Advances in Pure Mathematics, 2016, 6, 21-40
Published Online January 2016 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2016.61004

Scientific Research Publishing

The Space of Bounded $p(\cdot)$-Variation in the Sense Wiener-Korenblum with Variable Exponent

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Received 28 October 2015; accepted 16 January 2016; published 19 January 2016

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Abstract

In this paper we present the notion of the space of bounded $p(\cdot)$-variation in the sense of Wiener-Korenblum with variable exponent. We prove some properties of this space and we show that the composition operator *H*, associated with $h:\mathbb{R}\to\mathbb{R}$, maps the $\kappa BV^{W}_{p(\cdot)}([a,b])$ into itself, if and only if *h* is locally Lipschitz. Also, we prove that if the composition operator generated by $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ maps this space into itself and is uniformly bounded, then the regularization of *h* is affine in the second variable, *i.e.* satisfies the Matkowski's weak condition.

Keywords

Generalized Variation, $p(\cdot)$-Variation in the Sense of Wiener-Korenblum, Exponent Variable, Composition Operator, Matkowski's Condition

1. Introduction

A number of generalizations and extensions of variation of a function have been given in many directions since Camile Jordan in 1881 gave a first notion of bounded variation in the paper [1] devoted to the convergence of Fourier series. Consequently, the study of notions of generalized bounded variation forms an important direction in the field of mathematical analysis. Two well-known generalizations are the functions of bounded *p*-variation and the functions of bounded φ-variation, due to N. Wiener [2] and L. C. Young [3] respectively. In 1924 N. Wiener [2] generalized the Jordan notion and introduced the notion of *p*-variation (variation in the sense of Wiener). Later, in 1937, L. Young [3] introduced the notion of φ-variation of a function. The *p*-variation of a function *f* is the supremum of the sums of the *p*th powers of absolute increments of *f* over no overlapping inter-

How to cite this paper: Mejía, O., Merentes, N., Sánchez, J.L. and Valera-López, M. (2016) The Space of Bounded $p(\cdot)$-Variation in the Sense Wiener-Korenblum with Variable Exponent. *Advances in Pure Mathematics*, **6**, 21-40.
http://dx.doi.org/10.4236/apm.2016.61004

vals. Wiener mainly focused on the case $p = 2$, the 2-variation. For *p*-variations with $p \neq 2$, the first major work was done by Young [3], partly with Love [4]. After a long hiatus following Young's work, p^{th}-variations were reconsidered in a probabilistic context by R. Dudley [5] [6], in 1994 and 1997, respectively. Many basic properties of the variation in the sense of Wiener and a number of important applications of the concept can be found in [7] [8]. Also, the paper by V. V. Chistyakov and O. E. Galkin [9], in 1998, is very important in the context of *p*-variation. They study properties of maps of bounded *p*-variation $(p > 1)$ in the sense of Wiener, which are defined on a subset of the real line and take values in metric or normed spaces.

In 1997 while studying Poisson integral representations of certain class of harmonic functions in the unit disc of the complex plan B. Korenblum [10] introduced the notion of bounded κ-variation and proved that a function *f* is of bounded κ-variation if ot can be written as the difference of two κ-decreasing functions. This concept differs from others due to the fact that it introduces a distortion function κ that measures intervals in the domain of the function and not in the range. In 1986, S. Ki Kim and J. Kim [11], gave the notion of the space of functions of $\kappa\phi$-bounded variation on $[a,b]$, which is a combination of the notion of bounded ϕ-variation in the sense of Schramm and bounded κ-variation in the sense of Korenblum, and J. Park *et al.* [12] [13] proved some properties in this space. Considering $\phi_n(x) = x^p$ for $1 < p < \infty$ and $n \geq 1$, then it follows that this space generalized the space of functions of κp-bounded variation in the sense of Wiener-Korenblum. In 1990 S. Ki Kim and J. Yoon [14] showed the existence of the Riemann-Stieltjes integral of functions of bounded κ-variation and in 2011 W. Aziz, J. Guerrero, J. L. Sánchez and M. Sanoja, in [15], showed that the space of bounded κ-variation satisfies the Matkowski's weak condition. Also, in 2012, M. Castillo, M. Sanoja and I. Zea [16] presented the space of functions of bounded κ-variation in the sense of Riez-Korenblum, denoted by $\kappa BV_p([a,b])$, which is a combination of the notions of bounded *p*-variation in the sense of Riesz $(1 < p < \infty)$ and bounded κ-variation in the sense of Korenblum.

Recently, there has been an increasing interest in the study of various mathematical problems with variable exponents. With the emergency of nonlinear problems in applied sciences, standard Lebesgue and Sobolev spaces demonstrated their limitations in applications. The class of nonlinear problems with exponent growth is a new research field and it reflects a new kind of physical phenomena. In 2000 the field began to expand even further. Motivated by problems in the study of electrorheological fluids, L. Diening [17] raised the question of when the Hardy-Littlewood maximal operator and other classical operators in harmonic analysis are bounded on the variable Lebesgue spaces. These and related problems are the subject of active research nowadays. These problems are interesting in applications (see [18]-[21]) and give rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent, the origins of which can be traced back to the work of W. Orlicz in the 1930's [22]. In the 1950's, this study was carried on by H. Nakano [23] [24] who made the first systematic study of spaces with variable exponent. Later, Polish and Czechoslovak mathematicians investigated the modular function spaces (see for example J. Musielak [25] [26], O. Kovacik and J. Rakosnik [27]). We refer to books [21] for the detailed information on the theoretical approach to the Lebesgue and Sobolev spaces with variable exponents. In 2015, R. Castillo, N. Merentes and H. Rafeiro [28] studied a new space of functions of generalized bounded variation. There the authors introduced the notion of bounded variation in the Wiener sense with the exponent $p(\cdot)$-variable. In the same year, O. Mejia, N. Merentes and J. L. Sánchez in [29], proved some properties in this space, for the composition operator and showed a structural theorem for mappings of bounded variation in the sense of Wiener with the exponent $p(\cdot)$-variable.

The main purpose of this paper is threefold: First, we provide extension of the space of generalized bounded variation present in [28] and [29] in the sense Wiener-Korenblum and we give a detailed description of the new class formed by the functions of bounded variation in the sense of Wiener-Korenblum with the exponent $p(\cdot)$-variable. Second, we prove a necessary and sufficient condition for the acting of composition operator (Nemystskij) on the space $\kappa BV_{p(\cdot)}^{W}[a,b]$ and, third we show that any uniformly bounded composition operator that maps the space $\kappa BV_{p(\cdot)}^{W}[a,b]$ into itself necessarily satisfies the so called Matkowski's weak condition.

2. Preliminaries

We use throughout this paper the following notation: we will denote by

$$\kappa\omega_{p(x_{ts})}\left(f,[a,b]\right)=\sup\left\{\frac{\left|f(t)-f(s)\right|^{p(x_{ts})}}{k\left(\frac{t_i-t_{i-1}}{b-a}\right)}:t,s\in[a,b]\right\},$$

the diameter of the image $f([a,b])$ (or the oscillation of f on $[a,b]$) and by x_{ts} a number between $[t,s]$.

The class of bounded variation functions exhibit many interesting properties that it makes them a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics (see [8] and [30]). Since C. Jordan in 1881 (see [1]) gave the complete characterization of functions of bounded variation as a difference of two increasing functions, the notion of bounded variation functions has been generalized in different ways.

Definition 2.1. Let $f:[a,b]\to\mathbb{R}$ be a function. For each partition $\pi:a=t_0<t_1<\cdots<t_n=b$ of $[a,b]$, we define

$$V\left(f;[a,b]\right):=\sup_{\pi}\sum_{i=1}^{n}\left|f(t_i)-f(t_{i-1})\right|, \tag{1}$$

where the supremum is taken over all partitions π of the interval $[a,b]$. If $V\left(f;[a,b]\right)<\infty$, we say that f has bounded variation. The collection of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

A generalization of this notion was presented by *N*. Wiener (see [2]) who introduced the notion of *p*-variation as follows.

Definition 2.2. Given a real number $p\geq 1$, a partition $\pi:a=t_0<t_1<\cdots<t_n=b$ of $[a,b]$, and a function $f:[a,b]\to\mathbb{R}$. The nonnegative real number

$$V_p^W(f)=V_p^W\left(f;[a,b]\right):=\sup_{\pi}\sum_{j=1}^{n}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^p \tag{2}$$

is called the Wiener variation (or *p*-variation in Wiener's sense) of f on $[a,b]$ where the supremum is taken over all partitions π.

