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The Study on the (*L*,*M*)-Fuzzy Independent Set Systems

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Abstract

Independent sets play an important role in matroid theory. In this paper, the definitions of pre-independent fuzzy set system and independent fuzzy set system in *L*-fuzzy setting are presented. Independent *M*-fuzzifying set system is introduced and some of its properties are discussed. Further independent (*L*,*M*)-fuzzy set system is given and some of its properties are obtained. The relations of these independent set systems in the setting of fuzzy vector spaces and fuzzy graphs are showed.

Keywords

1. Introduction

As a generalization of both graphs and matrices, matroids were introduced by Whitney in 1935. It plays an important role in mathematics, especially in applied mathematics. Matroids are precisely the structures for which the simple and efficient greedy algorithm works.

In 1988, the concept of fuzzy matroids was introduced by R. Goetschel and W. Voxman [1]. Subsequently Goetschel-Voxman fuzzy matroids were researched by many scholars (see [2] [3] [4] [5], etc.). Recently, a new approach to fuzzification of matroids was introduced by Shi [6], namely *M*-fuzzifying matroids. In the study an *M*-fuzzifying matroid was defined as a mapping $\mathcal{I}: 2^E \to M$ satisfying three axioms. The approach to the fuzzification of matroids preserves many basic properties of crisp matroids, and an *M*-fuzzifying matroid and its fuzzy rank function are one-to-one

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corresponding. Further the concept of (*L*,*M*)-fuzzy matroid was presented by Shi [7], it is a wider generalization of *M*-fuzzifying matroids.

Independent set systems play an important role in matroids theory. In this paper, firstly, the pre-independent fuzzy set system, independent fuzzy set system to *L*-fuzzy setting, independent *M*-fuzzifying set system, and independent (*L*,*M*)-fuzzy set system are presented. Secondly, the properties of these independent set systems are discussed. Finally, the relevance of these independent set systems in the setting of fuzzy vector spaces and fuzzy graphs are given.

2. Preliminaries

Let E be a non-empty finite set. We denote the power set of E by 2^E and the cardinality of X by $|X|$ for any $X \subseteq E$. Let L be a lattice, an L-fuzzy set A on E is a mapping $A: E \to L$, we denote the family of L-fuzzy sets by L^E, $[0,1]$-fuzzy sets are called fuzzy sets for short, denote $[0,1]^E$. If $\mathfrak{J}$ is a non-empty subset of 2^E, then the pair $(E, \mathfrak{J})$ is called a (crisp) set system. Use $\mathfrak{F}$ insteads of $\mathfrak{J}$, where $\mathfrak{F}$ is non-empty subsets of L^E, then the pairs $(E, \mathfrak{F})$ is called an *L*-fuzzy set system, a $[0,1]$-fuzzy set system is called a fuzzy set system for short.

A set system $(E, \mathfrak{J})$ is called an independent set system if $\mathfrak{J}$ satisfies the following statement:

(**H**) $\forall A, B \in 2^E$ and $A \subseteq B$, if $B \in \mathfrak{J}$, then $A \in \mathfrak{J}$.

Use fuzzy sets on E instead of crisp sets, Novak [4] obtained the definition of independent fuzzy set systems as follows.

Definition 2.1. Let E be a finite set and $\mathfrak{F}$ be a fuzzy subset family on E. If $\mathfrak{F}$ satisfies the following condition:

(**FH**) $\forall A, B \in [0,1]^E$ and $A \le B$, if $B \in \mathfrak{F}$, then $A \in \mathfrak{F}$,

then the pairs $(E, \mathfrak{F})$ is called an independent fuzzy set system.

Throughout this paper, let E be a finite set, both L and M denote completely distributive lattices. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. We often do not distinguish a crisp subset of E and its characteristic function.

An element a in L is called a prime element if $a \le b \wedge c$ implies $a \le b$ or $a \le c$. a in L is called co-prime if $a \ge b \vee c$ implies $a \ge b$ or $a \ge c$ [8]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

The binary relation. $<$ in L is defined as follows: for $a, b \in L$, $a < b$ if and only if for every subset $D \subseteq L$, the relation $b \le \sup D$ always implies the existence of $d \in D$ with $a \le d$ [9]. $\{a \in L : a < b\}$ is called the greatest minimal family of b in the sense of [10], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap J(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L : a <^{op} b\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$. In a completely distributive lattice L, there exist $\alpha(b)$ and $\beta(b)$ for each $b \in L$, and $b = \vee \beta(b) = \wedge \alpha(b)$ (see [10]).

