Advances in Pure Mathematics, 2018, 8, 1-10
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

# The Estimates of the Upper Bounds of Hausdorff Dimensions for the Global Attractor for a Class of Nonlinear Coupled Kirchhoff-Type Equations

**Guoguang Lin, Ming Zhang**

Department of Mathematics, Yunnan University, Kunming, China
Email: gglin@ynu.edu.cn, 332026295@qq.com

**How to cite this paper:** Lin, G.G. and Zhang, M. (2018) The Estimates of the Upper Bounds of Hausdorff Dimensions for the Global Attractor for a Class of Nonlinear Coupled Kirchhoff-Type Equations. *Advances in Pure Mathematics*, **8**, 1-10.
https://doi.org/10.4236/apm.2018.81001

**Received:** December 7, 2017
**Accepted:** January 6, 2018
**Published:** January 9, 2018



## Abstract

This paper deals with the Hausdorff dimensions of the global attractor for a class of Kirchhoff-type coupled equations with strong damping and source terms. We obtain a precise estimate of upper bound of Hausdorff dimension of the global attractor.

## Keywords

Kirchhoff-Type Equations, The Global Attractor, Hausdorff Dimension

## 1. Introduction

Guohuang Lin, Ming Zhang [1] studied the initial boundary value problem for a class of Kirchhoff-type coupled equations and obtained the existence of the global attractor. Next, in this paper, we consider the Hausdorff dimensions for the global attractor for the following Kirchhoff-type equations:

$$u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u - \beta\Delta u_t + g_1(u,v) = f_1(x) \tag{1.1}$$

$$v_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v - \beta\Delta v_t + g_2(u,v) = f_2(x) \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega \tag{1.3}$$

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in \Omega \tag{1.4}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \tag{1.5}$$

where $\Omega$ is a bounded domain in $R^2$ with the smooth boundary $\partial\Omega$, $\beta > 0$

is a constant. $M(s)$ is a nonnegative $C^1$ function, $-\Delta u_t$ and $-\Delta v_t$ are strongly damping terms, $g_1(u,v)$ and $g_2(u,v)$ are nonlinear source terms, $f_1(x)$ and $f_2(x)$ are given forcing function.

Jingzhu Wu, Guoguang Lin [2] consider a class of damped Bossinesq equation:

$$u_{tt}+\alpha u_t-u_{xx}+u^{2k+1}=f(x),\ x\in\Omega,\ t>0 \tag{1.6}$$

$$u(x,0)=u_0(x) \tag{1.7}$$

$$u(x,t)=u(x+1,t) \tag{1.8}$$

where $\Omega\subset R$, $\alpha>0$, then they obtain the existence of the global attractor and the limited of Hausdorff dimension and the limited of Fractal dimension.

Xiaoming Fan, Shengfan Zhou [3] consider the following non-autonomous strongly damped wave equation of non-degenerate Kirchhoff-type:

$$u_{tt}-\alpha\Delta u_t-\left(\beta+\gamma\left(\int_\Omega|\nabla u|^2\,\mathrm{d}x\right)^p\right)\Delta u+h(u_t)+f(u,t)=g(x,t),\ x\in\Omega,\ t>\tau \tag{1.9}$$

$$u(x,t)\big|_{x\in\partial\Omega}=0,\ t\geq\tau \tag{1.10}$$

$$u(x,\tau)=u_{0\tau}(x),\ u_t(x,\tau)=u_{1\tau}(x),\ x\in\Omega \tag{1.11}$$

where $u=u(x,t)$ is a real-valued function on $\Omega\times[\tau,+\infty)$, $\tau\in R$, $\Omega$ is an open bounded set of $R^n$, $n=1,2,3$ with a smooth boundary $\partial\Omega$, $\alpha>0$ is called the strong damping, $\beta>0$, $\rho>-1$, $\gamma\geq 0$. $h\in C^1(R;R)$, $f\in C^1(R\times R;R)$, $g(\cdot,t)\in C_b(R,L^2(\Omega))$, $C_b(R,L^2(\Omega))$ is the set of continuous bounded functions from $R$ into $L^2(\Omega)$. And then, they obtained a precise estimate of upper bound of Hausdorff dimension of kernel sections, which decreases as the strong damping grows for large strong damping under some conditions, particularly in the autonomous case.

Guoguang Lin, Yunlong Gao [4] concerned the following nonlinear Higher-order Kirchhoff-type equations:

$$u_{tt}+(-\Delta)^m u_t+\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u+g(u)=f(x),\ (x,t)\in\Omega\times[0,+\infty) \tag{1.12}$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega \tag{1.13}$$

$$u(x,t)=0,\ \frac{\partial^i u}{\partial v^i}=0,\ i=1,\cdots,m-1,\ x\in\partial\Omega,\ t\in[0,+\infty) \tag{1.14}$$

where $m>1$ is an integer constant, $\alpha>0,\beta>0$ are constants and $q$ is a real number. $\Omega$ is a bounded domain of $R^n$ with a smooth boundary $\partial\Omega$ and $v$ is the unit outward normal on $\partial\Omega$. $g(u)$ is a nonlinear function specified later. And they obtained the existence of the global attractor. In this case, they considered that the estimation of the upper bounds of Hausdorff for the global attractors is obtained.