In case that $V_p^W\left(f;[a,b]\right)<\infty$, we say that f has bounded Wiener variation (or bounded *p*-variation in Wiener's sense) on $[a,b]$. The symbol $WBV_p([a,b])$ will denote the space of functions of bounded *p*-variation in Wiener's sense on $[a,b]$.

Other generalized version was given by B. Korenblum in 1975 [10]. He considered a new kind of variation, called κ-variation, and introduced a function κ for distorting the expression $\left|t_j-t_{j-1}\right|$ in the partition if self, rather than the expression $\left|f(t_j)-f(t_{j-1})\right|$ in the range. On advantage of this alternative approach is that a function of bounded κ-variation may be decomposed into the difference of two simpler functions called κ-decreasing functions.

Definition 2.3. A function $\kappa:[0,1]\to[0,1]$ is called a distortion function (κ-function) if κ satisfies the following properties:

1) κ is continuous with $\kappa(0)=0$ and $\kappa(1)=1$;
2) κ is concave and increasing;
3) $\lim_{y\to 0^+}\frac{\kappa(t)}{t}=\infty$.

B. Korenblum (see [10]), introduced the definition of bounded κ-variation as follows.

Definition 2.4. Let κ be a distortion function, f a real function $f:[a,b]\to\mathbb{R}$, and $\pi:a=t_0<t_1<\cdots<t_n=b$ a partition of the interval $[a,b]$. Let one consider

$$\kappa(f,\pi):=\frac{\sum_{i=1}^{n}\left|f(t_i)-f(t_{i-1})\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)}, \tag{3}$$
$$\kappa V(f)=\kappa V\left(f;[a,b]\right):=\sup_{\pi}\kappa(f,\pi),$$

where the supremum is taken over all partitions π of the interval $[a,b]$. In the case $\kappa V(f;[a,b])<\infty$ one says that f has bounded κ-variation on $[a,b]$ and one will denote by $\kappa BV[a,b]$ the space of functions of bounded κ-variation on $[a,b]$.

Some properties of κ-function cab be found in [12] [14] [16].

In 2013 R. Castillo, N. Merentes and H. Rafeiro [28] introduce the notation of bounded variation space in the Wiener sense with variable exponent on $[a,b]$ and study some of its basic properties.

Definition 2.5. Given a function $p:[a,b]\to(1,\infty)$, a partition $\pi: a=t_0<t_1<\cdots<t_n=b$ of the interval $[a,b]$, and a function $f:[a,b]\to\mathbb{R}$. The nonnegative real number

$$V_{p(\cdot)}^{W}(f)=V_{p(\cdot)}^{W}(f,[a,b]):=\sup_{\pi^*}\sum_{j=1}^{n}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})} \tag{4}$$

is called Wiener variation with variable exponent (or $p(\cdot)$-variation in Wiener's sense) of f on $[a,b]$ where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-1}$ subject to the conditions that for each i, $t_i\le x_i\le t_{i+1}$.

In case that $V_{p(\cdot)}^{W}(f;[a,b])<\infty$, we say that f has bounded Wiener variation with variable exponent (or bounded $p(\cdot)$-variation in Wiener's sense) on $[a,b]$. The symbol $WBV_{p(\cdot)}([a,b])$ will denote the space of functions of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent on $[a,b]$.

Remark 2.6. Given a function $p:[a,b]\to(1,\infty)$

1) If $p(x)=1$ for all x in $[a,b]$, then $WBV_{p(\cdot)}([a,b])=BV([a,b])$.

2) If $p(x)=p$ for all x in $[a,b]$ and $1<p<\infty$, then $WBV_{p(\cdot)}([a,b])=WBV_p([a,b])$.

In [29], O. Mejia, N. Merentes and J. L. Sánchez proved some properties in this space, for the composition operator and show a structural theorem for mappings of bounded variation in the sense of Wiener with the exponent $p(\cdot)$-variable.

Now, we generalized the notion of bounded variation space in the sense of Wiener-Korenblum with variable exponent on $[a,b]$. For this, we defined bellow the bounded $p(\cdot)$-variation in the sense of Wiener-Korenblum with exponent variable.

Definition 2.7. Given a function $p:[a,b]\to(1,\infty)$, a partition $\pi: a=t_0<t_1<\cdots<t_n=b$ of the interval $[a,b]$, κ be a distortion function and a function $f:[a,b]\to\mathbb{R}$. The nonnegative real number

$$\kappa V_{p(\cdot)}^{W}(f)=\kappa V_{p(\cdot)}^{W}(f,[a,b]):=\sup_{\pi^*}\frac{\sum_{j=1}^{n}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}. \tag{5}$$

is called Wiener-Korenblum variation with variable exponent (or $p(\cdot)$-variation in the sense of Wiener-Korenblum) of f on $[a,b]$ where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-1}$ subject to the conditions that for each i, $t_i\le x_i\le t_{i+1}$.

In case that $\kappa V_{p(\cdot)}^{W}(f;[a,b])<\infty$, we say that f has bounded Wiener-Korenblum variation with variable exponent (or bounded $p(\cdot)$-variation in the sense of Wiener-Korenblum) on $[a,b]$. The symbol $\kappa BV_{p(\cdot)}^{W}([a,b])$ will denote the space of functions of bounded $p(\cdot)$-variation in the sense Wiener-Korenblum with variable exponent on $[a,b]$.

Remark 2.8. Given a function $p:[a,b]\to(1,\infty)$

1) If $p(x)=1$ for all x in $[a,b]$, then $\kappa BV_{p(\cdot)}^{W}([a,b])=\kappa BV([a,b])$.

2) If $p(x)=p$ for all x in $[a,b]$ and $1<p<\infty$, then $\kappa BV_{p(\cdot)}^{W}([a,b])=\kappa BV_p([a,b])$.

Example 2.9. Let $u:[0,1]\to\mathbb{R}$ be a function such that $u\in C^1([0,1])$ and $|u'(x)|<L$ for $x\in[0,1]$. Then, from mean value theorem, we have

$$\frac{\sum_{i=1}^{n}\left|u(t_i)-u(t_{i-1})\right|^{p(x_{i-1})}}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)} \le \sum_{i=1}^{n}\left|u(t_i)-u(t_{i-1})\right|^{p(x_{i-1})} = \sum_{i=1}^{n}\left|u'(t_i)\right|^{p(x_{i-1})}\left|t_i-t_{i-1}\right|^{p(x_{i-1})}$$

$$\le L\sum_{i=1}^{n}\left|t_i-t_{i-1}\right|^{p(x_{i-1})} = L.$$

Therefore, $u \in \kappa BV_{p(\cdot)}^{W}([0,1])$.

3. Properties of the Space

Theorem 3.1. *Let* $p:[a,b]\to(1,\infty)$ *and* κ *be a distortion function then* $WBV_{p(\cdot)}([a,b]) \subset \kappa BV_{p(\cdot)}^{W}([a,b])$.

Proof. Let $p:[a,b]\to(1,\infty)$, $f\in WBV_{p(\cdot)}([a,b])$ and $\pi^*: a=t_0<t_1<\cdots<t_n=b$ be a partition of the interval $[a,b]$. Then, by the κ subadditivity, we have:

$$\sum_{j=1}^{n}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})} \le V_{p(\cdot)}^{W}(f) = V_{p(\cdot)}^{W}(f)\kappa\left(\sum_{i=1}^{n}\frac{t_j-t_{j-1}}{b-a}\right) \le V_{p(\cdot)}^{W}(f)\sum_{i=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right).$$

Thus,

$$\frac{\sum_{j=1}^{n}\left|f(t_j)-f(t_{j-1})\right|^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)} \le V_{p(\cdot)}^{W}(f). \tag{6}$$

Then considering the supremum of the left side we get

$$\kappa V_{p(\cdot)}^{W}(f) \le V_{p(\cdot)}^{W}(f), \tag{7}$$

therefore, $f\in WBV_{p(\cdot)}([a,b])$ and $WBV_{p(\cdot)}([a,b]) \subset \kappa BV_{p(\cdot)}^{W}([a,b])$. □

Remark 3.2. From this result we deduce that every function of bounded $p(\cdot)$-variation in of Wiener's sense with variable exponent on the interval $[a,b]$ is a bounded $p(\cdot)$-variation in the Wiener-Korenblum sense on the interval $[a,b]$.

