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On α-Weyl Operators

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Abstract

The purpose of this article is to present Schechter's manner to introduce α-Wayl operators and compare this definition with another one given by Yadav and Arora. Moreover, we introduce generalized Weyl operator in the way that we keep many properties of the class of Weyl operators.

Keywords

Fredholm Operators, α-Closed Subspaces, α-Weyl Operators

1. Introduction

Let H be a (complex) Hilbert space and $h = \dim(H)$ denote the Hilbert dimension of the space H, where $h > \aleph_0$. Then any nonzero proper closed two-side ideal in $B(H)$ is the form

$$\overline{\left\{K \in B(H) : \dim \overline{R(K)} < \alpha\right\}},$$

for some $0 \le \alpha < h$. Denote this ideal with $\mathcal{F}_\alpha$ (For more details see [1]). In the case when $\alpha = \aleph_0$ (cardinality of the natural numbers), we have $\mathcal{F}_{\aleph_0} = K(H)$, the ideal of all compact operators. We denote the kernel of an operator by $N(T)$, $n(T) = \dim N(T)$, and $n^*(T) = \operatorname{codim} R(T) (= \dim H/R(T))$, where $R(T)$ denotes the rank of an operator. Using duality we have $n^*(T) = n(T^*)$.

One of the classical (Atkinson) characterization of Fredholm operators is invertibility in Calkin algebra, *i.e.* $T \in B(H)$ is a Fredholm operator if and only if $\pi_0(T)$ is invertible in $B(H)/K_0(H)$ if and only if $\pi(T)$ is invertible in $B(H)/K(H)$, where π_0 and π are natural homomorphisms from $B(H)$ in the cocients $B(H)/K_0(H)$ and $B(H)/K(H)$ respectively ([2], Theorem 3.2.8). Another way to introduce Fredholm operators is using the dimensions of the kernel and the codimension of the rang of an operator: $T \in B(H)$ is called an upper semi-Fredholm operator if $n(T) < \infty$ and $R(T)$ is closed, and T is called lower semi-Fredholm if $n^*(T) < \infty$ (consequently the range of T is closed). The set of all upper (respectively, lower) semi-Fredholm operators will be denoted by $\Phi_+(H)$ (respectively $\Phi_-(H)$). The set of all semi-Fredholm operators is defined by

$\Phi_{s-F}(H) = \Phi_+(H) \cap \Phi_-(H)$ and the set of all Fredholm operators is defined by $\Phi(H) = \Phi_+(H) \cap \Phi_-(H)$. The index of a semi-Fredholm operator is defined as $i(T) = n(T) - n(T^*)$ and the Weyl operators with $\Phi_0(H) = \{T \in \Phi(H) : i(T) = 0\}$. The Weyl operator still conserves one of the basic properties for the operators between finite dimensional spaces: Fredholm alternative. Moreover, with some extra conditions (like finite ascent or descent), such operators have very nice property: there are Drazin invertible (For more details about generalized invertibility we suggest [3]). Now, the natural question appears: it is necessary to observe only finite dimensional situation for kernel, or co-dimension of range, or ascent and descent, etc.

We can find the very first investigation in this direction in the works of G. Edgar, J. Ernest and S. G. Lee. In papers [4] [5], they introduce the definition of an α-closed subspace which allowed them to give a new definition of an α-Fredholm operator. Accordingly ([4], Definition 2.7), $T \in B(H)$ is an α-Fredholm operator, $1 \le \alpha < \dim H$, if $\max\{n(T), n(T^*)\} < \alpha$ and $R(T)$ is α-closed.

S.C. Arora and P. Dharmarha, beside that observed joint weighted spectrum, introduced α-Weyl operators like α-Fredholm operator with abstract index equal to zero, or like intersection of perturbations with the operators of rank α (for more details see [6]-[11]). Beside that these two definitions of α-Weyl operators are not equivalent, the notion of abstract index is not very applicable, such that we need new ways of generalization of the index of an operator that widen Weyl operator theory in infinite dimensional case.

The main results of the paper are present in remaining two sections. In the next section we define β-index of an α-Fredholm operator, for $1 \le \beta < \alpha < h$, which we use in definition of α-Weyl operator and α-Weyl spectrum (Definition 2). In the theorems 3, 5 and 7 we give some basic properties of such operators. In the last section we define the generalized Weyl operator in the way that we widespread the definition given by D.S. Djordjević in [12]. The new class of the generalized Weyl operator conserves many properties of the class of Weyl operators (see Theorem 9).

