Advances in Pure Mathematics, 2016, 6, 903-914
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

# On the Prime Geodesic Theorem for Non-Compact Riemann Surfaces

**Muharem Avdispahić, Dženan Gušić**

Department of Mathematics, Faculty of Sciences and Mathematics, University of Sarajevo, Sarajevo, Bosnia and Herzegovina
Email: mavdispa@pmf.unsa.ba, dzenang@pmf.unsa.ba

**How to cite this paper:** Avdispahić, M. and Gušić, Dž. (2016) On the Prime Geodesic Theorem for Non-Compact Riemann Surfaces. *Advances in Pure Mathematics*, **6**, 903-914.
http://dx.doi.org/10.4236/apm.2016.612068

**Received:** October 23, 2016
**Accepted:** November 19, 2016
**Published:** November 22, 2016



## Abstract

We use B. Randol's method to improve the error term in the prime geodesic theorem for a noncompact Riemann surface having at least one cusp. The case considered is a general one, corresponding to a Fuchsian group of the first kind and a multiplier system with a weight on it.

## Keywords

Selberg Trace Formula, Selberg Zeta Function, Prime Geodesic Theorem

## 1. Introduction

The Selberg trace formula, introduced by A. Selberg in 1956, describes the spectrum of the hyperbolic Laplacian in terms of geometric data involving the lengths of geodesics on a Riemann surface. Motivated by analogy between this trace formula and the explicit formulas of number theory relating the zeroes of the Riemann zeta function to prime numbers, Selberg [1] introduced a zeta function whose analytic properties are encoded in the Selberg trace formula. By focusing on the Selberg zeta function, H. Huber ([2], p. 386; [3], p. 464), proved an analogue of the prime number theorem for compact Riemann surfaces with the error term $O\left(x^{\frac{3}{4}}(\log x)^{-\frac{1}{2}}\right)$ that agrees with Selberg's one. Using basically the same method as in [4], D. Hejhal ([5], p. 475), established also the prime geodesic theorem for non-compact Riemann surfaces with the remainder $O\left(x^{\frac{3}{4}}(\log x)^{-\frac{1}{2}}\right)$. However, in the compact case there exist several different proofs (see, B. Randol [6], p. 245; P. Buser [7], p. 257, Th. 9.6.1; M. Avdispahić and L. Smajlović

[8], Th. 3.1) that give the remainder $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$. Thanks to new integral representations of the logarithmic derivative of the Selberg zeta function (cf. [9], p. 185; [10], p. 128), M. Avdispahić and L. Smajlović ([11], p. 13) were in position to improve $O\left(x^{\frac{3}{4}}(\log x)^{-\frac{1}{2}}\right)$ error term in a non-compact, finite volume case up to $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$. Whereas the authors in [8] and [11] approached the prime number theorem in various settings via explicit formulas for the Jorgenson-Lang fundamental class of functions, our main goal is to obtain this improvement for non-compact Riemann surfaces with cusps following a more direct method of B. Randol [6].

## 2. Preliminaries

Let $X$ be a non-compact Riemann surface regarded as a quotient $\Gamma\backslash\mathcal{H}$ of the upper half-plane $\mathcal{H}$ by a finitely-generated Fuchsian group $\Gamma\subseteq\mathrm{PSL}(2,\mathbb{R})$ of the first kind, containing $n_1\geq 1$ cusps. Let $\Im$ denote the fundamental region of $\Gamma$. We shall assume that the fundamental region $\Im$ of $\Gamma$ has a finite non-Euclidean area $|\Im|$. We put

$$\bar{\Gamma}=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{R}):\frac{az+b}{cz+d}\in\Gamma\right\}$$

and denote by $v$ the multiplier system of the weight $m\in\mathbb{R}$ for $\bar{\Gamma}$. Let $\psi$ be an irreducible $r\times r$ unitary representation on $\Gamma$ and $W(T)=\psi(T)v(T)$, $T\in\bar{\Gamma}$. For an $r$ dimensional vector space $V$ over $\mathbb{C}$ we consider an essentially self-adjoint operator

$$\Delta_m=y^2\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right)-imy\frac{\partial}{\partial x}$$

on the space $\mathcal{D}_m$ of all twice continuously differentiable functions $f:\mathcal{H}\to V$, such that $f$ and $\Delta_m(f)$ are square integrable on $\Im$, and satisfy the equality

$$f(Sz)=\frac{(cz+d)^m}{|cz+d|^m}W(S)f(z),\ \text{for all } z\in\mathcal{H} \text{ and } \mathcal{S}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\bar{\Gamma}.$$

The operator $-\Delta_m$ has the unique self-adjoint extension $-\tilde{\Delta}_m$ to the space $\tilde{\mathcal{D}}_m$, a dense subspace of $L^2(\Gamma\backslash\mathcal{H})$. Let $T_j$, $j=1,\cdots,n_1$ be the set of parabolic transformations corresponding to $n_1$ cusps of $\Gamma$. $W(T_j)$ does not depend on the choice of a representative of the parabolic class $\{T_j\}$ and can be considered as a matrix from $\mathbb{C}^{r\times r}$. By $m_j$ we will denote the multiplicity of 1 as an eigen-value of the matrix $W(T_j)$, and $n_1^*=\sum_{j=1}^{n_1}m_j$ will be the degree of singularity of $W$. We mention that operator $-\tilde{\Delta}_m$ has both the discrete and continuous spectrum in the case $n_1^*\geq 1$, and only the discrete spectrum in the case $n_1^*=0$. The discrete spectrum will be denoted as $\{\lambda_n\}_{n>0}$ ($0=\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty$). The continuous spectrum is expressed through zeros (or equivalently poles) of the hyperbolic scattering determinant (see, [12]).