Now we will see that the class of function of bounded $p(\cdot)$-variation in the sense of Wiener-Korenblum has a structure of vector space.

Theorem 3.3. *Let* $p:[a,b]\to(1,\infty)$, *then the set* $\kappa BV_{p(\cdot)}^{W}$ *is a vector space.*

Proof. Let $f,g\in \kappa BV_{p(\cdot)}^{W}([a,b])$, then for each partition $\pi: a=t_0<t_1<\cdots<t_n=b$ of $[a,b]$, and π^* is a tagged partition of the interval $[a,b]$, we obtain:

$$\frac{\left[\left|(f+g)(t_j)-(f+g)(t_{j-1})\right|\right]^{p(x_{j-1})}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$

$$=\frac{\left[\left|f(t_j)-f(t_{j-1})\right|+\left|g(t_j)-g(t_{j-1})\right|\right]^{p(x_{j-1})}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$

$$\le 2^{p(x_{j-1})-1}\left[\frac{\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})}+\left[\left|g(t_j)-g(t_{j-1})\right|\right]^{p(x_{j-1})}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right].$$

Now adding from $j=1$ to $j=n$ we get

$$\frac{\sum_{j=1}^{n}\left[\left|(f+g)(t_j)-(f+g)(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$

$$\leq\left[\frac{\sum_{j=1}^{n}2^{p(x_{j-1})-1}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right]+\left[\frac{\sum_{j=1}^{n}2^{p(x_{j-1})-1}\left[\left|g(t_j)-g(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right].$$

Since $p(\cdot)$ is bounded, then there is a $M>0$ such that $2^{p(x_{j-1})-1}\leq M$ for all t_j, and we obtain

$$\frac{\sum_{j=1}^{n}\left[\left|(f+g)(t_j)-(f+g)(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$

$$\leq M\left[\frac{\sum_{j=1}^{n}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right]+M\left[\frac{\sum_{j=1}^{n}\left[\left|g(t_j)-g(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right].$$

In other word, if $f,g\in\kappa BV_{p(\cdot)}^{W}$, then the function $f+g$ is of bounded $p(\cdot)$-variation in the sense of Wiener-Korenblum with variable exponent on $[a,b]$ and

$$\kappa V_{p(\cdot)}^{W}\left(f+g,[a,b]\right)\leq M\left(\kappa V_{p(\cdot)}^{W}\left(f,[a,b]\right)+\kappa V_{p(\cdot)}^{W}\left(g,[a,b]\right)\right).$$

On the other hand, since $p(\cdot)$ is bounded, there exists $L>0$ such that

$$\frac{\sum_{j=1}^{n}\left[\left|(\alpha f)(t_j)-(\alpha f)(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$

$$=\left[\frac{\sum_{j=1}^{n}|\alpha|^{p(x_{j-1})}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right]$$

$$\leq L\left[\frac{\sum_{j=1}^{n}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right].$$

therefore, $\alpha f\in\kappa BV_{p(\cdot)}^{W}([a,b])$. So, $\kappa BV_{p(\cdot)}^{W}([a,b])$ is a vector space. □

Proposition 3.4. *Given a function* $p:[a,b]\to(1,\infty)$, *the variation* $\kappa V_{p(\cdot)}^{W}(f;[a,b])$ *is convex.*

Proof. Let $f,g\in\kappa BV_{p(\cdot)}^{W}([a,b])$ and $\alpha\in\mathbb{R}$. By Theorem 3.3 $\alpha f\in kBV_{p(\cdot)}^{W}([a,b])$. Since for $s>1$ the function $t\geq 0, t\to t^{s}$ is convex, then we get

$$\frac{\left[\left|\left((1-\alpha)f+\alpha g\right)(t_j)-\left((1-\alpha)f+\alpha g\right)(t_{j-1})\right|\right]^{p(t_j)}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$
$$=\frac{\left[\left|(1-\alpha)f(t_j)+\alpha g(t_j)-(1-\alpha)f(t_{j-1})+\alpha g(t_{j-1})\right|\right]^{p(t_j)}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$
$$\leq\frac{\left[(1-\alpha)\left|f(t_j)-f(t_{j-1})\right|+\alpha\left|g(t_j)+g(t_{j-1})\right|\right]^{p(t_j)}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}$$
$$\leq(1-\alpha)\frac{\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(t_j)}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}+\alpha\frac{\left[\left|g(t_j)+g(t_{j-1})\right|\right]^{p(t_j)}}{\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}.$$

Then,

$$\kappa V_{p(\cdot)}^{W}\left((1-\alpha)f+\alpha g;[a,b]\right)$$
$$\leq(1-\alpha)\kappa V_{p(\cdot)}^{W}\left(f;[a,b]\right)+\alpha\kappa V_{p(\cdot)}^{W}\left(g;[a,b]\right).$$

□

Definition 3.5. (*Norm in* $\kappa BV_{p(\cdot)}^{W}([a,b])$)

Let $f:[a,b]\to\mathbb{R}$ be a function that belongs to $\kappa BV_{p(\cdot)}^{W}([a,b])$. Then

$$\|f\|_{\kappa p(\cdot)}^{W}:=\left|f(a)\right|+\mu_{p(\cdot)}^{\kappa}(f),\ f\in\kappa BV_{p(\cdot)}^{W}([a,b]), \tag{8}$$

where $\mu_{p(\cdot)}^{\kappa}(f):=\inf_{\lambda>0}\left\{\lambda>0:\kappa V_{p(\cdot)}^{W}\left(\frac{f}{\lambda}\right)\leq 1\right\}$.

Theorem 3.6. $\left(\kappa BV_{p(\cdot)}^{W}([a,b]),\|\cdot\|_{\kappa p(\cdot)}^{W}\right)$ *is a normed space.*

Proof. Let $f,g\in\kappa BV_{p(\cdot)}^{W}([a,b])$, $\alpha\in\mathbb{R}$. Then, we have that:

a) $\|\cdot\|_{\kappa p(\cdot)}^{W}\geq 0$ since $\left|f(a)\right|\geq 0$ and $\mu_{p(\cdot)}(f)\geq 0$.

b)

$$\|\alpha f\|_{\kappa p(\cdot)}^{W}=\left|\alpha f(a)\right|+\inf_{\lambda>0}\left\{\lambda>0:\kappa V_{p(\cdot)}^{W}\left(\frac{|\alpha|f}{\lambda}\right)\leq 1\right\}$$
$$=|\alpha|\left|f(a)\right|+|\alpha|\inf_{\lambda>0}\left\{\frac{\lambda}{|\alpha|}>0:\kappa V_{p(\cdot)}^{W}\left(\frac{f}{\frac{\lambda}{|\alpha|}}\right)\leq 1\right\}$$
$$=|\alpha|\left|f(a)\right|+|\alpha|\inf_{\mu>0}\left\{\mu>0:\kappa V_{p(\cdot)}^{W}\left(\frac{f}{\mu}\right)\leq 1\right\}$$
$$=|\alpha|\left[\left|f(a)\right|+\mu_{p(\cdot)}(f)\right]=|\alpha|\|f\|_{\kappa p(\cdot)}^{W}.$$

Therefore, $\|\alpha f\|_{\kappa p(\cdot)}^{W}=|\alpha|\|f\|_{\kappa p(\cdot)}^{W}$.

c) Fix $\lambda_f > \|f\|_{\kappa p(\cdot)}^W$ and $\lambda_g > \|g\|_{\kappa p(\cdot)}^W$; then $\kappa V_{p(\cdot)}^W\left(\frac{f}{\lambda_f}\right) \le 1$ and $\kappa V_{p(\cdot)}^W\left(\frac{g}{\lambda_g}\right) \le 1$. Now let $\lambda = \lambda_f + \lambda_g$. Then by convexity of $\kappa V_{p(\cdot)}^W\left(f;[a,b]\right)$

$$\begin{aligned}\kappa V_{p(\cdot)}^W\left(\frac{f+g}{\lambda};[a,b]\right) &= \kappa V_{p(\cdot)}^W\left(\frac{\lambda_f}{\lambda}\frac{f}{\lambda_f} + \frac{\lambda_g}{\lambda}\frac{g}{\lambda_g};[a,b]\right)\\ &\le \frac{\lambda_f}{\lambda}\kappa V_{p(\cdot)}^W\left(\frac{f}{\lambda_f};[a,b]\right) + \frac{\lambda_g}{\lambda}\kappa V_{p(\cdot)}^W\left(\frac{g}{\lambda_g};[a,b]\right) \le 1.\end{aligned}$$