## 2. Hausdorff Dimensions of the Global Attractor

In this paper, some inner product, norms, abbreviations and some assumptions

($H_1$) - ($H_4$) and notations needs in the proof of our results in refer to [1].

## 2.1. Differentiability of the Semigroup

In order to estimate dimensions, we suppose:

($H_5$) For every $L>0$, there exist $k=k(L)$, such that:

$$\left\|g_{iu}(u,v)-g_{iu}(\gamma u+(1-\gamma)\bar{u},v)\right\|_{L^{\infty}(\Omega)}\le k\left\|\nabla\bar{u}-\nabla u\right\|^{\delta_1}+k\left\|\nabla\bar{v}-\nabla v\right\|^{\delta_1} \tag{2.1}$$

$$\left\|g_{iv}(u,v)-g_{iv}(u,\gamma v+(1-\gamma)\bar{v})\right\|_{L^{\infty}(\Omega)}\le k\left\|\nabla\bar{u}-\nabla u\right\|^{\delta_1}+k\left\|\nabla\bar{v}-\nabla v\right\|^{\delta_1} \tag{2.2}$$

where $u,\bar{u},v,\bar{v}\in H_0^1(\Omega)$ ; $\|\nabla u\|,\|\nabla\bar{u}\|,\|\nabla v\|,\|\nabla\bar{v}\|\le L$ ; $\lambda\in(0,1)$ ; $\delta_1>0$ ; $i=1,2$.

($H_6$) There exists constant $\mu_0,\mu_1,\mu_2$, such that

$$1<\mu_0\le M(s)\le\mu_1,\ \mu_2=\begin{cases}\mu_0, \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\theta\|^2+\|\nabla\omega\|^2\right)>0,\\ \mu_1, \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\theta\|^2+\|\nabla\omega\|^2\right)<0.\end{cases} \tag{2.3}$$

We define $A=-\Delta$, $E_0=H_0^1(\Omega)\times H_0^1(\Omega)\times L_2(\Omega)\times L_2(\Omega)$. The inner product and norm in $E_0$ space are defined as follows:

$$\forall\varphi_i=(u_i,v_i,p_i,q_i)\in E_0,(i=1,2)$$

we have

$$(\varphi_1,\varphi_2)_{E_0}=\left(A^{\frac{1}{2}}u_1,A^{\frac{1}{2}}u_2\right)+\left(A^{\frac{1}{2}}v_1,A^{\frac{1}{2}}v_2\right)+(p_1,p_2)+(q_1,q_2) \tag{2.4}$$

$$\|\varphi_1\|_{E_0}^2=(\varphi_1,\varphi_1)_{E_0}=\left\|A^{\frac{1}{2}}u_1\right\|^2+\left\|A^{\frac{1}{2}}v_1\right\|^2+\|p_1\|^2+\|q_1\|^2 \tag{2.5}$$

Setting $\forall\varphi=(u,v,p,q)^{\mathrm{T}}\in E_0$, $p=u_t+\varepsilon u$, $q=v_t+\varepsilon v$, the Equations (1.1)-(1.5) is equivalent to

$$\varphi_t+H(\varphi)=F(\varphi) \tag{2.6}$$

where

$$H(\varphi)=\begin{pmatrix}\varepsilon u-p\\ \varepsilon v-q\\ -\varepsilon p+\beta Ap+\varepsilon^2u+(1-\varepsilon\beta)Au\\ -\varepsilon q+\beta Aq+\varepsilon^2v+(1-\varepsilon\beta)Av\end{pmatrix} \tag{2.7}$$

$$F(\varphi)=\begin{pmatrix}0\\ 0\\ \left(1-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\right)Au-g_1(u,v)+f_1(x)\\ \left(1-M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\right)Av-g_2(u,v)+f_2(x)\end{pmatrix} \tag{2.8}$$

Lemma 2.1 For any $\varphi=(u,v,p,q)^{\mathrm{T}}\in E_0$, we have

$$\left(H(\varphi),\varphi\right)_{E_0} \geq \frac{\varepsilon}{4}\|\varphi\|_{E_0}^2 + \frac{\beta}{4}\left\|A^{\frac{1}{2}}p\right\|^2 + \frac{\beta}{4}\left\|A^{\frac{1}{2}}q\right\|^2 \tag{2.9}$$

Proof.

By (2.3), we get

$$\begin{aligned}
\left(H(\varphi),\varphi\right)_{E_0} &= \left(A^{\frac{1}{2}}(\varepsilon u-p),A^{\frac{1}{2}}u\right)+\left(A^{\frac{1}{2}}(\varepsilon v-q),A^{\frac{1}{2}}v\right)\\
&\quad+\left(-\varepsilon p+\beta Ap+\varepsilon^2 u+(1-\varepsilon\beta)Au,p\right)\\
&\quad+\left(-\varepsilon q+\beta Aq+\varepsilon^2 v+(1-\varepsilon\beta)Av,q\right)\\
&=\varepsilon\left\|A^{\frac{1}{2}}u\right\|^2-\varepsilon\|p\|^2+\beta\left\|A^{\frac{1}{2}}p\right\|^2+\varepsilon^2(u,p)-\varepsilon\beta\left(A^{\frac{1}{2}}u,A^{\frac{1}{2}}p\right)\\
&\quad+\varepsilon\left\|A^{\frac{1}{2}}v\right\|^2-\varepsilon\|q\|^2+\beta\left\|A^{\frac{1}{2}}q\right\|^2+\varepsilon^2(v,q)-\varepsilon\beta\left(A^{\frac{1}{2}}v,A^{\frac{1}{2}}q\right).
\end{aligned} \tag{2.10}$$