## 3. Selberg Zeta Function

Let $\mathrm{P}\Gamma_{\mathrm{h}}$ denotes the set of $\Gamma$-conjugacy classes of a primitive hyperbolic element $P_0$ in $\Gamma$, and $\Gamma_{\mathrm{h}}$ denotes the set of $\Gamma$-conjugacy classes of a hyperbolic element $P$ in $\Gamma$ that satisfy property $\mathrm{Tr}(P) > 2$. Assume that $|m| \le 1$. We define the Selberg zeta function associated to the pair $(\Gamma, W)$ by

$$Z_{\Gamma,W}(s) = \prod_{P_0 \in \mathrm{P}\Gamma_{\mathrm{h}}} \prod_{k=0}^{\infty} \det\left(I_r - W(P_0) N(P_0)^{-s-k}\right).$$

$Z_{\Gamma,W}(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Analytic considerations given in ([5], pp. 499-501) yield that the Selberg zeta function in this setting satisfies the functional equation

$$Z_{\Gamma,W}(s)\Psi(s) = Z_{\Gamma,W}(1-s)$$

with the fudge factor

$$\Psi(s) = \phi(s) \cdot \eta\left(\frac{1}{2}\right)\left(\int_{\frac{1}{2}}^{s} \frac{\eta'}{\eta}(u)\,\mathrm{d}u\right). \tag{1}$$

Here, $\phi$ denotes the hyperbolic scattering determinant. It can be represented in the form

$$\phi(s) = \left(\frac{\sqrt{\pi}\Gamma(s)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s+\frac{m}{2}\right)\Gamma\left(s-\frac{m}{2}\right)}\right)^{n_1^*} \sum_{n=1}^{\infty} \frac{a_n}{g_n^{2s}},$$

where the coefficients $a_n$ and $g_n$ depend on the group $\Gamma$ (see, [5], p. 437). Here, $n_1^*$ denotes the degree of singularity of $W$ (see Section 2). An explicit expression for the fudge factor $\eta$ in the Equation (1) is given in ([5], p. 501, Equation (5.10)).

The logarithmic derivative of the Selberg zeta function $Z_{\Gamma,W}(s)$ is given by

$$\frac{Z'_{\Gamma,W}(s)}{Z_{\Gamma,W}(s)} = \sum_{P \in \Gamma_{\mathrm{h}}} \frac{\Lambda(P)}{N(P)^s} \mathrm{Tr}(W(P)),$$

where $N(P)$ denotes the norm of the class $P$ and $\Lambda(P) = \dfrac{\log N(P_0)}{1 - N(P)^{-1}}$ for a primitive element $P_0$ such that $P = P_0^n$ for some $n \in \mathbb{N}$. We will omit the indices in $Z_{\Gamma,W}$ in the sequel.

## 4. Counting Functions $\psi_n(x,W)$

**Lemma 1.** *For* $\mathrm{Re}(s) > 1$,

$$\frac{Z'(s)}{Z(s)} = \sum_{P \in \Gamma_{\mathrm{h}}} \Lambda_1(P)\mathrm{Tr}(W(P)) N(P)^{-s} + \frac{Z'(s+1)}{Z(s+1)},$$

*where* $\Lambda_1(P) = \log N(P_0)$ *for a primitive element* $P_0$ *such that* $P = P_0^n$ *for some* $n \in \mathbb{N}$.

*Proof.*

$$\frac{Z'(s)}{Z(s)} = \sum_{P\in\Gamma_h} \Lambda_1(P)\operatorname{Tr}(W(P))\left(1-N(P)^{-1}\right)^{-1} N(P)^{-s}$$

$$= \sum_{P\in\Gamma_h} \Lambda_1(P)\operatorname{Tr}(W(P))N(P)^{-s} + \sum_{P\in\Gamma_h} \Lambda_1(P)\operatorname{Tr}(W(P))(N(P)-1)^{-1} N(P)^{-s}$$

$$= \sum_{P\in\Gamma_h} \Lambda_1(P)\operatorname{Tr}(W(P))N(P)^{-s} + \sum_{P\in\Gamma_h} \Lambda_1(P)\operatorname{Tr}(W(P))\left(1-N(P)^{-1}\right)^{-1} N(P)^{-(s+1)}.$$

We shall spend the rest of this section to derive a representation of $\psi_2(x,W)$ in the form (11) bellow. We choose not to write it in a separate statement because of the length of expressions involved. However, it will serve as a base for the proof of the prime geodesic theorem in Section 5.

Let us recall the following theorem given in ([13], p. 51, Th. 40).

**Theorem 1.** *If the Dirichlet's series* $f(s) = \Sigma a_n e^{-\lambda_n} = \Sigma a_n l_n^{-s}$ *is summable* $(l,k)$ *for* $s = \beta$ *and* $c > 0$, $c > \beta$, *then*

$$\omega^{-k} \sum_{l_n<\omega} a_n (\omega - l_n)^k = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{\Gamma(k+1)\Gamma(s)}{\Gamma(k+1+s)}\omega^s \mathrm{d}s. \tag{2}$$

By Lemma 1,

$$\frac{Z'(s)}{Z(s)} - \frac{Z'(s+1)}{Z(s+1)} = \sum_{P\in\Gamma_h} \Lambda_1(P)\operatorname{Tr}(W(P))N(P)^{-s}.$$