In [10], Wang thought that $\beta(0) = \{0\}$ and $\alpha(1) = \{1\}$. In fact, it should be that

$\beta(0) = \varnothing$ and $\alpha(1) = \varnothing$.

Definition 2.2 ([7]) Let $A \in L^E$ and $r \in L$. Define

$$A_{[r]} = \{x \in X : A(x) \geq r\}, \qquad A^{(r)} = \{x \in X : A(x) \not\geq r\}.$$

$$A_{(r)} = \{x \in X : a \in \beta(A(x))\}, \quad A^{[r]} = \{x \in X : r \notin \alpha(A(x))\}.$$

Some properties of these cut sets can be found in [7] [11] [12].

Let $\mathcal{A} \subseteq L^E$ and $\forall r \in L \setminus \{\bot\}$. We denote

$$\mathcal{A}[r] = \{A_{[r]} : A \in \mathcal{A}\}.$$

Similarly, $\forall r \in L \setminus \{\mathrm{T}\}$, we denote

$$\mathcal{A}(r) = \{A_{(r)} : A \in \mathcal{A}\}.$$

Let $A \in 2^E$ and $r \in L$, define $r \wedge A : E \to [0,1]$ by

$$r \wedge A(x) = \begin{cases} r, & x \in A, \\ \bot, & x \notin A. \end{cases}$$

3. Independent *L*-Fuzzy Set Systems and Theirs Properties

There is not a method such that we immediately believe which way of fuzzification of a crisp structure is more natural than others. Nevertheless, it seems to be widely accepted that any fuzzifying structures have an analogous crisp structures as theirs levels. Consequently, an *L*-fuzzy set system $(E, \mathcal{I})$ is a pre-independent *L*-fuzzy set system if and only if $\mathcal{I}[r]$ is an independent set system for each $r \in L \setminus \{\bot\}$, when $L = [0,1]$, a pre-independent $[0,1]$-fuzzy set system is an fuzzy pre-independent set system [4]. In this section, we introduce the concept of independent *L*-fuzzy set system and discuss theirs properties.

Definition 3.1. *Let E be a finite set. If a mapping $\mathcal{I} : L^E \to \{0,1\}$ satisfies the following condition:*

(LH) $\forall A, B \in L^E$, $A \leq B$, if $B \in \mathcal{I}$, then $A \in \mathcal{I}$,

then the pair $(E, \mathcal{I})$ is called an independent L-fuzzy set system. An independent $[0,1]$-fuzzy set system is precise a fuzzy independent set system [4].

Theorem 3.2. *Let $(E, \mathcal{I})$ be an independent L-fuzzy set system. Then $\mathcal{I}[r]$ is an independent set system for each $r \in L \setminus \{\bot\}$.*

For each $r \in L \setminus \{\bot\}$, we show that $\mathcal{I}[r]$ is an independent set system as follows. $\forall A, B \in 2^E$, $A \subseteq B$, if $B \in \mathcal{I}[r]$, there is $G \in \mathcal{I}$ such that $G_{[r]} = B$. Since $\mathcal{I}$ satisfies the condition (**LH**), then $r \wedge G_{[r]} \in \mathcal{I}$. We have $r \wedge A \leq r \wedge G_{[r]}$ since $A \subseteq B$, this implies $r \wedge A \in \mathcal{I}$. Thus $A = (r \wedge A)_{[r]} \in \mathcal{I}[r]$. Therefore $\mathcal{I}[r]$ is an independent set system.

By Theorem 3.2, it is easy to obtain the following.

Corollary 3.3. *Let $(E, \mathcal{I})$ be an independent L-fuzzy set system. Then $(E, \mathcal{I})$ be a pre-independent L-fuzzy set system.*

Conversely, given a family of independent set systems, we can obtain an independent *L*-fuzzy set system.