Hence

$$\begin{aligned}\|f+g\|_{\kappa p(\cdot)}^W &= \left|(f+g)(a)\right| + \mu_{p(\cdot)}(f+g)\\ &\le \left|f(a)+g(a)\right| + \mu_{p(\cdot)}(f) + \mu_{p(\cdot)}(g)\\ &\le \|f\|_{\kappa p(\cdot)}^W + \|g\|_{\kappa p(\cdot)}^W.\end{aligned}$$

Thus, $\|f+g\|_{\kappa p(\cdot)}^W \le \|f\|_{\kappa p(\cdot)}^W + \|g\|_{\kappa p(\cdot)}^W$.

d) Let us now prove that $\|f\|_{\kappa p(\cdot)}^W = 0$ if and only if $f = 0$. If $f \equiv 0$, then $\kappa V_{p(\cdot)}^W\left(\frac{f}{\lambda}\right) = 0 \le 1$ for all $\lambda > 0$, and so $\|f\|_{\kappa p(\cdot)}^W = 0$. Conversely, suppose that $\|f\|_{\kappa p(\cdot)}^W = 0$, *i.e.*,

$$\left|f(a)\right| + \mu_{p(\cdot)}(f) = 0$$

then $\left|f(a)\right| = 0$ and $\mu_{p(\cdot)}(f) = \inf_{\lambda>0}\left\{\lambda > 0 : \kappa V_{p(\cdot)}^W\left(\frac{f}{\lambda}\right) \le 1\right\} = 0$, we get

$$\kappa V_{p(\cdot)}^W\left(\frac{f}{\lambda}\right) = 0,$$

i.e.,

$$\frac{\sum_{j=1}^{n}\left[\left|\frac{f(t_j) - f(t_{j-1})}{\lambda}\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j - t_{j-1}}{b-a}\right)} = 0,$$

without loss of generality, considering the partition $\pi : a = t_1 < t_2 = x < t_3 = b$ we get

$$\frac{\left(\frac{\left|f(x) - f(a)\right|}{\lambda}\right)^{p(x)}}{\kappa\left(\frac{x-a}{b-a}\right)} + \frac{\left(\frac{\left|f(b) - f(x)\right|}{\lambda}\right)^{p(b)}}{\kappa\left(\frac{b-x}{b-a}\right)} = 0,$$

then

$$\frac{\left(\frac{\left|f(x) - f(a)\right|}{\lambda}\right)^{p(x)}}{\kappa\left(\frac{x-a}{b-a}\right)} = 0 \quad \text{and} \quad \frac{\left(\frac{\left|f(b) - f(x)\right|}{\lambda}\right)^{p(b)}}{\kappa\left(\frac{b-x}{b-a}\right)} = 0,$$

we get

$$\left[\left|f(x)-f(a)\right|\right]^{p(x)}=0 \quad \text{and} \quad \left[\left|f(b)-f(x)\right|\right]^{p(b)}=0.$$

Hence, $f(x)=f(a)=f(b)$ for all $x\in[a,b]$ and $f(a)=0$, therefore $f=0$. □

In the following, we show that $\kappa BV_{p(\cdot)}^{W}([a,b])$ endowed with the norm $\|\cdot\|_{\kappa p(\cdot)}^{W}$ is a Banach space.

Theorem 3.7. *Let* $p:[a,b]\to(1,\infty)$ *be a function, then* $\left(\kappa BV_{p(\cdot)}^{W}([a,b]),\|\cdot\|_{\kappa p(\cdot)}^{W}\right)$ *is a Banach space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(\kappa BV_{p(\cdot)}^{W}([a,b]),\|\cdot\|_{\kappa p(\cdot)}^{W}\right)$, then given $\varepsilon>0$, there is $N>0$ such that for $n,m\geq N$ we have

$$\|f_n-f_m\|_{\kappa p(\cdot)}^{W}<\varepsilon, \quad n,m\geq N,$$

i.e.

$$\left|f_n(a)-f_m(a)\right|+\mu_{p(\cdot)}(f_n-f_m)<\varepsilon, \quad n,m\geq N.$$

Then

$$\left|f_n(a)-f_m(a)\right|<\varepsilon \quad \text{and} \quad \mu_{p(\cdot)}(f_n-f_m)<\varepsilon, \quad n,m\geq N.$$

Thus, for all $t,s\in[a,b]$ and $\varepsilon>0$, we have that

$$\kappa V_{p(\cdot)}^{W}\left(\frac{f_n-f_m}{\varepsilon};[a,b]\right)\leq 1$$

then

$$\frac{\left(\dfrac{\left|(f_n-f_m)(t)-(f_n-f_m)(s)\right|}{\varepsilon}\right)^{p(x_{ts})}}{\kappa\left(\dfrac{t-s}{b-a}\right)}\leq \kappa V_{p(\cdot)}^{W}\left(\frac{f_n-f_m}{\varepsilon};[a,b]\right)\leq 1,$$

therefore

$$\left[\left|(f_n-f_m)(t)-(f_n-f_m)(s)\right|\right]^{p(x_{ts})}\leq \varepsilon^{p(x_{ts})}\kappa\left(\frac{t-s}{b-a}\right),$$

by properties of function $\log(t)$, we get

$$\begin{aligned}
&p(x_{ts})\log\left[\left|(f_n-f_m)(t)-(f_n-f_m)(s)\right|\right]\\
&\leq p(x_{ts})\log(\varepsilon)+\log\left[\kappa\left(\frac{t-s}{b-a}\right)\right]\\
&\leq p(x_{ts})\log(\varepsilon)+p(x_{ts})\log\left[\kappa\left(\frac{t-s}{b-a}\right)\right]\\
&\leq p(x_{ts})\left[\log(\varepsilon)+\log\left[\kappa\left(\frac{t-s}{b-a}\right)\right]\right]\\
&= p(x_{ts})\log\left[\varepsilon\kappa\left(\frac{t-s}{b-a}\right)\right]
\end{aligned}$$

then

$$\log\left[\left|(f_n-f_m)(t)-(f_n-f_m)(s)\right|\right]\leq \log\left[\varepsilon\kappa\left(\frac{t-s}{b-a}\right)\right],$$

hence

$$\left|\left(f_n - f_m\right)(t) - \left(f_n - f_m\right)(s)\right| \le \varepsilon\kappa\left(\frac{t-s}{b-a}\right).$$

In consequence, the sequence $\{f_n\}_{n\ge1}$, is a uniformly sequence of Cauchy, on the interval $[a,b]$. Since $\mathbb{R}$ is complete, there exists a function *f* defined on $[a,b]$ such that

$$\lim_{n\to\infty} f_n(t) = f(t), \quad t\in[a,b].$$

We will show that f_n converge on the norm $\|\cdot\|^{W}_{\kappa p(\cdot)}$.

Since the $\{f_n\}_{n\ge1}$ is a Cauchy sequence there is a $N>0$ such that

$$\|f_n - f_m\|^{W}_{\kappa p(\cdot)} < \varepsilon, \quad n,m \ge N.$$

From the fact that $\{f_n\}_{n\ge1}$ converge uniformly to the function *f* on the interval $[a,b]$, we get

$$\begin{aligned}\|f_n - f\|^{W}_{\kappa p(\cdot)} &= \left\|f_n - \lim_{m\to\infty} f_m\right\|^{W}_{\kappa p(\cdot)} \\ &= \lim_{m\to\infty}\|f_n - f_m\|^{W}_{\kappa p(\cdot)} < \varepsilon, \ n\ge N.\end{aligned}$$

Therefore, the sequence $\{f_n\}_{n\ge1}$ converge to the function *f* on the norm $\|\cdot\|^{W}_{\kappa p(\cdot)}$.

Thus $\left(\kappa BV^{W}_{p(\cdot)}([a,b]), \|\cdot\|^{W}_{\kappa p(\cdot)}\right)$ is a Banach space. □

The following properties of elements of $\kappa BV^{W}_{p(\cdot)}[a,b]$ allow us to get characterizations of them.