We have,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{Z'(s)}{Z(s)} - \frac{Z'(s+1)}{Z(s+1)}\right)s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^s \mathrm{d}s$$

$$= \frac{1}{k!}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\sum_{P\in\Gamma_h}\Lambda_1(P)\operatorname{Tr}(W(P))\left(\frac{N(P)}{x}\right)^{-s}\right)\frac{\Gamma(k+1)\Gamma(s)}{\Gamma(k+1+s)}\mathrm{d}s.$$

Therefore, substituting $\omega = 1$, $f(s) = \sum_{P\in\Gamma_h}\Lambda_1(P)\operatorname{Tr}(W(P))\left(\frac{N(P)}{x}\right)^{-s}$, and hence $a_n = \Lambda_1(P)\operatorname{Tr}(W(P))$, $l_n = \frac{N(P)}{x}$ in (2), we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{Z'(s)}{Z(s)} - \frac{Z'(s+1)}{Z(s+1)}\right)s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^s \mathrm{d}s$$

$$= \frac{1}{k!}\sum_{\frac{N(P)}{x}\le 1}\Lambda_1(P)\operatorname{Tr}(W(P))\left(1-\frac{N(P)}{x}\right)^{-s}$$

$$= \frac{1}{k!}\sum_{N(P)\le x}\Lambda_1(P)\operatorname{Tr}(W(P))\left(1-\frac{N(P)}{x}\right)^{-s}.$$

Then,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{Z'(s)}{Z(s)}s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^{s}\mathrm{d}s$$
$$=\frac{1}{k!}\sum_{N(P)\le x}\Lambda_1(P)\mathrm{Tr}(W(P))\left(1-\frac{N(P)}{x}\right)^{k} \tag{3}$$
$$+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{Z'(s+1)}{Z(s+1)}s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^{s}\mathrm{d}s.$$

Now, put

$$\psi_0(x,W)=\sum_{N(P)\le x}\Lambda_1(P)\mathrm{Tr}(W(P))$$

and

$$\psi_j(x,W)=\int_0^x\psi_{j-1}(t,W)\mathrm{d}t$$

for $j=1,2,\cdots$. Using ([14], p. 12, Th. 1.3.5), it is easy to get that

$$\psi_j(x,W)=\frac{1}{j!}\sum_{N(P)\le x}\Lambda_1(P)\mathrm{Tr}(W(P))(x-N(P))^{j}. \tag{4}$$

For $0<\lambda_n<\frac{1}{4}$, let $s_n=\frac{1}{2}-ir_n=\frac{1}{2}-i\sqrt{\lambda_n-\frac{1}{4}}$, $n=1,2,\cdots,K$, be the zeros of $Z(s)$ in $\left(\frac{1}{2},1\right)$. Let $\rho_k$, $k=0,1,\cdots,M_e$ denote all zeros of the hyperbolic scattering determinant in $\left(\frac{1}{2},1\right]$.

Assume $T>2$, $T\pm l\ne r_n$, $l\in\mathbb{N}$, where $s_n=\frac{1}{2}+ir_n$ and $\tilde{s}_n=\frac{1}{2}-ir_r$ for $r_n=-i\sqrt{\frac{1}{4}-\lambda_n}$, $\lambda_n>\frac{1}{4}$. Following ([5], p. 468), we may also assume $T\pm l\ne\gamma$, $l\in\mathbb{N}$, where, $1-\rho$, $1-\bar{\rho}$ are the zeros of the Selberg zeta function $Z(s)$ for each zero $\rho=\frac{1}{2}+\eta+i\gamma$, $\eta\ge 0$, $\gamma>0$, of the hyperbolic scattering determinant $\phi$. Let $A_0\ge\frac{3}{2}$ be a large constant such that $A_0\notin\mathbb{Z}$, $A_0+\frac{1}{2}\notin\mathbb{Z}$, $A_0\pm\frac{m}{2}\notin\mathbb{Z}$. We put $A=A_0+1$. Without loss of generality we may assume that $c=1+\varepsilon$, $\varepsilon>0$, $x\ge 2$. Let $R(A,T)=[-A,c]\times[-T,T]$. By the Cauchy residue theorem one has

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{Z'(s)}{Z(s)}s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^{s}\mathrm{d}s$$
$$=\frac{1}{2\pi i}\left(\int_{-A+iT}^{-A_0+iT}+\int_{-A_0+iT}^{\frac{1}{2}-\varepsilon+iT}+\int_{\frac{1}{2}-\varepsilon+iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{c+iT}+\int_{c-iT}^{\frac{1}{2}+\varepsilon-iT}+\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}-\varepsilon-iT}+\int_{\frac{1}{2}-\varepsilon-iT}^{-A_0-iT}+\int_{-A_0-iT}^{-A-iT}\right) \tag{5}$$
$$+\frac{1}{2\pi i}\int_{-A-iT}^{-A+iT}+\sum_{z\in R(A,T)}\mathrm{Res}_{s=z}\left(\frac{Z'(s)}{Z(s)}s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^{s}\right)$$

and

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{Z'(s+1)}{Z(s+1)}s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^{s}\mathrm{d}s$$
$$=\frac{1}{2\pi i}\left(\int_{-A+iT}^{-A_0+iT}+\int_{-A_0+iT}^{\frac{1}{2}-\varepsilon+iT}+\int_{\frac{1}{2}-\varepsilon+iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{c+iT}+\int_{c-iT}^{\frac{1}{2}+\varepsilon-iT}+\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}-\varepsilon-iT}+\int_{\frac{1}{2}-\varepsilon-iT}^{-A_0-iT}+\int_{-A_0-iT}^{-A-iT}\right)$$
$$+\frac{1}{2\pi i}\int_{-A-iT}^{-A+iT}+\sum_{z\in R(A,T)}\mathrm{Res}_{s=z}\left(\frac{Z'(s+1)}{Z(s+1)}s^{-1}(s+1)^{-1}\cdots(s+k)^{-1}x^{s}\right). \tag{6}$$