Theorem 3.4. Let E be a finite set and $\mathcal{I} \subseteq L^E$. If $\mathcal{I}[r]$ is an independent set system for each $r \in L \setminus \{\perp\}$, we define

$$\mathcal{J} = \left\{ A \in L^E : A_{[r]} \in \mathcal{I}[r] \quad \text{for each} \quad r \in L \setminus \{\perp\} \right\},$$

then $(E, \mathcal{J})$ is an independent L-fuzzy set system.

Proof. Since $\mathcal{I}[r]$ is an independent set system for each $r \in L \setminus \{\perp\}$, we have $\chi_\varnothing \in \mathcal{J}$. We show that $\mathcal{J}$ satisfies the property (**LH**) as follows. $\forall A, B \in L^E$, $A \leq B$, if $B \in \mathcal{J}$, it means $B_{[r]} \in \mathcal{I}[r]$ for each $r \in L \setminus \{\perp\}$. Since $A \leq B$, we have $A_{[r]} \subseteq B_{[r]}$ for each $r \in L \setminus \{\perp\}$. Because $\mathcal{I}[r]$ satisfies the condition (**H**), then $A_{[r]} \in \mathcal{I}[r]$ for each $r \in L \setminus \{\perp\}$. Thus $A \in \mathcal{J}$. Therefore $(E, \mathcal{J})$ is an independent L-fuzzy set system.

By Theorem 3.2, we get a family of independent set systems by an independent L-fuzzy set system $(E, \mathcal{I})$. Subsequently, Theorem 3.4 tells us the family of independent set systems can induce an independent L-fuzzy set system $(E, \mathcal{J})$. In general, $\mathcal{I} = \mathcal{J}$ is not true.

In the following, we will prove when $\mathcal{I}$ satisfies the condition (s) (which will be given in Theorem 3.5), we have $\mathcal{I} = \mathcal{J}$.

Theorem 3.5. Let E be a finite set and $(E, \mathcal{I})$ be an independent L-fuzzy set system. We suppose that $\mathcal{I}$ satisfies the statement:

(s) $\forall A \in L^E$, if $r \wedge A_{[r]} \in \mathcal{I}$ for each $r \in L \setminus \{\perp\}$, then $A \in \mathcal{I}$.

Define

$$\mathcal{J} = \left\{ A \in L^E : A_{[r]} \in \mathcal{I}[r] \quad \text{for each} \quad r \in L \setminus \{\perp\} \right\},$$

then $\mathcal{I} = \mathcal{J}$.

Proof. Obviously $\mathcal{I} \subseteq \mathcal{J}$. We show that $\mathcal{J} \subseteq \mathcal{I}$ as follows. $\forall A \in \mathcal{J}$, this implies $A_{[r]} \in \mathcal{I}[r]$ for each $r \in L \setminus \{\perp\}$. Since $\mathcal{I}$ is an independent L-fuzzy set system, then $r \wedge A_{[r]} \in \mathcal{I}$ for each $r \in L \setminus \{\perp\}$. By the condition (s), then $A \in \mathcal{I}$. Thus $\mathcal{J} = \mathcal{I}$.

We call $\mathcal{I}$ is strong if it satisfies the condition (**LH**) and (s), then the pair $(E, \mathcal{I})$ is called a strong independent L-fuzzy set system. Fuzzy matroids which are introduced by Goerschel and Voxman [1] are a subclass of strong independent L-fuzzy set systems.

For a strong independent L-fuzzy set system, we can obtain an equivalent description as follows.

Theorem 3.6. Let $(E, \mathcal{I})$ be an independent L-fuzzy set system. Then $\mathcal{I}$ is strong if and only if $\forall A \in L^E$, if $A_{[r]} \in \mathcal{I}[r]$ for each $r \in L \setminus \{\perp\}$, then $A \in \mathcal{I}$.

Proof. $(E, \mathcal{I})$ is an independent L-fuzzy set system. If $\mathcal{I}$ is strong, $\forall A \in L^E$, we suppose that $A_{[r]} \in \mathcal{I}[r]$ for each $r \in L \setminus \{\perp\}$, since $(E, \mathcal{I})$ is an independent L-fuzzy set system, we have $r \wedge A_{[r]} \in \mathcal{I}$ for each $r \in L \setminus \{\perp\}$. Thus $A \in \mathcal{I}$. Conversely, $\forall A \in L^E$, if $r \wedge A_{[r]} \in \mathcal{I}$ for each $r \in L \setminus \{\perp\}$, since $A_{[r]} = \left(r \wedge A_{[r]} \right)_{[r]} \in \mathcal{I}[r]$, then $A \in \mathcal{I}$.