Lemma 3.8. (General properties of the *p*(·)-variation) Let $f:[a,b]\to\mathbb{R}$ be a arbitrary map and κ be a distortion function. We have

(P1) Minimality: if $t,s\in[a,b]$, then

$$\begin{aligned}|f(t)-f(s)|^{p(x_{ts})} &\le \kappa\omega_{p(x_{ts})}(f,[a,b]) \\ &\le \kappa V_{p(\cdot)}(f,[a,b]).\end{aligned}$$

(P2) Change of variable: if $[c,d]\subset\mathbb{R}$ and $\varphi:[c,d]\to[a,b]$ is a (not necessarily strictly) monotone function, then $\kappa V^{W}_{p(\cdot)}(f,\varphi[c,b]) = \kappa V^{W}_{p(\cdot)}(f\circ\varphi,[c,d])$.

(P3) Regularity: $\kappa V^{W}_{p(\cdot)}(f,[a,b]) = \sup\left\{\kappa V^{W}_{p(\cdot)}(f,[s,t]); s,t\in[a,b], a\le b\right\}$.

Proof. (P1) Let $a,t,s,b\in[a,b]$, $a\le t\le s\le b$.

$$\begin{aligned}|f(t)-f(s)|^{p(x_{ts})} &\le \sup_{\pi^*}\left\{\frac{|f(t)-f(s)|^{p(x_{ts})}}{\kappa\left(\frac{t-s}{b-a}\right)} : t,s\in[a,b], a\le b\right\} = \kappa\omega_{p(x_{ts})}(f,[a,b]) \\ &\le \sup_{\pi^*}\frac{\sum_{j=1}^{n}\left[\left|f(t_j)-f(t_{j-1})\right|\right]^{p(x_{j-1})}}{\sum_{j=1}^{n}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)} = \kappa V^{W}_{p(\cdot)}(f,[a,b]).\end{aligned}$$

(P2) Let $[c,d]\subset\mathbb{R}$, $\varphi:[c,d]\to[a,b]$ a (not necessary strictly) monotone function, π_0 a tagged partition of the interval $[c,d]$, $T_l = \{\tau_j\}_{j=0}^{m} \in \pi_0$ and $T=\{t_j\}_{j=0}^{m}$ with $t_j = \varphi(\tau_j)$, then

$$\kappa V_{p(\cdot)}^{W}\left(f\circ g,T_l\right)=\sup_{T_l}\left\{\frac{\sum_{j=1}^{m}\left|f\left(\varphi\left(\tau_j\right)\right)-f\left(\varphi\left(\tau_{j-1}\right)\right)\right|^{p\left(x_{j-1}\right)}}{\sum_{j=1}^{m}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right\}=\sup_{T}\left\{\frac{\sum_{j=1}^{m}\left|f\left(t_j\right)-f\left(t_{j-1}\right)\right|^{p\left(x_{j-1}\right)}}{\sum_{j=1}^{m}\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)}\right\}$$
$$=\kappa V_{p(\cdot)}^{W}\left(f,T\right)\le\kappa V_{p(\cdot)}^{W}\left(f,\varphi\left(\left[c,d\right]\right)\right).$$

On the other hand, if a partition $T=\left\{t_j\right\}_{j=0}^{m}$ of $\varphi\left(\left[c,b\right]\right)$ is such that $t_{j-1}<t_j$ for $j=1,\cdots,m$, then there exist $\tau_j\in\left[c,d\right]$ such that $t_j=\varphi\left(\tau_j\right)$ and again by the monotonicity of φ:

$$\kappa V_{p(\cdot)}^{W}\left(f,T\right)=\kappa V_{p(\cdot)}^{W}\left(f\circ\varphi,T_l\right)\le\kappa V_{p(\cdot)}^{W}\left(f\circ\varphi,\left[c,d\right]\right).$$

(P3) By monotonocity of $\kappa V_{p(\cdot)}^{W}$ we get

$$\kappa V_{p(\cdot)}\left(f,\left[c,b\right]\right)\ge\sup\left\{\kappa V_{p(\cdot)}^{W}\left(f,\left[s,t\right]\right):s,t\in\left[a,b\right],a\le b\right\}.$$

On the other hand, for any number $\alpha<\kappa V_{p(\cdot)}^{W}\left(f,\left[a,b\right]\right)$ there is a partition $T=\left\{t_j\right\}_{j=0}^{m}\in\pi^{*}$, $t_j<t_{j+1}$ with $\kappa V_{p(\cdot)}^{W}\left(f,T\right)\ge\alpha$. We define $\hat{\pi}$ a partition of the interval $\left[t_0,t_m\right]$, then $T\in\hat{\pi}$ and $\kappa V_{p(\cdot)}^{W}\left(f,\left[a,b\right]\right)\ge\sup\left\{\kappa V_{p(\cdot)}^{W}\left(f,\left[s,t\right]\right),s,t\in\left[a,b\right],a\le b\right\}$. □

In the next section we will be dealing with the composition operator (Nemitskij).

4. Composition Operator between the Space $\kappa BV_{p(\cdot)}^{W}\left(\left[a,b\right]\right)$

In any field of nonlinear analysis composition operators (Nemytskij), the superposition operators generated by appropriate functions, play a crucial role in the theory of differential, integral and functional equations. Their analytic properties depend on the postulated properties of the defining function and on the function space in which they are considered. A rich source of related questions is the monograph by J. Appell and P. P. Zabrejko [31] and J. Appell, J. Banas, N. Merentes [8].

The composition operator problem refers to determining the conditions on a function $h:\mathbb{R}\to\mathbb{R}$, such that the composition operator, associated with the function *h*, maps a space $\mathbb{X}$ of functions $u:\left[a,b\right]\to\mathbb{R}$ into itself [32] [33]. There are several spaces where the composition operator problem has been resolved. In 1961, A. A. Babaev [34] showed that the composition *H*, associated with the function $h:\mathbb{R}\to\mathbb{R}$, maps the space $Lip\left[a,b\right]$ of the Lipschitz functions into itself if and only if *h* is locally Lipschitz; in 1967, K. S. Mukhtarov [35] obtained the same result for the space $Lip\left[a,b\right]$ of the Hölder functions of order α $\left(0<\alpha<1\right)$.

The first work on the composition operator problem in the space of functions of bounded variation $BV\left[a,b\right]$ was made by M. Josephy in 1981, [36]. Other work of this type have been preformed over $BV_{\varphi}^{W}\left[a,b\right]$, $HBV\left[a,b\right]$, $AC\left[a,b\right]$, $RV_{\varphi}\left[a,b\right]$, $RV_{p}\left[a,b\right]$, $\phi BV\left[a,b\right]$, $\Lambda BV_{\varphi}\left[a,b\right]$, $\kappa BV\left[a,b\right]$ and $\kappa BV_{\phi}\left[a,b\right]$ (see [8]).

Now, we define the composition operator. Given a function $h:\mathbb{R}\to\mathbb{R}$, the composition operator *H*, associated to a function *f* (autonomous case) maps each function $f:\left[a,b\right]\to\mathbb{R}$ into the composition function $Hf:\left[a,b\right]\to\mathbb{R}$, given by

$$Hf\left(t\right):=h\left(f\left(t\right)\right),\quad\left(t\in\left[a,b\right]\right). \tag{9}$$

More generally, given $f:\left[a,b\right]\times\mathbb{R}\to\mathbb{R}$, we consider the operator *H*, defined by

$$Hf\left(t\right):=h\left(t,f\left(t\right)\right),\quad\left(t\in\left[a,b\right]\right). \tag{10}$$

This operator is also called *superposition operator* or *susbtitution operator* or *Nemytskij operator*. In what follows, will refer (9) as the *autonomus case* and to (10) as the *non-autonomus case*.

In order to obtain the main result of this section, we will use a function of the zig-zag type such as the employed by J. Appell *et al.* [8] [37] that the locally Lipschitz condition of the function *h* is a necessary and sufficient condition such that $H\left(Lip[a,b]\right) \subset \kappa BV_{p(\cdot)}^{W}[a,b]$ and that in this situation *H* is bounded.

One of our main goals is to prove a result in the case when *h* is locally Lipschitz if and only if the composition operator maps the space of functions of bounded $p(\cdot)$-variation into itself.

The following lemma, established in [38], will be useful in the proof of our main Theorem (Theorem 4.2).