Arguing as in [5] (p. 474) and [4] (pp. 105-108), we easily find that the sum of the first eight integrals on the right hand side of (5) is $O\left(\frac{x^{1+\varepsilon}}{\varepsilon T^{k}}\right)$. Similarly, taking into account that $\frac{Z'(s)}{Z(s)}$ is bounded for $\mathrm{Re}(s)>1$, we obtain that the sum of the first eight integrals on the right hand side of (6) is $O\left(\frac{x^{1+\varepsilon}}{\varepsilon T^{k}}\right)$. Following [5] (p. 474) and [4] (p. 85, Prop. 5.7), we obtain that the ninth resp. the third integral on the right hand side of (5) resp. (6) are $O\left(x^{-A}\right)$. Now, if we take $k=2$, (5) and (6) will give us

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{Z'(s)}{Z(s)}\frac{x^{s}}{s(s+1)(s+2)}\mathrm{d}s$$
$$=O\left(x^{-A}\right)+O\left(\frac{x^{1+\varepsilon}}{\varepsilon T^{2}}\right)+\sum_{z\in R(A,T)}\mathrm{Res}_{s=z}\left(\frac{Z'(s)}{Z(s)}\frac{x^{s}}{s(s+1)(s+2)}\right) \tag{7}$$

and

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{Z'(s+1)}{Z(s+1)}\frac{x^{s}}{s(s+1)(s+2)}\mathrm{d}s$$
$$=O\left(x^{-A}\right)+O\left(\frac{x^{1+\varepsilon}}{\varepsilon T^{2}}\right)+\sum_{z\in R(A,T)}\mathrm{Res}_{s=z}\left(\frac{Z'(s+1)}{Z(s+1)}\frac{x^{s}}{s(s+1)(s+2)}\right). \tag{8}$$

Bearing in mind location of the poles of $\frac{Z'(s)}{Z(s)}$ given in ([5], p. 439, Th. 2.16; or [5], p. 498, Th. 5.3) and the fact that $|m|\le 1$, we may assume without loss of generality that

$$-1-\frac{m}{2},-1+\frac{m}{2}\in\left(-\frac{5}{4},-\frac{1}{2}\right],-3-\frac{m}{2},-3+\frac{m}{2}\notin\left[-A,-\frac{5}{2}\right].$$