In Theorem 3.6, when $L = [0,1]$, the strong independent $[0,1]$-fuzzy set systems are the perfect independent set systems which are defined by Novak [4].

Lemma 3.7 ([13]). If G is a finite L-fuzzy set and $G_{(a)} \neq \varnothing$ for $a \in \beta(\text{T})$, then

there exists $b \in L$ such that $a \in \beta(b)$ and $G_{(a)} = G_{[b]}$.

Theorem 3.8. Let E be a finite set and $(E, \mathcal{I})$ be an independent L-fuzzy set system. For each $r \in L \setminus \{\mathrm{T}\}$, we have then $\mathcal{I}(r)$ is an independent set system.

Proof. For each $r \in L \setminus \{\mathrm{T}\}$, we show that $\mathcal{I}(r)$ is an independent set system as follows. $\forall A, B \in 2^E$, $A \subseteq B$, if $B \in \mathcal{I}(r)$, there is $G \in \mathcal{I}$ such that $G_{(r)} = B$. By Lemma 3.7, there exists $a \in L$ such that $r \in \beta(a)$ and $G_{(r)} = G_{[a]}$. Since $\mathcal{I}$ satisfies the condition (**LH**), then $a \wedge G_{[a]} \in \mathcal{I}$. Hence

$$a \wedge A \le a \wedge G_{[a]} \in \mathcal{I}.$$

This implies $a \wedge A \in \mathcal{I}$. Thus $A = (a \wedge A)_{(r)} \in \mathcal{I}(r)$. Therefore $\mathcal{I}(r)$ is an independent set system.

4. Independent *M*-Fuzzifying Set Systems

In crisp independent set system $(E, \mathfrak{I})$, we can regard $\mathfrak{I}$ as a mapping $2^E \to \{0, 1\}$ satisfies the property (**H**). Use fuzzy sets instead of crisp sets, Novak [4] presented an approach to the fuzzification of independent set systems, which is called fuzzy independent set system. In fact, we may consider such a mapping $\mathcal{I} : 2^E \to M$ satisfies some conditions.

Definition 4.1. Let E be a finite set. A mapping $\mathcal{I} : 2^E \to M$ satisfies $\mathcal{I}(\chi_\varnothing) = \mathrm{T}$ and the following statement:

(**MH**) $VA, B \in 2^E$, if $A \subseteq B$, then $\mathcal{I}(A) \ge \mathcal{I}(B)$,

then the pair $(E, \mathcal{I})$ is called an independent *M*-fuzzifying set system. Specially, when an independent $[0, 1]$-fuzzifying set system is also called an independent fuzzifying set system for short.

Theorem 4.2. Let $\mathcal{I} : 2^E \to M$ be a mapping. Then the following statements are equivalent:

(i) $(E, \mathcal{I})$ is an independent *M*-fuzzifying set system;

(ii) For each $r \in L \setminus \{\bot\}$, $\mathcal{I}_{[r]}$ is an independent set system;

(iii) For each $r \in P(L)$, $\mathcal{I}^{(r)}$ is an independent set system;

(iv) For each $r \in L \setminus \{\bot\}$, $\mathcal{I}_{(r)}$ is an independent set system;

(v) For each $r \in L \setminus \{\mathrm{T}\}$, $\mathcal{I}^{[r]}$ is an independent set system;

(vi) For each $r \in P(L)$, $\mathcal{I}^{[r]}$ is an independent set system.

Proof. $(i) \Rightarrow (ii)$ For each $r \in L \setminus \{\bot\}$, $\forall A, B \in 2^E$, $A \subseteq B$, if $B \in \mathcal{I}_{[r]}$, it means $\mathcal{I}(B) \ge r$. Since $\mathcal{I}$ satisfies (**MH**), then $\mathcal{I}(A) \ge \mathcal{I}(B) \ge r$. Thus $A \in \mathcal{I}_{[r]}$, *i.e.* $\mathcal{I}_{[r]}$ satisfies (**H**). Therefore $\mathcal{I}_{[r]}$ is an independent set system.