By using holder inequality and Young's inequality and Poincare inequality, we deal with the terms in (2.9) by as follows:

$$\varepsilon^2(u,p)\geq-\frac{\varepsilon^2}{2\lambda_1}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{\varepsilon^2}{2}\|p\|^2 \tag{2.11}$$

$$-\varepsilon\beta\left(A^{\frac{1}{2}}u,A^{\frac{1}{2}}p\right)\geq-\frac{\varepsilon^2\beta}{2}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{\beta}{2}\left\|A^{\frac{1}{2}}p\right\|^2 \tag{2.12}$$

$$\varepsilon^2(v,q)\geq-\frac{\varepsilon^2}{2\lambda_1}\left\|A^{\frac{1}{2}}v\right\|^2-\frac{\varepsilon^2}{2}\|q\|^2 \tag{2.13}$$

$$-\varepsilon\beta\left(A^{\frac{1}{2}}v,A^{\frac{1}{2}}q\right)\geq-\frac{\varepsilon^2\beta}{2}\left\|A^{\frac{1}{2}}v\right\|^2-\frac{\beta}{2}\left\|A^{\frac{1}{2}}q\right\|^2 \tag{2.14}$$

By $0<\varepsilon<\min\left\{\frac{3}{4}\cdot\frac{2\lambda_1}{1+\beta\lambda_1},\frac{-5+\sqrt{25+8\lambda_1\beta}}{4}\right\}$ and substituting (2.11)-(2.14) into (2.10), we obtain

$$\begin{aligned}
\left(H(\varphi),\varphi\right)_{E_0} &\geq\left(\varepsilon-\frac{\varepsilon^2}{2\lambda_1}-\frac{\varepsilon^2\beta}{2}\right)\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2\right)\\
&\quad+\left(-\varepsilon-\frac{\varepsilon^2}{2}\right)\left(\|p\|^2+\|q\|^2\right)+\frac{\beta}{2}\left(\left\|A^{\frac{1}{2}}p\right\|^2+\left\|A^{\frac{1}{2}}q\right\|^2\right)\\
&\geq\left(\varepsilon-\frac{\varepsilon^2}{2\lambda_1}-\frac{\varepsilon^2\beta}{2}\right)\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2\right)+\left(-\varepsilon-\frac{\varepsilon^2}{2}\right)\left(\|p\|^2+\|q\|^2\right)\\
&\quad+\frac{\beta}{4}\left(\left\|A^{\frac{1}{2}}p\right\|^2+\left\|A^{\frac{1}{2}}q\right\|^2\right)+\frac{\beta}{4}\left(\left\|A^{\frac{1}{2}}p\right\|^2+\left\|A^{\frac{1}{2}}q\right\|^2\right)
\end{aligned}$$

$$\geq\left(\varepsilon-\frac{\varepsilon^2}{2\lambda_1}-\frac{\varepsilon^2\beta}{2}\right)\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2\right)$$
$$+\left(\frac{\lambda_1\beta}{4}-\varepsilon-\frac{\varepsilon^2}{2}\right)\left(\|p\|^2+\|q\|^2\right)+\frac{\beta}{4}\left(\left\|A^{\frac{1}{2}}p\right\|^2+\left\|A^{\frac{1}{2}}q\right\|^2\right) \tag{2.15}$$
$$\geq\frac{\varepsilon}{4}\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2+\|p\|^2+\|q\|^2\right)+\frac{\beta}{4}\left(\left\|A^{\frac{1}{2}}p\right\|^2+\left\|A^{\frac{1}{2}}q\right\|^2\right).$$

Proof finished.

The linearized equations of (1.1)-(1.5), the above equations as follows:

$$U_{tt}+M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)2\left[(\nabla u,\nabla U)+(\nabla v,\nabla V)\right]Au$$
$$+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)AU+\beta AU_t+g_{1u}(u,v)U+g_{1v}(u,v)V=0, \tag{2.16}$$

$$V_{tt}+M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)2\left[(\nabla u,\nabla U)+(\nabla v,\nabla V)\right]Av$$
$$+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)AV+\beta AV_t+g_{2u}(u,v)U+g_{2v}(u,v)V=0, \tag{2.17}$$

$$U(x,0)\big|_{x\in\partial\Omega}=V(x,0)\big|_{x\in\partial\Omega}=0,\ t>0 \tag{2.18}$$

$$U(x,0)=\xi_1,\ U_t(x,0)=\xi_2 \tag{2.19}$$

$$V(x,0)=\zeta_1,\ V_t(x,0)=\zeta_2 \tag{2.20}$$

where $(\xi_1,\zeta_1,\xi_2,\zeta_2)\in E_0$, $(u,v,u_t,v_t)=S(t)(u_0,v_0,u_1,v_1)$ is the solution of with $(u_0,v_0,u_1,v_1)\in \mathrm{A}$

Given $(u_0,v_0,u_1,v_1)\in \mathrm{A}$ and $S(t):E_0\to E_0$, the solution $S(t)(u_0,v_0,u_1,v_1)\in E_0$, by stand methods we can show that for any $(\xi_1,\zeta_1,\xi_2,\zeta_2)\in E_0$, the linear initial boundary value problem (2.16)-(2.20) possess a unique solution $(U(t),V(t),U_t(t),V_t(t))\in L^\infty(0,+\infty;E_0)$.