2. α-Weyl Operators

For Hilbert spaces, L. A Coburn, in [13], defined the Weyl spectrum of an operator as

$$\omega_1(T) = \bigcap_{K \in \mathcal{K}(H)} \sigma(T + K),$$

where $\sigma(S)$ denoted usual spectrum of $S \in B(H)$.

On the other hand, in the same year, M. Schechter (see [14]) defined the Weyl spectrum of T by

$$\omega_2(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm of index } 0\}.$$

S. K. Berberian in [15] established the equivalence between both definitions and we will use $\omega(T)$ to denote the Weyl spectrum.

The notion of Fredholm operators can be extended to an arbitrary dimension (less then the dimension of the space *H*) of the null space of *T* and T^* using the α-closedness. In this way, in [4], G. Edgar, J. Ernest and S. G. Lee defined the α-Fredholm operator, for some $1 \le \alpha < \dim H$, like an operator $T \in B(H)$ that has α-closed range and both of $n(T)$ and $n(T^*)$ are less than α. It is worth mentioning that a subset *K* in *H* is α-closed if there exists a closed subset $M \subset H$ such that $M \subset K$ and $\dim(K \cap M^{\perp}) < \alpha$.

If $d(T)$ denotes the approximate nullity of *T* (for the definition and basic property see [4]), then, by ([4], Theorem 2.6) and ([16], Theorem 3.1), we have nice (Atkison type) characterization for an α-Fredholm operator.

Theorem 1. *Let T be an operator on* $B(H)$ *and* α *be a cardinal number such that* $\aleph_0 \le \alpha < h$. *Then the following conditions are equivalent*:

1) *T* is an α- Fredholm operator (in notation $T \in \Phi_\alpha(H)$).

2) $\max\{d(T), d(T^*)\} < \alpha$.

3) *T* is invertible modulo $\mathcal{F}_\alpha$.

For more properties of α-Fredholm operators we specially refer to [4] [16] [17].

Later, Yadav and Arora, in [11], for non separable Hilbert spaces, defined the Weyl spectrum of wight α for some operator $T \in B(H)$ as

$$\alpha\omega_1(T) = \bigcap_{K \in \mathcal{F}_\alpha} \sigma(T + K).$$

Let $T \in B(H)$ be an α-Fredholm operator, $1 \le \alpha < h$, then we can extend the definition of index, for all $\beta < \alpha$, using slightly modification of definition in [18] (for more details see [16]):

$$\operatorname{ind}_\beta(T) = \begin{cases} n(T) - n(T^*), & \text{if } \beta = \aleph_0 \text{ or } \beta > \aleph_0 \text{ and } \max\{n(T), n(T^*)\} \ge \beta; \\ 0, & \text{if } \beta > \aleph_0 \text{ and } \max\{n(T), n(T^*)\} < \beta. \end{cases}$$

In the way of Schechter definition of the Weyl operators, we defined α-Weyl operators next.

Definition 2. For an operator $T \in B(H)$ we say that it is α-Weyl operator, for some cardinal α, $\aleph_0 \le \alpha < h$, if T is an α-Fredholm operator with $\operatorname{ind}_\beta(T) = 0$, for all cardinals β, $\aleph_0 \le \beta < \alpha$.

We can define the α-Weyl spectrum in (one of the usual) way(s):

$$\begin{aligned} \alpha\omega_2(T) &= \{\lambda \in \mathbb{C} : \lambda - T \text{ is not an } \alpha\text{-Weyl operator}\} \\ &= \{\lambda \in \mathbb{C} : \lambda \notin \sigma_\alpha(T) \text{ or } \operatorname{ind}_\beta(T - \lambda) \ne 0, \text{ for some } \beta, \aleph_0 \le \beta < \alpha\}. \end{aligned}$$

Now arise natural question about equivalency of two ways of definition of α-Weyl spectrums. It is easy to see that

$$_{\aleph_0}\omega_1(T) = {}_{\aleph_0}\omega_2(T) = \omega(T).$$

The next theorem gives us the answer for all α, $\aleph_0 \le \alpha < h$.

Theorem 3. *Let T be an operator on $B(H)$ and let α be a cardinal number such that $\aleph_0 \le \alpha < h$. Then $\alpha\omega_1(T) = \alpha\omega_2(T)$.*

Proof. If $\alpha = \aleph_0$, then the results follows from the classical Fredholm theory.