Calculating residues and passing to the limit $T\to+\infty, A\to+\infty$ in (7) and (8) we get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{Z'(s)}{Z(s)}\frac{x^s}{s(s+1)(s+2)}\mathrm{ds}=\frac{1}{6}x+\Bigg(\sum_{n=1}^{K}\frac{x^{s_n}}{s_n(s_n+1)(s_n+2)}+\sum_{n=1}^{K}\frac{x^{\tilde{s}_n}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}+\sum_{k=0}^{M_e}\frac{x^{1-\rho_k}}{(1-\rho_k)(2-\rho_k)(3-\rho_k)}$$
$$+A_0\frac{x^{-\frac{m}{2}}}{-\frac{m}{2}\left(-\frac{m}{2}+1\right)\left(-\frac{m}{2}+2\right)}+A_1\frac{x^{-1-\frac{m}{2}}}{\left(-1-\frac{m}{2}\right)\left(-\frac{m}{2}\right)\left(-\frac{m}{2}+1\right)}+B_0\frac{x^{\frac{m}{2}}}{\frac{m}{2}\left(\frac{m}{2}+1\right)\left(\frac{m}{2}+2\right)}$$
$$+B_1\frac{x^{-1+\frac{m}{2}}}{\left(-1+\frac{m}{2}\right)\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right)}+\frac{8}{15}\mathrm{Tr}\left(I_r-\Phi\left(\frac{1}{2}\right)\right)x^{\frac{1}{2}}-\frac{8}{3}n_1^*x^{-\frac{1}{2}}-g_1n_1^*x^{-1}-n_1^*x^{-1}\log x$$
$$+\left(-\frac{3}{2}+h_1\right)\frac{n_1^*}{2}+\frac{n_1^*}{2}\log x\Bigg)+\Bigg(\sum_{r_n>0}\frac{x^{s_n}}{s_n(s_n+1)(s_n+2)}+\sum_{r_n>0}\frac{x^{\tilde{s}_n}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)} \tag{9}$$
$$+\sum_{\gamma>0}\frac{x^{\frac{1}{2}-\eta-i\gamma}}{\left(\frac{1}{2}-\eta-i\gamma\right)\left(\frac{3}{2}-\eta-i\gamma\right)\left(\frac{5}{2}-\eta-i\gamma\right)}+\sum_{\gamma>0}\frac{x^{\frac{1}{2}-\eta+i\gamma}}{\left(\frac{1}{2}-\eta+i\gamma\right)\left(\frac{3}{2}-\eta+i\gamma\right)\left(\frac{5}{2}-\eta+i\gamma\right)}\Bigg)$$
$$+\left(A_2\frac{x^{-2-\frac{m}{2}}}{\left(-2-\frac{m}{2}\right)\left(-1-\frac{m}{2}\right)\left(-\frac{m}{2}\right)}+B_2\frac{x^{-2+\frac{m}{2}}}{\left(-2+\frac{m}{2}\right)\left(-1+\frac{m}{2}\right)\frac{m}{2}}+\frac{8}{3}n_1^*x^{-\frac{3}{2}}-\frac{8}{15}n_1^*x^{-\frac{5}{2}}+\left(\frac{3}{2}+f_1\right)\frac{n_1^*}{2}x^{-2}+\frac{n_1^*}{2}x^{-2}\log x\right)$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{Z'(s+1)}{Z(s+1)}\frac{x^s}{s(s+1)(s+2)}\mathrm{ds}=\Bigg(\sum_{n=1}^{K}\frac{x^{s_n-1}}{(s_n-1)s_n(s_n+1)}+\sum_{n=1}^{K}\frac{x^{\tilde{s}_n-1}}{(\tilde{s}_n-1)\tilde{s}_n(\tilde{s}_n+1)}+A_0\frac{x^{-\frac{m}{2}-1}}{\left(-\frac{m}{2}-1\right)\left(-\frac{m}{2}\right)\left(-\frac{m}{2}+1\right)}$$
$$+B_0\frac{x^{-1+\frac{m}{2}}}{\left(-1+\frac{m}{2}\right)\frac{m}{2}\left(\frac{m}{2}+1\right)}-\frac{8}{3}\mathrm{Tr}\left(I_r-\Phi\left(\frac{1}{2}\right)\right)x^{-\frac{1}{2}}+\sum_{k=0}^{M_e}\frac{x^{-\rho_k}}{(-\rho_k)(-\rho_k+1)(-\rho_k+2)}-g_1'n_1^*x^{-1}-n_1^*x^{-1}\log x$$
$$+\left(-\frac{3}{2}+h_1'\right)\frac{1}{2}+\frac{1}{2}\log x\Bigg)+\Bigg(\sum_{r_n>0}\frac{x^{s_n-1}}{(s_n-1)s_n(s_n+1)}+\sum_{r_n>0}\frac{x^{\tilde{s}_n-1}}{(\tilde{s}_n-1)\tilde{s}_n(\tilde{s}_n+1)}+\sum_{\gamma>0}\frac{x^{-\frac{1}{2}-\eta-i\gamma}}{\left(-\frac{1}{2}-\eta-i\gamma\right)\left(\frac{1}{2}-\eta-i\gamma\right)\left(\frac{3}{2}-\eta-i\gamma\right)} \tag{10}$$
$$+\sum_{\gamma>0}\frac{x^{-\frac{1}{2}-\eta+i\gamma}}{\left(-\frac{1}{2}-\eta+i\gamma\right)\left(\frac{1}{2}-\eta+i\gamma\right)\left(\frac{3}{2}-\eta+i\gamma\right)}\Bigg)+\Bigg(A_1\frac{x^{-2-\frac{m}{2}}}{\left(-2-\frac{m}{2}\right)\left(-1-\frac{m}{2}\right)\left(-\frac{m}{2}\right)}$$
$$+B_1\frac{x^{-2+\frac{m}{2}}}{\left(-2+\frac{m}{2}\right)\left(-1+\frac{m}{2}\right)\frac{m}{2}}+\left(\frac{3}{2}+f_1'\right)\frac{n_1^*}{2}x^{-2}+\frac{n_1^*}{2}x^{-2}\log x+\frac{8}{3}n_1^*x^{-\frac{3}{2}}-\frac{8}{15}n_1^*x^{-\frac{5}{2}}\Bigg).$$

The implied constants on the right sides of (9) and (10) depend solely on $\Gamma$, $m$ and $W$. With $k=2$ in (3), $j=2$ in (4), Equations (4), (3), (9) and (10) yield