Theorem 2.1 For any $t>0$, $R>0$, the mapping $S(t):E_0\to E_0$ is Frechet differentiable on. Its differential at $\varphi=(u_0,v_0,u_1,v_1)^{\mathrm{T}}$ is the linear operator on

$$E:(\xi_1,\zeta_1,\xi_2,\zeta_2)^{\mathrm{T}}\to(U(t),V(t),P(t),Q(t))^{\mathrm{T}}$$

where $U(t)$, $V(t)$ is the solution of (2.16)-(2.20).

Proof.

Let $\varphi_0=(u_0,v_0,u_1,v_1)^{\mathrm{T}}\in E_0$, $\overline{\varphi_0}=(u_0+\xi_1,v_0+\zeta_1,u_1+\xi_2,v_1+\zeta_2)^{\mathrm{T}}\in E_0$ with $\|\varphi_0\|_{E_0}\leq R$, $\left\|\overline{\varphi_0}\right\|_{E_0}\leq R$, we denote $(u,u_t)^{\mathrm{T}}=S(t)\varphi_0$, $(\overline{u},\overline{u_t})^{\mathrm{T}}=S(t)\overline{\varphi_0}$. We can get the Lipchitz property of $S(t)$ on the bounded sets of $E_0$, that is

$$\left\|S(t)\varphi_0-S(t)\overline{\varphi_0}\right\|_{E_0}^2\leq \mathrm{e}^{C_1t}\left\|(\xi_1,\zeta_1,\xi_2,\zeta_2)\right\|_{E_0}^2 \tag{2.21}$$

Let $\theta=\overline{u}-u-U$, $\omega=\overline{v}-v-V$ is the solution of problem

$$\theta_{tt}+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)A\theta+\beta A\theta_t=h_1 \tag{2.22}$$

$$\omega_{tt}+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)A\omega+\beta A\omega_t=h_2 \tag{2.23}$$

$$\theta(0)=\theta_t(0)=0 \tag{2.24}$$

$$\omega(0)=\omega_t(0)=0 \tag{2.25}$$

where

$$\begin{aligned} h_1 = & \left[M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-M\left(\|\nabla \bar{u}\|^2+\|\nabla \bar{v}\|^2\right)\right]A\bar{u} \\ & +2M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left[(\nabla u,\nabla U)+(\nabla v,\nabla V)\right]Au \\ & -g_1(\bar{u},\bar{v})+g_1(u,v)+g_{1u}(u,v)U+g_{1v}(u,v)V, \end{aligned} \tag{2.26}$$

$$\begin{aligned} h_2 = & \left[M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-M\left(\|\nabla \bar{u}\|^2+\|\nabla \bar{v}\|^2\right)\right]A\bar{v} \\ & +2M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left[(\nabla u,\nabla U)+(\nabla v,\nabla V)\right]Av \\ & -g_2(\bar{u},\bar{v})+g_2(u,v)+g_{2u}(u,v)U+g_{2v}(u,v)V. \end{aligned} \tag{2.27}$$

Let $s=\|\nabla u\|^2+\|\nabla v\|^2$, $s_1=\|\nabla \bar{u}\|^2+\|\nabla \bar{v}\|^2$, so we can get

$$\begin{aligned} & \left[M(s)-M(s_1)\right]A\bar{u}+2M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left[(\nabla u,\nabla U)+(\nabla v,\nabla V)\right]Au \\ & =M''(\xi)(-\gamma)\left[(\nabla\bar{u}+\nabla u,\nabla\bar{u}-\nabla u)+(\nabla\bar{v}+\nabla v,\nabla\bar{v}-\nabla v)\right]^2 A\bar{u} \\ & \quad -M'(s)\left[(\nabla\bar{u}+\nabla u,\nabla\bar{u}-\nabla u)+(\nabla\bar{v}+\nabla v,\nabla\bar{v}-\nabla v)\right]A(\bar{u}-u) \\ & \quad -M'(s)\left[(\nabla\bar{u}-\nabla u,\nabla\bar{u}-\nabla u)+(\nabla\bar{v}-\nabla v,\nabla\bar{v}-\nabla v)\right]Au \\ & \quad -2M'(s)\left[(\nabla u,\nabla\theta)+(\nabla v,\nabla\omega)\right]Au \end{aligned} \tag{2.28}$$