Lemma 4.1. Let $u:[a,b] \to \mathbb{R}$, $a \le s < \eta < t \le b$, then

$$\frac{\left|u(t)-u(s)\right|}{t-s} \le \frac{\left|u(\eta)-u(s)\right|}{\eta - s} + \frac{\left|u(t)-u(\eta)\right|}{t-\eta}.$$

Theorem 4.2. *Let H be a composition operator associated to* $h:\mathbb{R} \to \mathbb{R}$. *H maps the space* $\kappa BV_{p(\cdot)}^{W}(f)$ *into itself if and only if h is locally Lipschitz.*

Proof. We may suppose without loss generality that $[a,b]=[0,1]$. First, let $u:\mathbb{R} \to \mathbb{R}$ be locally Lipschitz on $\mathbb{R}$, and let $u \in \kappa BV_{p(\cdot)}^{W}([0,1])$. Then $\kappa V_{p(\cdot)}^{W}(\lambda u;[0,1]) < \infty$ for some $\lambda > 0$. Considering the local Lipschitz condition

$$\left|h(u)-h(v)\right| \le k(r)\left|u-v\right| \quad \left(u,v \in \mathbb{R}, |u|,|v| \le \mathbb{R}\right) \tag{11}$$

for $r := \|f\|_\infty$, for any partition $\pi: 0 = t_0 < t_1 < \cdots < t_n = 1$ we obtain the estimate

$$\begin{aligned}
&\frac{\sum_{j=1}^{n}\left[\frac{\lambda}{k\left(\|f\|_\infty\right)}\left|h\left(u\left(t_j\right)\right)-h\left(u\left(t_{j-1}\right)\right)\right|\right]^{p\left(t_{j-1}\right)}}{\sum_{j=1}^{n}\kappa\left(t_j - t_{j-1}\right)} \\
&\le \frac{\sum_{j=1}^{n}\left[\frac{\lambda}{k\left(\|f\|_\infty\right)}k\left(\|f\|_\infty\right)\left|u\left(t_j\right)-u\left(t_{j-1}\right)\right|\right]^{p\left(t_{j-1}\right)}}{\sum_{j=1}^{n}\kappa\left(t_j - t_{j-1}\right)} \\
&= \frac{\sum_{j=1}^{n}\left[\lambda\left|u\left(t_j\right)-u\left(t_{j-1}\right)\right|\right]^{p\left(t_{j-1}\right)}}{\sum_{j=1}^{n}\kappa\left(t_j - t_{j-1}\right)} \le \kappa V_{p(\cdot)}^{W}\left(\lambda u,[0,1]\right) < \infty.
\end{aligned}$$

This shows that for $\mu := \frac{\lambda}{k\left(\|f\|_\infty\right)}$, $\kappa V_{p(\cdot)}^{W}\left(\mu Hu,[0,1]\right) < \infty$, and hence $Hu \in \kappa BV_{p(\cdot)}^{W}([0,1])$ as claimed.

The proof of the only if direction will be by contradiction, that is we assume $H\left(Lip[0,1]\right) \subset \kappa BV_{p(\cdot)}^{W}([0,1])$ and h is not locally Lipschitz. Since the identity function $I_d:[0,1] \to [0,1]$ belong to $Lip[0,1]$, then $h \circ I_d \in \kappa BV_\phi[0,1]$ and therefore *h* is bounded in the interval $[0,1]$. Without loss of generality we may assume that

$$\left\|h|_{[0,1]}\right\|_\infty \le \frac{1}{4}. \tag{12}$$

Since *h* is not locally Lipschitz in $\mathbb{R}$ there is a closed interval *I* such that *h* does not satisfy any Lipschitz condition. In order to simplify the proof we can assume that $I=[0,1]$. In this way for any increasing sequence of positive real numbers $\{k_n\}_{n\ge1}$ that converge to infinite, that we will define later, we can choose sequences

$\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, such that

$$\left|h(b_n)-h(a_n)\right| > k_n\left|b_n-a_n\right|, \quad (n\in\mathbb{N}). \tag{13}$$

In addition choose a_n, b_n such that

$$a_n < b_n, \quad (n\in\mathbb{N}).$$

Considering subsequence if it necessary, we can assume without loss of generality that the sequence $\{a_n\}_{n\geq 1}$ is monotone increasing.

Since $[0,1]$ is compact, from inequality (13) we have that exist subsequences of $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ that we will denote in the same way, and that converge to $a_\infty\in[0,1]$.

Since the sequence $\{a_n\}_{n\geq 1}$ is a Cauchy sequence we can assume (taking subsequence if it is necessary) that

$$\left|a_m-a_n\right| < \frac{1}{k_n}, \quad (m>n). \tag{14}$$

Again considering subsequences if needed and using the properties of the function κ we can assume that

$$\max\left\{\kappa(b_n-a_n),\kappa(a_m-a_n)\right\} < \frac{1}{k_n}, \quad (n\in\mathbb{N}, m\geq n). \tag{15}$$

Consider the new sequence $\{m_n\}_{n\geq 1}$ defined by

$$m_n := \frac{1}{k_n(b_n-a_n)}, \quad (n\in\mathbb{N}).$$

From of inequalities (12) and (13) it follows that $m_n > 2$, therefore

$$\frac{m_n}{2} < [m_n] \leq m_n, \quad (n\in\mathbb{N}).$$

Consider the sequence defined recursively $\{t_n\}_{n\geq 1}$ by

$$t_1 := 0, \quad t_{n+1} := t_n + a_{n+1} - a_n + 2[m_n](b_n-a_n), \quad (n\in\mathbb{N}).$$

This sequence is strictly increasing and from the relations (14) and (15), we get

$$t_n \to t_\infty := \sum(t_{n+1}-t_n) = \sum_{n=1}^{\infty}(a_{n+1}-a_n) + 2\sum_{n=1}^{\infty}[m_n](b_n-a_n) \leq 3\sum_{n=1}^{\infty}\frac{1}{k_n}.$$

Then to ensure that $t_\infty\in[0,1]$, is sufficient to suppose that $\sum_{n=1}^{\infty}\frac{1}{k_n}\leq\frac{1}{3}$.

We define the continuous zig-zag function $u:[0,1]\to\mathbb{R}$, as shown below

$$u(t) := \begin{cases} a_n, & t = t_n + 2i(b_n-a_n),\ i=0,\cdots,[m_n], \\ b_n, & t = t_n + (2i+1)(b_n-a_n),\ i=0,\cdots,[m_n]-1, \\ a_\infty, & t_\infty \leq t \leq 1 \\ \text{affine}, & \text{othercase.} \end{cases}$$

Put

$$t_{n,i} := t_n + i(b_n-a_n), \quad n\in\mathbb{N}, i=0,\cdots,2[m_n].$$

We can write each interval $I_n = [t_n, t_{n+1}]$, $n\in\mathbb{N}$, as the union of the family of non-overlapping intervals

$$I_{n,i} := [t_{n.i}, t_{n,i+1}],\ i=0,\cdots,[m_n]-1, \quad I_{n,2[m_n]} := \left[t_{n,2[m_n]}, t_{n+1}\right].$$

And function u is defined on $I_{n,i}, i=0,\cdots,2[m_n]$, as follows

$$u(t) = t - (t_n + 2i(b_n - a_n)) + a_n, \left(t \in I_{n,2i}\right), \tag{16}$$

$$u(t) = t - t_n + (2i+1)(b_n - a_n) + b_n, \left(t \in I_{n,2i+1}\right), \tag{17}$$

and

$$u(t) = t - t_{n+1} + a_n, \left(t \in I_{n,2[m]}\right). \tag{18}$$

In all these situations the slopes of these segments of lines is 1.

Hence, we have for $n \in \mathbb{N}$, the absolute value of the slope of the line segments in these ranges are bounded by 1, as shown below

$$2^{-n} \frac{|b_n - a_n|}{\kappa^{-1}(b_n - a_n)} \le 2^{-n} k_n (b_n - a_n) \le 1,$$

$$2^{-(n+1)} \frac{a_{n+1} - a_n}{t_n + a_{n+1} - a_n} \le 1.$$

We will show that $u \in Lip[0,1]$.

Let $0 \le s < t \le 1$, then there are the following possibilities for the location of *s* and *t* on $[0,1]$.

Case 1: If $s,t \in I_n, (n \in \mathbb{N})$ are in the same interval $I_{n,i}, i = 0,\cdots,2[m_n]$.

From relations (16), (17) and (18) follows $\frac{|u(t)-u(s)|}{|t-s|} = 1$.

Case 2: If $s,t \in I_n, (n \in \mathbb{N})$ are in two different intervals $I_{n,i}, i = 0,\cdots,2[m_n]$.