$$
\begin{aligned}
\psi_2(x,W) &= \frac{1}{2!}\sum_{N(P)\le x}\Lambda_1(P)\,\mathrm{Tr}\big(W(P)\big)\big(x-N(P)\big)^2 \\
&= \frac{1}{2\pi i}\left(\int_{c-i\infty}^{c+i\infty}\frac{Z'(s)}{Z(s)}\frac{x^{s+2}}{s(s+1)(s+2)}\mathrm{d}s-\int_{c-i\infty}^{c+i\infty}\frac{Z'(s+1)}{Z(s+1)}\frac{x^{s+2}}{s(s+1)(s+2)}\mathrm{d}s\right) \\
&= \frac{1}{6}x^3+\left(\sum_{n=1}^{K}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}+\sum_{n=1}^{K}\frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}+\sum_{k=0}^{M_e}\frac{x^{3-\rho_k}}{(1-\rho_k)(2-\rho_k)(3-\rho_k)}\right. \\
&\quad +A_0\frac{x^{-\frac{m}{2}+2}}{-\frac{m}{2}\left(-\frac{m}{2}+1\right)\left(-\frac{m}{2}+2\right)}+A_1\frac{x^{1-\frac{m}{2}}}{\left(-1-\frac{m}{2}\right)\left(-\frac{m}{2}\right)\left(-\frac{m}{2}+1\right)}+B_0\frac{x^{\frac{m}{2}+2}}{\frac{m}{2}\left(\frac{m}{2}+1\right)\left(\frac{m}{2}+2\right)} \\
&\quad +B_1\frac{x^{1+\frac{m}{2}}}{\left(-1+\frac{m}{2}\right)\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right)}+\frac{8}{15}\mathrm{Tr}\left(I_r-\Phi\left(\frac{1}{2}\right)\right)x^{\frac{5}{2}}-\frac{8}{3}n_1^*x^{\frac{3}{2}}-g_1n_1^*x+\left(-\frac{3}{2}+h_1\right)\frac{n_1^*}{2}x^2 \\
&\quad +\frac{n_1^*}{2}x^2\log x-\sum_{n=1}^{K}\frac{x^{s_n+1}}{(s_n-1)s_n(s_n+1)}-\sum_{n=1}^{K}\frac{x^{\tilde{s}_n+1}}{(\tilde{s}_n-1)\tilde{s}_n(\tilde{s}_n+1)}-A_0\frac{x^{-\frac{m}{2}+1}}{\left(-\frac{m}{2}-1\right)\left(-\frac{m}{2}\right)\left(-\frac{m}{2}+1\right)} \\
&\quad -B_0\frac{x^{1+\frac{m}{2}}}{\left(-1+\frac{m}{2}\right)\frac{m}{2}\left(\frac{m}{2}+1\right)}+\frac{8}{3}\mathrm{Tr}\left(I_r-\Phi\left(\frac{1}{2}\right)\right)x^{\frac{3}{2}}-\frac{1}{2}x^2\log x-\sum_{k=0}^{M_e}\frac{x^{-\rho_k+2}}{(-\rho_k)(-\rho_k+1)(-\rho_k+2)} \\
&\quad \left.+g_1'n_1^*x-\left(-\frac{3}{2}+h_1'\right)\frac{1}{2}x^2\right)+\left(\sum_{r_n>0}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}+\sum_{r_n>0}\frac{x^{\tilde{s}_n+2}}{\tilde{s}_n(\tilde{s}_n+1)(\tilde{s}_n+2)}\right. \\
&\quad +\sum_{\gamma>0}\frac{x^{\frac{5}{2}-\eta-i\gamma}}{\left(\frac{1}{2}-\eta-i\gamma\right)\left(\frac{3}{2}-\eta-i\gamma\right)\left(\frac{5}{2}-\eta-i\gamma\right)}+\sum_{\gamma>0}\frac{x^{\frac{5}{2}-\eta+i\gamma}}{\left(\frac{1}{2}-\eta+i\gamma\right)\left(\frac{3}{2}-\eta+i\gamma\right)\left(\frac{5}{2}-\eta+i\gamma\right)} \\
&\quad -\sum_{r_n>0}\frac{x^{s_n+1}}{(s_n-1)s_n(s_n+1)}-\sum_{r_n>0}\frac{x^{\tilde{s}_n+1}}{(\tilde{s}_n-1)\tilde{s}_n(\tilde{s}_n+1)}-\sum_{\gamma>0}\frac{x^{\frac{3}{2}-\eta-i\gamma}}{\left(-\frac{1}{2}-\eta-i\gamma\right)\left(\frac{1}{2}-\eta-i\gamma\right)\left(\frac{3}{2}-\eta-i\gamma\right)} \\
&\quad \left.-\sum_{\gamma>0}\frac{x^{\frac{3}{2}-\eta+i\gamma}}{\left(-\frac{1}{2}-\eta+i\gamma\right)\left(\frac{1}{2}-\eta+i\gamma\right)\left(\frac{3}{2}-\eta+i\gamma\right)}\right)+\left(A_2\frac{x^{-\frac{m}{2}}}{\left(-2-\frac{m}{2}\right)\left(-1-\frac{m}{2}\right)\left(-\frac{m}{2}\right)}\right. \\
&\quad +B_2\frac{x^{\frac{m}{2}}}{\left(-2+\frac{m}{2}\right)\left(-1+\frac{m}{2}\right)\frac{m}{2}}+\left(\frac{3}{2}+f_1\right)\frac{n_1^*}{2}+\frac{n_1^*}{2}\log x-A_1\frac{x^{-\frac{m}{2}}}{\left(-2-\frac{m}{2}\right)\left(-1-\frac{m}{2}\right)\left(-\frac{m}{2}\right)} \\
&\quad \left.-B_1\frac{x^{\frac{m}{2}}}{\left(-2+\frac{m}{2}\right)\left(-1+\frac{m}{2}\right)\frac{m}{2}}-\left(\frac{3}{2}+f_1'\right)\frac{n_1^*}{2}-\frac{n_1^*}{2}\log x\right) \\
&= \frac{1}{6}x^3+\sum_{\mathrm{I}}+\sum_{\mathrm{II}}+\sum_{\mathrm{III}},
\end{aligned}
\tag{11}
$$

where the first sum ranges over the finite set of poles $s$ of

$$\frac{Z'(s)}{Z(s)}\frac{x^s}{s(s+1)(s+2)}, \frac{Z'(s+1)}{Z(s+1)}\frac{x^s}{s(s+1)(s+2)}$$

with $\mathrm{Re}(s)>-\frac{5}{4}$, $\mathrm{Im}(s)=0$, the second sum ranges over the set of poles s of the same functions with $\mathrm{Im}(s)>0$, and the third sum ranges over the finite set of their poles $s$ with $\mathrm{Re}(s)<-\frac{5}{4}$.

## 5. Prime Geodesic Theorem

In our setting, the prime geodesic counting function is defined by

$$\pi_0(x,W)=\sum_{N(P_0)\le x}\mathrm{Tr}\left(W\left(P_0\right)\right), x\ge 1,$$

where the sum on the right is taken over all primitive hyperbolic classes $P_0\in \mathrm{P}\Gamma_{\mathrm{h}}$ with respect to $\bar{\Gamma}$ (see, [5], p. 473, [11], p. 13).