And

$$\begin{aligned} & g_1(u,v)-g_1(\bar{u},\bar{v})+g_{1u}(u,v)U+g_{1v}(u,v)V \\ & =g_1(u,v)-g_1(\bar{u},v)+g_{1u}(u,v)(\bar{u}-u)-g_{1u}(u,v)\theta \\ & \quad +g_1(\bar{u},v)-g_1(\bar{u},\bar{v})+g_{1v}(u,v)(\bar{v}-v)-g_{1v}(u,v)\omega. \end{aligned} \tag{2.29}$$

Then, we have

$$\begin{aligned} & \left(M''(s_2)(-\gamma)\left[(\nabla\bar{u}+\nabla u,\nabla\bar{u}-\nabla u)+(\nabla\bar{v}+\nabla v,\nabla\bar{v}-\nabla v)\right]^2 A\bar{u},\theta_t\right) \\ & \le C_1\left(\|\nabla\bar{u}-\nabla u\|^2+\|\nabla\bar{v}-\nabla v\|^2\right)\left\|A^{\frac{1}{2}}\theta_t\right\|, \end{aligned} \tag{2.30}$$

$$\begin{aligned} & \left(-M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left[(\nabla\bar{u}-\nabla u,\nabla\bar{u}+\nabla u)+(\nabla\bar{v}-\nabla v,\nabla\bar{v}+\nabla v)\right]A(\bar{u}-u),\theta_t\right) \\ & \le C_2\left(\|\nabla\bar{u}-\nabla u\|^2+\|\nabla\bar{v}-\nabla v\|^2\right)\cdot\left\|A^{\frac{1}{2}}\theta_t\right\|, \end{aligned} \tag{2.31}$$

$$\begin{aligned} & \left(-M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left[(\nabla\bar{u}-\nabla u,\nabla\bar{u}-\nabla u)+(\nabla\bar{v}-\nabla v,\nabla\bar{v}-\nabla v)\right]Au,\theta_t\right) \\ & \le C_3\left(\|\nabla\bar{u}-\nabla u\|^2+\|\nabla\bar{v}-\nabla v\|^2\right)\left\|A^{\frac{1}{2}}\theta_t\right\|, \end{aligned} \tag{2.32}$$

$$\begin{aligned} & \left(-M'\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\left[(\nabla u,\nabla\theta)+(\nabla v,\nabla\omega)\right]Au,\theta_t\right) \\ & \le C_4\left(\|\nabla\theta\|+\|\nabla\omega\|\right)\left\|A^{\frac{1}{2}}\theta_t\right\|. \end{aligned} \tag{2.33}$$

By using (2.30)-(2.33), we have

$$\begin{aligned}&\left(\left[M(s)-M(s_1)\right]A\bar{u}+2M'(s)\left[(\nabla u,\nabla U)+(\nabla v,\nabla V)\right]Au,\theta_t\right)\\&\le C_5\left(\|\nabla\bar{u}-\nabla u\|^2+\|\nabla\bar{v}-\nabla v\|^2\right)\left\|A^{\frac{1}{2}}\theta_t\right\|+C_4\left(\|\nabla\theta\|+\|\nabla\omega\|\right)\left\|A^{\frac{1}{2}}\theta_t\right\|.\end{aligned}\tag{2.34}$$

Similarly

$$\begin{aligned}&\left(\left[M(s)-M(s_1)\right]A\bar{v}+2M'(s)\left[(\nabla u,\nabla U)+(\nabla v,\nabla V)\right]Av,\omega_t\right)\\&\le C_6\left(\|\nabla\bar{u}-\nabla u\|^2+\|\nabla\bar{v}-\nabla v\|^2\right)\left\|A^{\frac{1}{2}}\omega_t\right\|+C_7\left(\|\nabla\theta\|+\|\nabla\omega\|\right)\left\|A^{\frac{1}{2}}\omega_t\right\|.\end{aligned}\tag{2.35}$$

And by using ($H_5$)

$$\begin{aligned}&\left(g_1(u,v)-g_1(\bar{u},v)+g_{1u}(u,v)(\bar{u}-u)-g_{1u}(u,v)\theta,\theta_t\right)\\&+\left(g_1(\bar{u},v)-g_1(\bar{u},\bar{v})+g_{1v}(u,v)(\bar{v}-v)-g_{1v}(u,v)\omega,\theta_t\right)\\&\le C_8\left(\|\nabla\bar{u}-\nabla u\|^{\delta_1+1}+\|\nabla\bar{v}-\nabla v\|^{\delta_1}\|\nabla\bar{u}-\nabla u\|\right)\cdot\|\theta_t\|+C_{10}\left(\|\theta\|\cdot\|\theta_t\|+\|\omega\|\cdot\|\theta_t\|\right)\\&\quad+C_9\left(\|\nabla\bar{u}-\nabla u\|^{\delta_1}\|\nabla\bar{v}-\nabla v\|+\|\nabla\bar{v}-\nabla v\|^{\delta_1+1}\right)\cdot\|\theta_t\|\\&\le C_{10}\left(\|\theta\|\cdot\|\theta_t\|+\|\omega\|\cdot\|\theta_t\|\right)+C_{11}\left(\|\nabla\bar{u}-\nabla u\|^{\delta_1+1}+\|\nabla\bar{v}-\nabla v\|^{\delta_1+1}\right)\cdot\|\theta_t\|.\end{aligned}\tag{2.36}$$