$(ii) \Rightarrow (i)$ $\forall A, B \in 2^E$, $A \subseteq B$, let $a = \mathcal{I}(B)$, then $B \in \mathcal{I}_{[a]}$. Since $\mathcal{I}_{[b]}$ satisfies (**H**), we have $A \in \mathcal{I}_{[a]}$, it implies that $\mathcal{I}(A) \ge a = \mathcal{I}(B)$. Thus $\mathcal{I}$ is an independent *M*-fuzzifying set system on *E*.

$(i) \Rightarrow (iii)$ For each $r \in P(L)$, $\forall A, B \in 2^E$, $A \subseteq B$, if $B \in \mathcal{I}^{(r)}$, it means $\mathcal{I}(B) \nleq r$. Since $\mathcal{I}$ satisfies (**MH**), then $\mathcal{I}(A) \nleq r$. Thus $(E, \mathcal{I}^{(r)})$ is an independent set system.

$(iii) \Rightarrow (i)$ $\forall A, B \in 2^E$, $A \subseteq B$, let $a = \mathcal{I}(B)$, then $B \in \mathcal{I}^{(b)}$ for any

$a \nleq b \in P(M)$. Since $\mathcal{I}^{(b)}$ satisfies the condition (**H**), we have $A \in \mathcal{I}^{(b)}$ for any $a \nleq b \in P(M)$, it implies that $\mathcal{I}(A) \nleq b$ for any $a \nleq b \in P(M)$. Then $\mathcal{I}(B) = a = \bigvee \{b \in P(M) : a \nleq b\} \leq \mathcal{I}(A)$.

$(i) \Rightarrow (iv)$ For each $r \in L \setminus \{\perp\}$, $\forall A, B \in 2^E$, $A \subseteq B$, if $B \in \mathcal{I}_{(r)}$, it means $r \in \beta(\mathcal{I}(B))$. We have $r \in \beta(\mathcal{I}(A))$ since $\mathcal{I}(A) \geq \mathcal{I}(B)$, it implies $A \in \mathcal{I}_{(r)}$.

$(iv) \Rightarrow (i)$ $\forall A, B \in 2^E$, $A \subseteq B$, let $a = \mathcal{I}(B)$, then for each $b \in \beta(a)$ we have $B \in \mathcal{I}_{(b)}$. Since $\mathcal{I}_{(b)}$ is an independent set system we have $A \in \mathcal{I}_{(b)}$ for each $b \in \beta(a)$, *i.e.* $b \in \beta(\mathcal{I}(A))$ for each $b \in \beta(a)$. Then $\mathcal{I}(B) = a = \bigvee \beta(a) \leq \mathcal{I}(A)$. Therefore $(E, \mathcal{I})$ is an independent *M*-fuzzifying set system on E.

We can similarly prove the remainder statements are also equivalent.

Corollary 4.3. Let $\mathcal{I} : 2^E \to [0,1]$ be a mapping. Then the following statements are equivalent:

(i) $(E, \mathcal{I})$ is an independent *M*-fuzzifying set system;

(ii) For each $r \in (0,1]$, $\mathcal{I}_{[r]}$ is an independent set system;

(iii) For each $r \in [0,1)$, $\mathcal{I}_{(r)}$ is an independent set system.

Remark 4.4. In Proposition 2 of [4], Novak has illuminated that when $L = M = [0,1]$, a closed fuzzy independent set system is equivalent with an independent fuzzifying set system. M-fuzzifying matroids [13] are precise a subclass of the independent *M*-fuzzifying set system.

5. Independent (*L,M*)-Fuzzy Set Systems

In this section, we obtain the definition of independent *M*-fuzzifying set systems and discuss theirs properties.

Definition 5.1. Let E be a finite set and *L,M* be lattices. A mapping $\mathcal{I} : L^E \to M$ satisfies the following statement:

(**LMH**) $\forall A, B \in L^E$, if $A \leq B$, we have $\mathcal{I}(A) \geq \mathcal{I}(B)$,

then the pair $(E, \mathcal{I})$ is called an independent (*L,M*)-fuzzy set system.

Obviously, an independent (2,*M*)-fuzzy set system can be viewed as an independent *M*-fuzzifying set system, where $2 = \{\perp, \top\}$. Moreover, an independent (*L*,2)-fuzzy set system is called an independent *L*-fuzzy set system. An crisp independent set system can be regarded as an independent (2,2)-fuzzy set system.