Let $\alpha > \aleph_0$ and $\lambda \notin \alpha\omega_1(T)$. Then there exists a $K \in \mathcal{F}_\alpha$ such that $T - \lambda + K$ is invertible. Hence, $T - \lambda + K$ is an (invertible) α-Fredholm operator with $\operatorname{ind}_\beta(T - \lambda + K) = 0$, for all $\beta < \alpha$. Since

$$T - \lambda = (T - \lambda + K) - K,$$

by ([17], Theorem 2.6) and ([18], Corollary 1), $T - \lambda$ is an α-Fredholm operator and $\operatorname{ind}_\beta(T - \lambda) = 0$, for all $\beta < \alpha$, i.e. $\lambda \notin \alpha\omega_2(T)$.

Let $\lambda \notin \alpha\omega_2(T)$. Then $T - \lambda$ is an α-Fredholm operator and, for every cardinal $\aleph_0 \le \beta < \alpha$, we have $\operatorname{ind}_\beta(T - \lambda) = 0$. In this case we have that $0 \le n(T) = n(T^*) = \gamma < \alpha$. If $\gamma \le \aleph_0$, then results follows by the properties of (usual) Weyl spectrum.

Let $\aleph_0 < \gamma < \alpha$. Suppose that $T - \lambda$ has closed range. Let i be an isometry from $N(T - \lambda)$ onto $N((T - \lambda)^*)$ (that are same dimension γ) and let $K \in B(X)$ is definite by:

$$K = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(T - \lambda) \\ R(T - \lambda)^* \end{pmatrix} \to \begin{pmatrix} N(T - \lambda)^* \\ R(T - \lambda) \end{pmatrix}.$$

In the same decomposition of H we can present $T - \lambda$ in the way:

$$T - \lambda = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix} : \begin{pmatrix} N(T - \lambda) \\ R(T - \lambda)^* \end{pmatrix} \to \begin{pmatrix} N(T - \lambda)^* \\ R(T - \lambda) \end{pmatrix},$$

where $T_1 \in B(R(T - \lambda)^*, R(T - \lambda))$ is invertible. It is easy to see that the operator $T - \lambda + K$ is invertible and $K \in \mathcal{F}_\alpha$.

In the case when $T - \lambda$ is α-Fredholm operator with no closed range, by ([4], Theorem 2.6), for each $\epsilon > 0$ small enough, there exists a closed subspace W_ϵ of H that contains $N(T - \lambda)$ and $\|(T - \lambda)y\| < \epsilon\|y\|$ for any non-zero vector in W_ϵ. Moreover, $\|(T - \lambda)y\| \ge \epsilon\|y\|$, for every $y \in W_\epsilon^\perp$, $\dim W_\epsilon = d(T - \lambda)$ and

$$\dim\left(\left(T - \lambda\right)W_\epsilon^\perp\right)^\perp = d\left(\left(T - \lambda\right)^*\right).$$

Additionally, by ([16], Lemma 4.8), $d(T - \lambda) = d((T - \lambda)^*)\left(= n(T - \lambda) = n((T - \lambda)^*)\right)$. Since $T - \lambda$ is bounded below in $W_\epsilon^\perp$, follows that $(T - \lambda)(W_\epsilon^\perp)$ is a closed subspace that together with $N(T - \lambda) \subset W_\epsilon$ implies that $T_3 = (T - \lambda)_{|W_\epsilon^\perp} : W_\varepsilon^\perp \to (T - \lambda)(W_\epsilon^\perp)$ is an invertible operator. Set

$$T_\epsilon = \begin{pmatrix} Iso & 0 \\ 0 & T_3 \end{pmatrix} : \begin{pmatrix} W_\epsilon \\ W_\epsilon^\perp \end{pmatrix} \to \begin{pmatrix} \left((T-\lambda)W_\epsilon^\perp\right)^\perp \\ (T-\lambda)W_\epsilon^\perp \end{pmatrix},$$

where $Iso \in B\left(W_\epsilon, \left((T-\lambda)W_\epsilon^\perp\right)^\perp\right)$ is an arbitrary isomorphism between γ-dimensional (closed sub)spaces. Then T_ϵ is an invertible operator and $K_\varepsilon = T_\varepsilon - (T-\lambda) \in \mathcal{F}_\alpha$. To proof last, let

$$T-\lambda = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix} : \begin{pmatrix} W_\varepsilon \\ W_\varepsilon^\perp \end{pmatrix} \to \begin{pmatrix} \left((T-\lambda)W_\epsilon^\perp\right)^\perp \\ (T-\lambda)W_\epsilon^\perp \end{pmatrix},$$

then

$$K_\epsilon = \begin{pmatrix} Iso - T_1 & 0 \\ T_2 & 0 \end{pmatrix} : \begin{pmatrix} W_\epsilon \\ W_\epsilon^\perp \end{pmatrix} \to \begin{pmatrix} \left((T-\lambda)W_\epsilon^\perp\right)^\perp \\ (T-\lambda)W_\epsilon^\perp \end{pmatrix}$$

and $R(K_\epsilon) \subset \left((T-\lambda)W_\epsilon^\perp\right)^\perp$ that implies $\dim R(K_\epsilon) \le \dim\left((T-\lambda)W_\epsilon^\perp\right)^\perp = \gamma < \alpha$. Hence, $(T-\lambda)+K_\epsilon = T_\epsilon$ is an invertible operator, *i.e.* $\lambda \notin \alpha\omega_1(T)$. □

Remark 4. (1) In the future, we will use the notation $\alpha\omega(T)$ for the α-Weyl spectrum of *T*.