**Theorem 2.** *For* $x\ge 2$, *the formula*

$$\pi_0(x,W)=\sum_{n=0}^{K} li\left(x^{s_n}\right)+O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$$

*holds true, where,* $s_n=\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda_n}$ *for* $0\le\lambda_n<\frac{1}{4}$, *and the implied constant depends solely on* $\Gamma$, *m and W.*

*Proof.* Following [6] (p. 245) and [15] (p. 11), for a positive number $d>0$, we define the second difference operator $\Delta_2^+$ by

$$\Delta_2^+ f(x)=\int_x^{x+d}\int_y^{y+d} f''(t)\mathrm{d}t\mathrm{d}y. \tag{12}$$

Here, $d$ is a constant which will be fixed later. By the mean value theorem, we have

$$\Delta_2^+ x^{\theta}=d^2\theta(\theta-1)\tilde{x}^{\theta-2} \tag{13}$$

for some $\tilde{x}\in[x,x+2d]$. It is easy to verify that

$$\Delta_2^+ f(x)=f(x+2d)-2f(x+d)+f(x). \tag{14}$$

Reasoning as in [5] (p. 475), we may assume without loss of generality that $\psi_0(x,W)$ is non-decreasing. Hence, (12) implies

$$\psi_0(x,W)\le d^{-2}\Delta_2^+\psi_2(x,W)\le\psi_0(x+2d,W). \tag{15}$$

Since (14) holds true, one can easily deduce that $\Delta_2^+ Cf(x)=C\Delta_2^+ f(x)$, $\Delta_2^+ C=0$, $\Delta_2^+ x=0$, $d^{-2}\Delta_2^+ x^2=O(1)$, $d^{-2}\Delta_2^+ x^2\log x=O\left(x^{\frac{1}{4}}\log x\right)=O\left(x^{\frac{3}{4}}\right)$, $d^{-2}\Delta_2^+\log x=O(1)$, $d^{-2}\Delta_2^+\left(\frac{1}{6}x^3\right)=x+O(d)$. Thus, (13) and finiteness of the sums contained in $\sum_{\mathrm{I}}$ on the right hand side (11) yield

$$d^{-2}\Delta_2^{+}\left(\frac{1}{6}x^3+\sum_{\mathrm{I}}\right)=x+O(d)+\sum_{n=1}^{K}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right). \tag{16}$$

Similarly,

$$d^{-2}\Delta_2^{+}\sum_{\mathrm{III}}=O\left(x^{-\frac{3}{2}}\right). \tag{17}$$

In order to estimate $d^{-2}\Delta_2^{+}\sum_{\mathrm{II}}$, we will first consider

$$d^{-2}\Delta_2^{+}\sum_{r_n>0}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}.$$

By (14) it is evident that

$$d^{-2}\Delta_2^{+}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}=O\left(d^{-2}|s_n|^{-3}x^{\frac{5}{2}}\right). \tag{18}$$

On the other hand, the mean value theorem (13) gives us

$$d^{-2}\Delta_2^{+}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}=O\left(|s_n|^{-1}x^{\frac{1}{2}}\right). \tag{19}$$

Let $N(t)$ be the number of roots of $Z(s)$ on the critical line $\frac{1}{2}+ix$ in the interval $0<x\le t$. It is known ([5], p. 477, Th. 3.8) that $N(t)\sim\frac{r|\mathfrak{I}|}{4\pi}t^2$. Taking $M>1$ and following ([3], pp. 463-464; [6], p. 246), we use (19) resp. (18) in the sums over $s_n$, $\frac{1}{2}<|s_n|<1$, $1\le|s_n|<M$ resp. sum over $s_n$, $|s_n|>M$ (below) to get

$$\begin{aligned}
&\left|d^{-2}\Delta_2^{+}\sum_{r_n>0}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}\right| \\
&\le\sum_{\frac{1}{2}<|s_n|<1}\left|d^{-2}\Delta_2^{+}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}\right|+\sum_{1\le|s_n|<M}\left|d^{-2}\Delta_2^{+}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}\right| \\
&\quad+\sum_{|s_n|>M}\left|d^{-2}\Delta_2^{+}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}\right| \\
&\le C_1x^{\frac{1}{2}}\sum_{\frac{1}{2}<|s_n|<1}|s_n|^{-1}+C_1x^{\frac{1}{2}}\sum_{1\le|s_n|<M}|s_n|^{-1}+C_2d^{-2}x^{\frac{5}{2}}\sum_{|s_n|>M}|s_n|^{-3} \\
&=O\left(x^{\frac{1}{2}}\right)+C_1x^{\frac{1}{2}}\int_1^M t^{-1}\mathrm{d}N(t)+C_2d^{-2}x^{\frac{5}{2}}\int_M^{+\infty}t^{-3}\mathrm{d}N(t) \\
&=O\left(x^{\frac{1}{2}}\right)+O\left(Mx^{\frac{1}{2}}\right)+O\left(d^{-2}x^{\frac{5}{2}}M^{-1}\right).
\end{aligned}$$

Thus,

$$d^{-2}\Delta_2^{+}\sum_{r_n>0}\frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)}=O\left(x^{\frac{1}{2}}\right)+O\left(Mx^{\frac{1}{2}}\right)+O\left(d^{-2}x^{\frac{5}{2}}M^{-1}\right). \tag{20}$$

Similarly,

$$d^{-2}\Delta_2^{+}\sum_{r_n>0}\frac{x^{\tilde{s}_n+2}}{\tilde{s}_n\left(\tilde{s}_n+1\right)\left(\tilde{s}_n+2\right)}=O\left(x^{\frac{1}{2}}\right)+O\left(Mx^{\frac{1}{2}}\right)+O\left(d^{-2}x^{\frac{5}{2}}M^{-1}\right). \tag{21}$$

Observe that $\sum_{|\gamma|\le t}1=O\left(t^2\right)$ (see, [5], p. 437, Prop. 2.13). Thus, application of $d^{-2}\Delta_2^{+}$ to the third and the fourth sum in $\sum_{\mathrm{II}}$ gives us

$$O\left(x^{\frac{1}{2}}\right)+O\left(Mx^{\frac{1}{2}}\right)+O\left(d^{-2}x^{\frac{5}{2}}M^{-1}\right).$$