There are several possibilities:

a) $s \in I_{n.i}$, $t \in I_{n,j}$, $i < j < 2[m_n]$.

a_1) $j = i+1$. By Lemma 4.1 and relations (16) and (17) we have

$$\frac{|u(t)-u(s)|}{|t-s|} \le \frac{|u(t_{n,i+1})-u(s)|}{t_{n,i+1}-s} + \frac{|u(t)-u(t_{n,i+1})|}{t-t_{n,i+1}} \le 2.$$

a_2) $j > i+1$. Then

$$\frac{|u(t)-u(s)|}{|t-s|} \le \frac{b_n - a_n}{t_{n,i+2} - t_{n,,i+1}} = 1.$$

b) $s \in I_{n.i}, t \in I_{n,j}, i < j = 2[m_n]$.

If $j = i+1$ proceed as a_1).

If $j > i+1$, again using the Lemma 4.1 and relations (16), (17) and (18) we obtain

$$\frac{|u(t)-u(s)|}{|t-s|} \le \frac{|u(t_{n,2[m_n]})-u(s)|}{t_{n,2[m_n]}-s} + \frac{|u(t)-u(t_{n,2[m_n]})|}{t_{n,2[m_n]}-t} \le \frac{b_n - a_n}{t_{n,2[m_n]} - t_{n,2[m_n]-1}} + 1 \le 2.$$

Case 3: If $s \in I_n, t \in I_m, n,m \in \mathbb{N}, n < m$.

From Lemma 4.1 and the second case, we conclude

$$\frac{|u(t)-u(s)|}{t-s} \le \frac{|u(t_{n+1})-u(s)|}{t_{n+1}-s} + \frac{|u(t)-u(t_m)|}{t-t_m} \le 4.$$

Case 4: If $s \in I_n, n \in \mathbb{N}, t = t_\infty$.

Then from Lemma 4.1

$$\frac{|u(t_\infty)-u(s)|}{t_\infty - s} \le \frac{|u(t_{n,i+1})-u(s)|}{t_{n,i+1}-s} + \frac{|a_\infty - u(t_{n,i+1})|}{b_n - a_n} \le 1 + \frac{a_\infty - a_n}{b_n - a_n} \le 2.$$

Case 5: If $s < t_\infty < t \le 1$.
From Lemma 4.1 and Case 4

$$\frac{|u(t)-u(s)|}{t-s} \le \frac{|u(t_\infty)-u(s)|}{t_\infty - s} \le 2.$$

Case 6: If $t_\infty \le s < t \le 1$.
In this circumstance $u(s)=u(t)=a_\infty$ and the situation is trivial. Therefore we have that

$$|u(t)-u(s)| \le |t-s|, \quad (s,t \in [0,1]).$$

So u is Lipschitz in $[0,1]$. Moreover, for each partition of interval $[0,1]$ of the form

$$\pi : 0 = t_1 < t_1 + (b_1 - a_1) < \cdots < t_1 + 2[m_1](b_1 - a_2) < t_2$$
$$< t_2 + (b_2 - a_2) < \cdots < t_k < \cdots < t_k + 2[m_k](b_k - a_k) < 1,$$

and $c > 0$, using the inequality (13), convexity of the function $\varphi_n, n \ge 1$ and definition of $m_n, n \in \mathbb{N}$, we have

$$\kappa V_\phi(h \circ u;[0,1]) = \frac{\sum_{n=1}^{k} \left|h\circ u(t_j) - h\circ u(t_{j-1})\right|^{p(x_{j-1})}}{\sum_{n=1}^{k} \kappa\left(u(t_j)-u(t_{j-1})\right)} \ge \frac{\sum_{n=1}^{k}\left[2[m_n]\left|h(b_n)-h(a_n)\right|\right]^{p(x_{j-1})}}{\sum_{n=1}^{k}\left[2[m_n]\kappa(b_n-a_n)+\kappa(a_{n+1}-a_n)\right]}$$
$$\ge \frac{\sum_{n=1}^{k}\left[2[m_n]k_n|b_n-a_n|\right]^{p(x_{j-1})}}{\sum_{n=1}^{k}\left[2[m_n]\kappa(b_n-a_n)+\kappa(a_{n+1}-a_n)\right]} \ge \frac{\sum_{n=1}^{k}\left[2[m_n]\frac{|b_n-a_n|}{m_n|b_n-a_n|}\right]^{p(x_{j-1})}}{\sum_{n=1}^{k}\frac{1}{k_n}}$$
$$\ge \sum_{n=1}^{k}\left(\frac{2[m_n]}{m_n}\right)^{p(x_{j-1})} \ge \sum_{n=1}^{k}(1)^{p(x_{j-1})}.$$

As the serie $\sum_{n=1}^{\infty}(1)^{p(x_{j-1})}$ diverge, $h\circ u \notin \kappa BV^W_{p(\cdot)}[0,1]$, which is a contradiction. □

5. Uniformly Continuous Composition Operator

In a seminal article of 1982, J. Matkowski [39] showed that if the composition operator H, associated with the function $h:[a,b]\times\mathbb{R}\to\mathbb{R}$, maps the space $Lip[a,b]$ of the Lipschitzian functions into itself and is a globally Lipschitzian map, then the function h has the form

$$h(t,x) = \alpha(t)x\beta(t), \quad t\in[a,b], x\in\mathbb{R}, \tag{19}$$

for some $\alpha,\beta \in Lip[a,b]$.

There are a variety of spaces besides $Lip[a,b]$ that verify this result [37]. The spaces of Banach $(\mathbb{X},\|\cdot\|)$ that fulfill this property are said to satisfy the Matkowski property [32].

In 1984, J. Matkowski and J. Miś [40] considered the same hypotheses on the operator H for the space $BV[a,b]$ of the function of bounded variation and concluded that (19) is true for the regularization h^- of the function h with respect of the first variable; that is,

$$h^-(t,x) = \alpha(t)x + \beta(t), \quad t\in[a,b], x\in\mathbb{R},$$

where $\alpha,\beta\in BV^-[a,b]$. The spaces that satisfy this condition are said to verify *weak Matkowski property*, [32].

In this section, we give the other main result of this paper, namely, we show that any uniformly bounded composition operator that maps the space $\kappa BV^W_{p(\cdot)}[a,b]$ into itself necessarily satisfies the so called Matkowski's weak condition.

First of all we will give the definition of left regularization of a function.

Definition 5.1. *Let* $f \in WBV_{p(\cdot)}([a,b])$, *its left regularization* $f^-:(a,b]\to\mathbb{R}$ *of mapping f is the function given as*

$$f^-(t) := \begin{cases} \lim_{s\to t^-} f(s) & t\in(a,b]; \\ f(a) & t=a. \end{cases}$$

We will denote by $WBV^-_{p(\cdot)}([a,b])([a,b])$ the subset in $WBV_{p(\cdot)}([a,b])$ which consists of those functions that are left continuous on $(a,b]$.

Lemma 5.2. *If* $f\in WBV_{p(\cdot)}([a,b])$, *then* $f^-\in WBV_{p(\cdot)}([a,b])$.

Thus, if a function $f\in WBV_{p(\cdot)}([a,b])$, then its left regularization is a left continuous function, *i.e.*, $f^-\in WBV^-_{p(x_{j-1})}$.

Also, we will denote by $\kappa BV^{W-}_{p(\cdot)}([a,b])$ the subset in $\kappa BV^{W}_{p(\cdot)}([a,b])$ which consists of those functions that are left continuous on $(a,b]$.

Lemma 5.3. *If* $f\in WBV_{p(\cdot)}([a,b])$, *then* $f^-\in \kappa BV^W_{p(\cdot)}([a,b])$.

Proof. By Lemma 5.2, we have $f^-\in WBV_{p(\cdot)}([a,b])$. Then, by Theorem 3.1, $f^-\in \kappa BV^W_{p(\cdot)}([a,b])$.

Thus, if a function $f\in WBV_{p(\cdot)}([a,b])$, then its left regularization is a left continuous function, *i.e.*, $f^-\in WBV^-_{p(\cdot)}([a,b])$. In consequence, $f^-\in \kappa BV^{W-}_{p(\cdot)}([a,b])$.

Another lemma useful for the follow theorem is developed below:

Lemma 5.4. *Let* $\kappa:[0,1]\to[0,1]$, *be a distortion function,* $u\in\kappa BV^W_{p(\cdot)}([a,b])$ *and* $\lambda>0$. *Then* $\mu^{\kappa}_{p(\cdot)}(u)<\lambda$ *if and only if* $\kappa V^W_{p(\cdot)}\left(\frac{u}{\lambda}\right)\le 1$.