(2) In the case when $\aleph_0 < \alpha$, if we slightly modificated the proof of Theorem 3 with additional condition that $\|Iso\| \le \epsilon$, we can see that an α-Weyl operator that is not invertible can be approximated (in the norm) with an invertible operator, *i.e.* its belong to $\partial B^{-1}(H)$.

Let $\aleph_0 \le \alpha < h$, then we can define family of α-Weyl operators, in notation $\Phi_\alpha^0(H)$, like:

$$\Phi_\alpha^0(H) = \left\{T \in B(H) : T \in \Phi_\alpha(H) \text{ and } \operatorname{ind}_\beta(T) = 0, \text{ for all } \aleph_0 \le \beta < \alpha\right\}.$$

Theorem 5. *Let T be an operator on* $B(H)$ *and let α be a cardinal number such that* $\aleph_0 \le \alpha < h$. *Then the following conditions are equivalent*:

1) $T \in \Phi_\alpha^0(H)$;

2) $n(T) = n(T^*) < \alpha$ and $R(T)$ is α-closed;

3) $d(T) = d(T^*) < \alpha$.

Proof. (1) ⇔ (2) follows directly from definition of α-Fredholm operator and definition of $\operatorname{ind}_\beta(T)$.

(2) ⇔ (3) follows from ([4], Theorem 2.6) and ([16], Theorem 3.1 and Lemma 4.8). □

Remark 6. Any of equivalent condition (1)-(3) of Theorem 5 implies that there exists a closed subspaces *M* and *N* of *H* such that $M \cap N(T) = \{0\}$, $N \subset R(T)$ and $\dim M^\perp = \dim N^\perp = \gamma < \alpha$. Really, let

$$d(T) = d(T^*)(= \gamma < \alpha).$$

Then, by ([16], p. 221), there exists a $\epsilon_0 > 0$ such that $d_\epsilon(T) = d(T) = d(T^*) = d_\varepsilon(T^*)$, for any $\epsilon \in (0, \epsilon_0]$. For some fix $\epsilon \in (0, \epsilon_0]$, let $M = H_\epsilon$ from definition of d_ϵ for *T* and *N* we define in similar way, only using operator T^*.

Theorem 7. *Let T be an operator on* $B(H)$ *and let α be a cardinal number such that* $\aleph_0 \le \alpha < h$.

1) $\Phi_\alpha^0(H)$ is open.

2) If $T, S \in \Phi_\alpha^0(H)$, then $TS \in \Phi_\alpha^0(H)$.

3) If $T \in \Phi_\alpha^0(H)$ and $K \in \mathcal{F}_\alpha$, then $T + K \in \Phi_\alpha^0(H)$.

Proof. (1) Let $T \in \Phi_\alpha^0(H)$. By Theorem 5 (3), $d(T) = d(T^*) < \alpha$ and, for small enough $\epsilon > 0$, there exists a closed subspace H_ϵ such that

$$\|Tx\| < \epsilon\|x\|, \text{ for all } x \in H_\epsilon \setminus \{0\}, N(T) \subset H_\epsilon, \|Tx\| \ge \epsilon\|x\|,$$

for all $x \in H_\epsilon^\perp$ and

$$N(T^*) \subset \left(T(H_\epsilon^\perp)\right)^\perp, \|T^*x\| < \epsilon\|x\|, \text{ for all } x \in \left(T(H_\epsilon^\perp)\right)^\perp \setminus \{0\},$$

$$\text{and } \|T^*x\| \ge \epsilon\|x\|, \text{ for all } x \in T(H_\epsilon^\perp).$$

Moreover, $\dim H_\epsilon = d(T) = d(T^*) = \dim\left(T(H_\epsilon^\perp)\right)^\perp < \alpha$, where $T(H_\epsilon^\perp)$ is a closed subspace of *H*. For more details see ([16], pp. 220-221).