Theorem 5.2. Let E be a finite set and $\mathcal{I} : L^E \to M$ be a mapping. Then the following statements are equivalent:

(i) $(E, \mathcal{I})$ is an independent (*L,M*) -fuzzy set system;

(ii) For each $r \in M \setminus \{\perp_M\}$, $(E, \mathcal{I}_{[r]})$ is an independent *L*-fuzzy set system;

(iii) For each $r \in \alpha(\perp_M)$, $(E, \mathcal{I}_{[r]})$ is an independent *L*-fuzzy set system;

(iv) For each $r \in \alpha^*(\perp_M)$, $(E, \mathcal{I}_{(r)})$ is an independent *L*-fuzzy set system;

(v) For each $r \in P(M)$, $(E, \mathcal{I}_{(r)})$ is an independent *L*-fuzzy set system.

The prove is trivial and omitted.

Corollary 5.3. Let E be a finite set and $\mathcal{I} : [0,1]^E \to [0,1]$ be a mapping. Then the following conditions are equivalent:

(i) $(E, \mathcal{I})$ is an independent $([0,1],[0,1])$ -fuzzy set system;

(ii) $\forall r \in (0,1]$, $\mathcal{I}_{[r]}$ is an independent fuzzy set system;

(ii) $\forall r \in [0,1)$, $\mathcal{I}_{(r)}$ is an independent fuzzy set system.

6. Some Examples of Independent (*L,M*)-Fuzzy Set Systems

Example 6.1. Let $\tilde{G} = (G, \mu)$ be a fuzzy graph, where $G = (V, E)$. We define a mapping $\mathcal{I} : 2^E \to [0,1]$ by

$$\mathcal{I}(A) = \vee \left\{ a \in (0,1] : A \subseteq \mu_{[a]} \text{ and } A \text{ is non-circuit} \right\},$$

then the pair $(E, \mathcal{I})$ is an independent fuzzifying set system.

Obviously, $\mathcal{I}(\chi_\varnothing) = \mathrm{T}$. $\forall A, B \in 2^E$, if $A \subseteq B$, it is easy to obtain $\mathcal{I}(A) \geq \mathcal{I}(B)$.

Example 6.2. Let $\tilde{G} = (G, \mu)$ be a fuzzy graph, where $G = (V, E)$. We define a subfamily of $[0,1]^E$ by

$$\mathcal{I}_\mu = \left\{ A \in [0,1]^E : A \leq \mu \text{ and } A_{[a]} \text{ does not contain any cycles of } \left(E, \mu_{[a]} \right) \text{ for any } a \in (0,1] \right\},$$

then the pair $(E, \mathcal{I})$ is an independent fuzzy set system.

The prove is trivial and omitted.

Similarly, we can obtain easily the followings.

Example 6.3. Let $\tilde{V} = (V, \lambda)$ be a fuzzy vector space. If E is a subset of V, we define a mapping $\mathcal{I} : 2^E \to [0,1]$ by

$$\mathcal{I}_\lambda = \vee \left\{ a \in (0,1] : A \subseteq \lambda_{[a]} \text{ and } A \text{ is linear independent} \right\},$$

then the pair $(E, \mathcal{I})$ is an independent fuzzifying set system.

Example 6.4. Let $\tilde{V} = (V, \lambda)$ be a fuzzy vector space. If E is a subset of V, we define a subfamily of $[0,1]^E$ by

$$\mathcal{I}_\lambda = \left\{ A \in [0,1]^E : A \leq \lambda \text{ and } A_{[a]} \text{ is linear independent in } \lambda_{[a]} \text{ for any } a \in (0,1] \right\},$$

then the pair $(E, \mathcal{I})$ is an independent fuzzy set system.

7. Conclusion

In this paper, pre-independent fuzzy set system and independent fuzzy set system to *L*-fuzzy setting are defined. Independent *M*-fuzzifying set system is introduced and obtained its some properties. Further the definition of independent *M*-fuzzifying set system is generalized to independent (*L,M*)-fuzzy set system, and its some properties are proved. Finally, the relevance of generalized independent set systems are presented in the setting of fuzzy vector spaces and fuzzy graphs.

Funds

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