Similar

$$\begin{aligned}&\left(\left(g_{2u}(u,v)-g_{2u}(x,v)\right)(\bar{u}-u)-g_{2u}(u,v)\theta,\omega_t\right)\\&+\left(\left(g_{2v}(u,v)-g_{2v}(\bar{u},y)\right)(\bar{v}-v)-g_{2v}(u,v)\omega,\omega_t\right)\\&\le C_{12}\left(\|\nabla\bar{u}-\nabla u\|^{\delta_1+1}+\|\nabla\bar{v}-\nabla v\|^{\delta_1+1}\right)\cdot\|\omega_t\|+C_{13}\left(\|\theta\|\cdot\|\omega_t\|+\|\omega\|\cdot\|\omega_t\|\right).\end{aligned}\tag{2.37}$$

So, we can get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\theta_t\|^2+\|\omega_t\|^2+\mu_2\left(\|\nabla\theta\|^2+\|\nabla\omega\|^2\right)\right]\\&\le C_{15}\left(\|\nabla\bar{u}-\nabla u\|^{2\delta_1+2}+\|\nabla\bar{v}-\nabla v\|^{2\delta_1+2}+\|\nabla\bar{u}-\nabla u\|^4+\|\nabla\bar{v}-\nabla v\|^4\right)\\&\quad+C_{14}\left(\|\theta_t\|^2+\|\omega_t\|^2+\mu_2\left(\|\nabla\theta\|^2+\|\nabla\omega\|^2\right)\right).\end{aligned}\tag{2.38}$$

Then, by using Gronwall's inequality and (2.21), we obtain

$$\begin{aligned}&\|\theta_t\|^2+\|\omega_t\|^2+\beta_3\left(\|\nabla\theta\|^2+\|\nabla\omega\|^2\right)\\&\le C_{15}\mathrm{e}^{C_{14}t}\int_0^t\|\nabla\bar{u}-\nabla u\|^{2\delta_1+2}+\|\nabla\bar{v}-\nabla v\|^{2\delta_1+2}+\|\nabla\bar{u}-\nabla u\|^4+\|\nabla\bar{v}-\nabla v\|^4\,\mathrm{d}t\\&\le C_{17}\mathrm{e}^{C_{16}t}\left(\|(\xi_1,\zeta_1,\xi_2,\zeta_2)\|_{E_0}^4+\|(\xi_1,\zeta_1,\xi_2,\zeta_2)\|_{E_0}^{2\delta_1+2}\right).\end{aligned}\tag{2.39}$$

Then, we get

$$\frac{\|\bar{\varphi}(t)-\varphi(t)-U(t)\|_{E_0}^2}{\|(\xi_1,\zeta_1,\xi_2,\zeta_2)\|_{E_0}^2}\le C_{17}\mathrm{e}^{C_{16}t}\left(\|(\xi_1,\zeta_1,\xi_2,\zeta_2)\|_{E_0}^2+\|(\xi_1,\zeta_1,\xi_2,\zeta_2)\|_{E_0}^{2\delta_1}\right)\to 0\tag{2.40}$$

as $(\xi_1,\zeta_1,\xi_2,\zeta_2)^{\mathrm{T}}\to 0$ in $E_0$. The proof is competed.

### 2.2. The Upper Bounds of Hausdorff Dimensions for the Global Attractor

Consider the first variation of (2.6) with initial condition:

$$\psi'_t + P(\varphi)\psi = \Gamma_1(\varphi)\psi + \Gamma_2(\varphi)\psi, \psi(0) = (\xi_1, \zeta_1, \xi_2, \zeta_2)^{\mathrm{T}} \in E_0, t > 0 \tag{2.41}$$

where $\psi = (U, V, P, Q)^{\mathrm{T}} \in E_0$, $P = U_t + \varepsilon U$, $Q = V_t + \varepsilon V$ and $\varphi = (u, v, p, q)^{\mathrm{T}} \in E_0$ is a solution of (2.41),

$$P(\varphi) = \begin{pmatrix} \varepsilon I & 0 & -I & 0 \\ 0 & \varepsilon I & 0 & -I \\ \varepsilon^2 I - (1-\varepsilon\beta)\Delta & 0 & -\varepsilon I - \beta\Delta & 0 \\ 0 & \varepsilon^2 I - (1-\varepsilon\beta)\Delta & 0 & -\varepsilon I - \beta\Delta \end{pmatrix} \tag{2.42}$$

$$\Gamma_1(\varphi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -g_{1u}(u,v) & -g_{1v}(u,v) & 0 & 0 \\ -g_{2u}(u,v) & -g_{2v}(u,v) & 0 & 0 \end{pmatrix} \tag{2.43}$$

$$\Gamma_2(\varphi) = \begin{pmatrix} 0 \\ 0 \\ \left[M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - 1\right]\Delta u + M'\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)2\left[(\nabla u, \nabla U) + (\nabla v, \nabla V)\right]\Delta u \\ \left[M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - 1\right]\Delta v + M'\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)2\left[(\nabla u, \nabla U) + (\nabla v, \nabla V)\right]\Delta v \end{pmatrix} \tag{2.44}$$

