consider and obtain an uniformly asymptotically stability criterion for solutions of a linear system with coefficient matrix of diagonal dominance conditions, and this result applies to meaningful example of a linear discrete system.

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