Let us write

$$\sum_{\mathrm{II}}=\sum_{1}-\sum_{2}, \tag{22}$$

where $\sum_1$ denotes the sum of the first four sums in $\sum_{\mathrm{II}}$ and $\sum_2$ denotes the sum of the last four sums in $\sum_{\mathrm{II}}$. Now, Equations (11), (16), (17), (20), (21) and (22) give us

$$\begin{aligned}d^{-2}\Delta_2^{+}\psi_2\left(x,W\right)&=x+O\left(d\right)+\sum_{n=1}^{K}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right)+O\left(Mx^{\frac{1}{2}}\right)\\&\quad+O\left(d^{-2}x^{\frac{5}{2}}M^{-1}\right)-d^{-2}\Delta_2^{+}\sum_{2}.\end{aligned} \tag{23}$$

Putting $M=x^{\frac{1}{4}}$, $d=x^{\frac{3}{4}}$, the Equation (23) becomes

$$d^{-2}\Delta_2^{+}\psi_2\left(x,W\right)=x+\sum_{n=1}^{K}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right)-d^{-2}\Delta_2^{+}\sum_{2}. \tag{24}$$

Since the left sides of Equations (20), (21) are $O\left(x^{\frac{3}{4}}\right)$ for such choice of $M$ and $d$, we get $d^{-2}\Delta_2^{+}\sum_2=O\left(x^{\frac{3}{4}}\right)$. Now, it is obvious that $d^{-2}\Delta_2^{+}\sum_1=O\left(x^{-\frac{1}{4}}\right)$. Finally, Equation (24) gives us

$$d^{-2}\Delta_2^{+}\psi_2\left(x,W\right)=x+\sum_{n=1}^{K}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right).$$

Returning to (15), we conclude that inequality

$$\psi_0\left(x,W\right)\le x+\sum_{n=1}^{K}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right)$$

holds true. Following ([15], p. 11), we analogously obtain that

$$x+\sum_{n=1}^{K}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right)\le\psi_0\left(x,W\right).$$

Hence,

$$\psi_0\left(x,W\right)=x+\sum_{n=1}^{K}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right). \tag{25}$$

Arguing as in [5] (p. 475) and [4] (p. 113), one immediately sees that equality (25) proves the theorem.

## Acknowledgements

We thank the Editor and the referee for their comments.

## References

[1] Selberg, A. (1956) Harmonic Analysis and Discontinuous Groups in Weakly Symmetric Riemannian Spaces with Applications to Dirichlet Series. *Journal of the Indian Mathematical Society*, **20**, 47-87.

[2] Huber, H. (1961) Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen II. *Mathematische Annalen*, **142**, 385-398. https://doi.org/10.1007/BF01451031

[3] Huber, H. (1961) Nachtrag zu. *Mathematische Annalen*, **143**, 463-464. https://doi.org/10.1007/BF01470758

[4] Hejhal, D. (1973) The Selberg Trace Formula for $\mathrm{PSL}(2,\mathbb{R})$, Vol. I. *Lecture Notes in Mathematics*, Volume 548. Springer-Verlag, Berlin-Heidelberg.

[5] Hejhal, D. (1983) The Selberg Trace Formula for $\mathrm{PSL}(2,\mathbb{R})$, Vol. II. *Lecture Notes in Mathematics*, Volume 1001. Springer-Verlag, Berlin-Heidelberg.

[6] Randol, B. (1977) On the Asymptotic Distribution of Closed Geodesics on Compact Riemann Surfaces. *Transactions of the American Mathematical Society*, **233**, 241-247. https://doi.org/10.1090/S0002-9947-1977-0482582-9

[7] Buser, P. (1992) Geometry and Spectra of Compact Riemann Surfaces, Progress in Mathematics, Vol. 106. Birkhäuser, Boston-Basel-Berlin.

[8] Avdispahić, M. and Smajlović, L. (2009) On the Prime Number Theorem for a Compact Riemmann Surface. *Rocky Mountain Journal of Mathematics*, **39**, 1837-1845. https://doi.org/10.1216/RMJ-2009-39-6-1837

[9] Avdispahić, M. and Smajlović, L. (2006) An explicit Formula and Its Application to the Selberg Trace Formula. *Monatshefte für Mathematik*, **147**, 183-198. https://doi.org/10.1007/s00605-005-0317-0

[10] Avdispahić, M. and Smajlović, L. (2008) Euler Constants for a Fuchsian Group of the First Kind. *Acta Arithmetica*, **131**, 125-143. https://doi.org/10.4064/aa131-2-2

[11] Avdispahić, M. and Smajlović, L. (2016) Selberg Trace Formula as an Explicit Formula and the Prime Geodesic Theorem. (Submitted)

[12] Fischer, J. (19760 An Approach to the Selberg Trace Formula via Selberg Zeta-Function. *Lecture Notes in Mathematics*, Volume 1253. Springer-Verlag, Berlin-Heidelberg.

[13] Hardy, G.H. and Riesz. M. (1915) *T*he General Theory of Dirichlet's Series. Cambridge University Press, Cambridge.

[14] Jameson, G.J. (2003) The Prime Number Theorem. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9781139164986

[15] Park, J. (2010) Ruelle Zeta Function and Prime Geodesic Theorem for Hyperbolic Manifolds with Cusps. In: van Dijk, G. and Wakayama, M., Eds., *Casimir Force*, *Casimir Operators and Riemann Hypothesis*, de Gruyter, Berlin, 89-104.