It is easy to show form Theorem 2.1 that (2.41) is a well-posed problem in $E_0$, the mapping $S_\varepsilon(\tau)$: $\{u_0, v_0, p_1 = u_1 + \varepsilon u_0, q_1 = v_1 + \varepsilon v_0\} \to \{u(\tau), v(\tau), p(\tau) = u_t(\tau) + \varepsilon u(\tau), q(\tau) = v_t(\tau) + \varepsilon v(\tau)\}$ is frechet differentiable on $E_0$ for any $t \ge 0$, its differential at $\varphi = (u_0, v_0, p_1, q_1)^{\mathrm{T}}$ is the linear operator on

$$E_0 : (\xi_1, \zeta_1, \xi_2, \zeta_2) \to (U(t), V(t), P(t), Q(t))^{\mathrm{T}}$$

where $(U(t), V(t), P(t), Q(t))^{\mathrm{T}}$ is the solution of (2.40).

Lemma 2.2 [5] For any orthonormal family of elements of $\left(E_0, \|\cdot\|_{E_0}\right)$, $(\xi_{1j}, \zeta_{1j}, \xi_{2j}, \zeta_{2j})^{\mathrm{T}}$, $j = 1, 2, \cdots, n_1$, we have

$$\sum_{j=1}^{n_1}\left(\left\|\nabla^\nu \xi_{1j}\right\|^2 + \left\|\nabla^\nu \zeta_{1j}\right\|^2\right) \le 2\sum_{j=1}^{n_1}\left(\lambda_j^{\nu-1} + \lambda_j^{\nu-1}\right), \nu \in [0,1) \tag{2.45}$$

Proof. This is a direct consequence of Lemma VI. 6.3 of [5].

Theorem 2.2 If $(H_1)$-$(H_6)$ hold, satisfying, then there exists $\beta > 1$, such that the Hausdorff dimension of global attractor $\mathrm{A}$ in $E_0$ satisfies

$$d_H(\mathrm{A}) \le \min\left\{n_1 \middle| n_1 \in N, \frac{1}{n_1}\sum_{j=1}^{n_1}\left(\lambda_j^{-1} + \lambda_j^{-1}\right) < \frac{\varepsilon}{16C_{24}}\right\} \tag{2.46}$$

where $R_0$ is as in Lemma 2.6 in [1].

Proof. Let $n_1 \in N$ be fixed. Consider $m_1$ solutions $\psi_1, \psi_2, \cdots, \psi_{n_1}$ of (2.41).

At a given time $\tau$, let $B_{n_1}(\tau)$ denote the orthogonal projection in $E_0$ onto span $\{\psi_1(s),\psi_2(s),\cdots,\psi_{n_1}(s)\}$.

Let $y_j(s)=(\xi_j,\zeta_j,\xi_j,\zeta_j)^{\mathrm{T}}\in E_0$, $j=1,2,\cdots,n_1$, be an orthonormal basis of

$$B_{n_1}(s)E_0=span\{\psi_1(s),\psi_2(s),\cdots,\psi_{n_1}(s)\} \tag{2.47}$$

with respect to the inner product $(\cdot,\cdot)_{E_0}$ and norm $\|\cdot\|_{E_0}$.

Suppose

$$\varphi(\tau)=(u(\tau),v(\tau),p(\tau),q(\tau))^{\mathrm{T}}\in \mathrm{A} \tag{2.48}$$

then $\|\varphi(\tau)\|_{E_0}\le M_0$, $\forall s>\tau$. By $\|y_j\|_{E_0}=1$ and Lemma 2.1, we have

$$-(P(\varphi(s))y_j(s),y_j(s))_{E_0}\le -\frac{\varepsilon}{4}-\frac{\beta}{4}\|\nabla\xi_{2_j}\|^2-\frac{\beta}{4}\|\nabla\zeta_{2_j}\|^2 \tag{2.49}$$

$$\begin{aligned}&(\Gamma_1(\varphi(s))y_j(s),y_j(s))_{E_0}\\ &\le C_{18}\|\nabla^{-1}\xi_{1j}\|\cdot\|\nabla\xi_{2j}\|+C_{19}\|\nabla^{-1}\zeta_{1j}\|\cdot\|\nabla\xi_{2j}\|\\ &\quad+C_{20}\|\nabla^{-1}\xi_{1j}\|\cdot\|\nabla\zeta_{2j}\|+C_{21}\|\nabla^{-1}\zeta_{1j}\|\cdot\|\nabla\zeta_{2j}\|.\end{aligned} \tag{2.50}$$

Then, by the Sobolev embedding theorem:

$$H_0^1(\Omega)\subset H^1(\Omega)\subset H^{-1}(\Omega) \tag{2.51}$$

Therefore

$$\begin{aligned}&(\Gamma_1(\varphi(s))y_j(s),y_j(s))_{E_0}\\ &\le C_{22}\|\xi_{1j}\|\cdot\|\nabla\xi_{2j}\|+C_{23}\|\zeta_{1j}\|\cdot\|\nabla\xi_{2j}\|\\ &\quad+C_{24}\|\xi_{1j}\|\cdot\|\nabla\zeta_{2j}\|+C_{25}\|\zeta_{1j}\|\cdot\|\nabla\zeta_{2j}\|\\ &\le C_{26}\left(\|\xi_{1j}\|^2+\|\zeta_{1j}\|^2\right)+\frac{\beta}{8}\|\nabla\xi_{2j}\|^2+\frac{\beta}{8}\|\nabla\zeta_{2j}\|^2.\end{aligned} \tag{2.52}$$