Let $S \in B(H)$ such that $\|T - S\| < \epsilon/2$. Suppose that there is a non-zero $x \in N(S) \cap H_\epsilon^\perp$. Then $\|Tx\| \le \|(T - S)x\| \le \frac{\epsilon}{2}\|x\|$, which is contradictory with selection of subspace H_ϵ. Hence, $N(S) \subset H_\epsilon$ and $n(S) < \alpha$. Also, by $N(S^*) \subset \left(T\left(H_\epsilon^\perp\right)\right)^\perp$, we have $n(S^*) < \dim\left(T\left(H_\epsilon^\perp\right)\right)^\perp < \alpha$.

Moreover, S is bounded below on $H_\epsilon^\perp$,

$$\|Sx\| \ge \left\|\|Tx\| - \|(T - S)x\|\right\| \ge \frac{\epsilon}{2}\|x\|,$$

that implies $S\left(H_\epsilon^\perp\right)$ is a closed subspace contained in $R(S)$. Using the matrix representation of S in respect of decomposition $H = H_\epsilon \oplus H_\epsilon^\perp$ and $H = \left(S\left(H_\epsilon^\perp\right)\right)^\perp \oplus S\left(H_\epsilon^\perp\right)$, we get

$$\left(S\left(H_\epsilon^\perp\right)\right)^\perp \cap R(S) \subset S(H_\epsilon) \text{ and } \dim\left(\left(S\left(H_\epsilon^\perp\right)\right)^\perp \cap R(S)\right) < \alpha$$

which implies $R(S)$ is an α-closed subspace.

Hence, for any $S \in B(H)$ such that $\|T - S\| < \epsilon/2$, $R(S)$ is an α-closed subspace and $n(T) = n(T^*) < \alpha$, *i.e.* $S \in \Phi_\alpha^0(H)$.

(2) Let $T, S \in \Phi_\alpha^0(H)$, then $0 \notin \alpha\omega(T) \cup \alpha\omega(S)$. By Theorem 3, there exists a $K_1, K_2 \in \mathcal{F}_\alpha$ such that $T + K_1, S + K_2 \in B(H)^{-1}$. Then,

$$B(H)^{-1} \ni (T + K_1)(S + K_2) = TS + (TK_2 + K_1S + K_1K_2)$$
$$\text{and } TK_2 + K_1S + K_1K_2 \in \mathcal{F}_\alpha,$$

i.e. $TS \in \Phi_\alpha^0(H)$.

(3) By Theorem 3 and ([18], Corollary 1) (also see ([11], Theorem 3)).

3. Generalized Weyl Operators

Let ϑ and α be cardinality such that $\aleph_0 \le \vartheta < \alpha < h$. By ([17], Renmark 3.2), we have that $\Phi_\vartheta(H) \subset \Phi_\alpha(H)$. Moreover, it is easy to see that, if for some $T \in \Phi_\vartheta(H)$, $\operatorname{ind}_\beta(T) = 0$, for all $\aleph_0 \le \beta < \vartheta$, then $\operatorname{ind}_\beta(T) = 0$, for all $\vartheta \le \beta < \alpha$, *i.e.* $\Phi_\vartheta^0(H) \subset \Phi_\alpha^0(H)$.

Definition 8. The set of *the generalized Weyl operator*, in notation $\Phi_g^0(H)$, is

$$\Phi_g^0(H) = \bigcup_{\aleph_0 \le \alpha < h} \Phi_\alpha^0(H).$$

In [12], D. S. Djordjević defines the class of generalized Weyl operator like

$$\Phi_0^g(H) = \left\{T \in B(H) : R(T) \text{ is closed and } n(T) = n(T^*)\right\}.$$

Since a closed subspace in H is α-closed, for any cardinal α, $1 \le \alpha < h$, and $n(T) = n(T^*)$ implies that $\operatorname{ind}_\beta(T) = 0$, we have that $\Phi_0^g(H) \subset \Phi_g^0(H)$. Besides that, the properties of $\Phi_g^0(H)$ are more similar to $\Phi_0(H)$ than those of $\Phi_0^g(H)$. The set $\Phi_0^g(H)$ is not open, but, by Theorem 7, the set $\Phi_g^0(H)$ is open (like $\Phi_0(H)$). Similar, the set $\Phi_0^g(H)$ is closed in respect of composition of operators (by Theorem 7 (2)) and compact perturbations. Hence, we have next theorem that generalizes results from [12].

Theorem 9. (1) *If* $T, S \in \Phi_g^0(H)$, *then* $TS \in \Phi_g^0(H)$.

(2) If $T \in \Phi_g^0(H)$ and $K \in K(H)$, then $T + K \in \Phi_g^0(H)$.

(3) $\Phi_g^0(H)$ is open in $B(H)$.

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