By Young's inequality, we have

$$\begin{aligned}&(\Gamma_2(\varphi(s))y_j(s),y_j(s))_{E_0}\\ &\le(1-\mu_0)\|\nabla\xi_{1j}\|\cdot\|\nabla\xi_{2j}\|+2R_0^2k_0\|\nabla\xi_{1j}\|\cdot\|\nabla\xi_{2j}\|+2R_0^2k_0\|\nabla\zeta_{1j}\|\cdot\|\nabla\xi_{2j}\|\\ &\quad+(1-\mu_0)\|\nabla\zeta_{1j}\|\cdot\|\nabla\zeta_{2j}\|+2R_0^2k_0\|\nabla\xi_{1j}\|\cdot\|\nabla\zeta_{2j}\|+2R_0^2k_0\|\nabla\zeta_{1j}\|\cdot\|\nabla\zeta_{2j}\|\\ &\le\frac{\lambda_j^{\frac{1}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{1}{2}}R_0^2k_0+\frac{\lambda_j^{\frac{1}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{1}{2}}R_0^2k_0.\end{aligned} \tag{2.53}$$

So exist $\beta$ satisfying

$$-\frac{\beta}{8}\|\nabla\xi_{2j}\|^2+\frac{\lambda_j^{\frac{1}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{1}{2}}R_0^2k_0\le\frac{\varepsilon}{16} \tag{2.54}$$

$$-\frac{\beta}{8}\|\nabla\zeta_{2j}\|^2+\frac{\lambda_j^{\frac{1}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{1}{2}}R_0^2k_0\le\frac{\varepsilon}{16} \tag{2.55}$$

We obtain

$$
\begin{aligned}
p_{n_1}(s) &= \sum_{j=1}^{n_1}\left(-P(\varphi(s))+\Gamma_1(\varphi(s))+\Gamma_2(\varphi(s))y_j(s),y_j(s)\right)_{E_0} \\
&\le \sum_{j=1}^{n_1}\left(-\frac{\varepsilon}{4}-\frac{\beta}{4}\|\nabla\xi_{2j}\|^2-\frac{\beta}{4}\|\nabla\zeta_{2j}\|^2+\frac{\beta}{8}\|\nabla\xi_{2j}\|^2+\frac{\beta}{8}\|\nabla\zeta_{2j}\|^2\right) \\
&\quad+\sum_{j=1}^{n_1}\left(\frac{\lambda_j^{\frac{1}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{1}{2}}R_0^2k_0+\frac{\lambda_j^{\frac{1}{2}}(1-\mu_0)}{2}+2\lambda_j^{\frac{1}{2}}R_0^2k_0\right) \\
&\quad+\sum_{j=1}^{n_1}C_{26}\left(\|\xi_{1j}\|^2+\|\zeta_{1j}\|^2\right) \\
&\le -\frac{\varepsilon}{8}n_1+2C_{27}\sum_{j=1}^{n_1}\left(\lambda_j^{-1}+\lambda_j^{-1}\right).
\end{aligned} \tag{2.56}
$$

If $\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\lambda_j^{-1}+\lambda_j^{-1}\right)\le\frac{\varepsilon}{16C_{27}}$, then

$$
\begin{aligned}
q_{n_1}(s) &= \lim_{t\to\infty}\inf\sup_{\tau\in R}\sup_{\phi\in E_0}\sup_{\varphi(\tau)\in A}\frac{1}{t}\int_{\tau}^{\tau+1}p_{n_1}(s)\,\mathrm{d}s \\
&\le -2n_1C_{27}\left(\frac{\varepsilon}{16C_{27}}-\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\lambda_j^{-1}+\lambda_j^{-1}\right)\right)<0.
\end{aligned} \tag{2.57}
$$

Proof finish.

## References


[1] Lin, G.G. and Zhang, M. (2017) The Global Attractors for a Class of Nonlinear Coupled Kirchhoff-Type Equations. *European Journal of Mathematics and Computer Science*, **4**.

[2] Wu, J.Z. and Lin, G.G. (2009) The Global Attractor and Its Dimensions Estimation of Bossinesq Equation with Damping Term. *Journal of Yunnan University*, **31**, 335-340.

[3] Fan, X.M. and Zhou, S.F. (2004) Kernel Sections for Non-Autonomous Strongly Damped Wave Equations of Non-Degenerate Krichhoff-Type. *Applied Mathematics and Computation*, **158**, 253-266. https://doi.org/10.1016/j.amc.2003.08.147

[4] Lin, G.G. and Gao, Y.L. (2017) The Global and Exponential Attractors for the Higher-Order Kirchhoff-Type Equation with Strong Linear Damping. *Journal of Mathematics Research*, **9**.

[5] Teman, R. (1988) Infinite-Dimensional Dynamical Systems in Mechanics and Physics. *Applied Mathematical Sciences*, **68**, Springer-Verlag, New York. https://doi.org/10.1007/978-1-4684-0313-8