Proof. Let $u\in\kappa BV^W_{p(\cdot)}([a,b])$. Suppose that $\mu^{\kappa}_{p(\cdot)}<\lambda$; then by definition of $\mu^{\kappa}_{p(\cdot)}(u)$ there exists κ such that $\lambda>\kappa>\mu^{\kappa}_{p(\cdot)}(u)$ and $\kappa V^W_{p(\cdot)}\left(\frac{u}{\lambda}\right)\le 1$. Since, for $s>1$ the function $t\ge 0, t\to t^s$ is convex, we have:

$$\kappa V^W_{p(\cdot)}\left(\frac{u}{\lambda}\right)=\kappa V^W_{p(\cdot)}\left(\frac{u}{k}\frac{k}{\lambda}\right)=\frac{k}{\lambda}\kappa V^W_{p(\cdot)}\left(\frac{u}{k}\right)\le\frac{k}{\lambda}\le 1.$$

Conversely, assume $\kappa V^W_{p(\cdot)}\left(\frac{u}{\lambda}\right)<1$, then $\lambda\in\left\{\lambda>0:\kappa V^W_{p(\cdot)}\left(\frac{u}{\lambda}\right)\le 1\right\}$; hence $\mu^{\kappa}_{p(\cdot)}(u)<\lambda$.

Theorem 5.5. *Suppose that the composition operator H generated by* $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ *maps* $\kappa BV^W_{p(\cdot)}([a,b])$ *into itself and satisfies the following inequality*

$$\|Hf_1-Hf_2\|^W_{\kappa p(\cdot)}\le\gamma\left[\|f_1-f_2\|^W_{\kappa p(\cdot)}\right]\quad\left(f_1,f_2\in\kappa BV^W_{p(\cdot)}([a,b])\right)\tag{20}$$

for some function $\gamma:[0,\infty)\to[0,\infty)$. *Then, there exist functions* $\alpha,\beta\in\kappa BV^W_{p(\cdot)}([a,b])$ *such that*

$$h^-(t,x)=\alpha(t)x+\beta(t),\ t\in[a,b],\ x\in\mathbb{R}\tag{21}$$

where $h^-(\cdot,x):(a,b]\to\mathbb{R}$ *is the left regularization of* $h(\cdot,x)$ *for all* $x\in\mathbb{R}$.

Proof. By hypothesis, for $x\in\mathbb{R}$ fixed, the constant function $f(t)=x, t\in[a,b]$ belongs to $\kappa BV^W_{p(\cdot)}([a,b])$. Since H maps $\kappa BV^W_{p(\cdot)}([a,b])$ into itself, we have $(Hf)(t)=h(t,f(t))\in\kappa BV^W_{p(\cdot)}([a,b])$. By Lemma 5.2 the left regularization $h^-(\cdot,x)\in\kappa BV^{W-}_{p(\cdot)}([a,b])$ for every $x\in\mathbb{R}$.

From the inequality (20) and definition of the norm $\|\cdot\|^W_{\kappa p(\cdot)}$ we obtain for $f_1,f_2\in\kappa BV^W_{p(\cdot)}([a,b])$,

$$\mu_{p(\cdot)}\left(H\left(f_1\right)-H\left(f_2\right)\right)\le\left\|H\left(f_1\right)-H\left(f_2\right)\right\|_{\kappa p(\cdot)}^{W}\le\gamma\left[\left\|f_1-f_2\right\|_{\kappa p(\cdot)}^{W}\right]. \tag{22}$$

From the inequality (22) and Lemma 5.2, if $\gamma\left[\left\|f_1-f_2\right\|_{\kappa p(\cdot)}^{W}\right]>0$ then

$$\kappa V_{p(\cdot)}^{W}\left(\frac{H\left(f_1\right)-H\left(f_2\right)}{\gamma\left[\left\|f_1-f_2\right\|_{\kappa p(\cdot)}^{W}\right]}\right)\le 1. \tag{23}$$

Let $a\le s<t\le b$, and let $\pi_m:=\{t_0,t_1,\cdots,t_{2m}\}\in\pi$ be the equidistant partition defined by

$$t_0=s,\ t_j-t_{j-1}=\frac{t-s}{2m}\quad\left(j=1,2,\cdots,2m\right).$$

Given $u,v\in\mathbb{R}$ with $u\ne v$, define $f_1,f_2:[a,b]\to\mathbb{R}$ by

$$f_1(x):=\begin{cases}v, & \text{if } x=t_j \text{ for some even } j,\\ \dfrac{u+v}{2}, & \text{if } x=t_j \text{ for some odd } j,\\ \text{linear}, & \text{otherwise}\end{cases} \tag{24}$$

and

$$f_2(x):=\begin{cases}\dfrac{u+v}{2}, & \text{if } x=t_j \text{ for some even } j,\\ u, & \text{if } x=t_j \text{ for some odd } j,\\ \text{linear}, & \text{otherwise.}\end{cases} \tag{25}$$

Then the difference f_1-f_2 satisfies

$$\left|f_1(x)-f_2(x)\right|\equiv\frac{|u-v|}{2}\quad\left(a\le x\le b\right).$$

Consequently, by the inequality (20)

$$\left\|Hf_1-Hf_2\right\|_{\kappa p(\cdot)}^{W}\le\gamma\left[\left\|f_1-f_2\right\|_{\kappa p(\cdot)}^{W}\right]\le\gamma\left(\frac{|u-v|}{2}\right).$$

From the inequality (23) and the definition of $p(\cdot)$-variation in the sense of Wiener-Korenblum we have

$$\frac{\displaystyle\sum_{j=1}^{m}\left(\frac{\left|\left(h^-\circ f_1\right)\left(t_{2j}\right)-\left(h^-\circ f_2\right)\left(t_{2j}\right)-\left(h^-\circ f_1\right)\left(t_{2j-1}\right)+\left(h^-\circ f_2\right)\left(t_{2j-1}\right)\right|}{\gamma\left(2^{-1}|u-v|\right)}\right)^{p\left(x_{j-1}\right)}}{\displaystyle\sum_{n=1}^{m}\kappa\left(\frac{t_{2j}-t_{2j-1}}{b-a}\right)}\le 1.$$

However, by definition of the functions f_1 and f_2,

$$\begin{aligned}&\left|\left(h^-\circ f_1\right)\left(t_{2j}\right)-\left(h^-\circ f_2\right)\left(t_{2j}\right)-\left(h^-\circ f_1\right)\left(t_{2j-1}\right)+\left(h^-\circ f_2\right)\left(t_{2j-1}\right)\right|\\&=\left|h^-(v)-h^-\left(\frac{u+v}{2}\right)-h^-\left(\frac{u+v}{2}\right)+h^-(u)\right|\\&=\left|h^-(v)-2h^-\left(\frac{u+v}{2}\right)+h^-(u)\right|.\end{aligned}$$

Then

$$\frac{\sum_{j=1}^{m}\left(\frac{\left|h^{-}(v)-2h^{-}\left(\frac{u+v}{2}\right)+h^{-}(u)\right|}{\gamma\left(2^{-1}|u-v|\right)}\right)^{p(x_{j-1})}}{\sum_{n=1}^{m}\kappa\left(\frac{t_{2j}-t_{2j-1}}{b-a}\right)}\leq 1. \tag{26}$$

Since $1\leq p(x_{j-1})<\infty$ for all $j=1,2,\cdots,2m$, $\sum_{n=1}^{m}\kappa\left(\frac{t_{2j}-t_{2j-1}}{b-a}\right)>1$, and passing to the limit as $m\to\infty$, then

$$\left(\frac{\left|h^{-}(v)-2h^{-}\left(\frac{u+v}{2}\right)+h^{-}(u)\right|}{\gamma\left(2^{-1}|u-v|\right)}\right)^{p(x_{j-1})}=0,$$

hence,

$$h^{-}(v)-2h^{-}\left(\frac{u+v}{2}\right)+h^{-}(u)=0.$$

So, we conclude that $h^{-}(s,\cdot)$ satisfies the Jensen equation in $\mathbb{R}$ (see [41], page 315). The continuity of h^{-} with respect of the second variable implies that for every $t\in[a,b]$ there exist $\alpha,\beta:[a,b]\to\mathbb{R}$ such that

$$h^{-}(t,x)=\alpha(t)x+\beta(t),\quad t\in[a,b],\ x\in\mathbb{R}.$$

Because $\beta(t)=h^{-}(t,0), t\in[a,b]$, $\alpha(t)=h^{-}(t,1)-\beta(t)$ and $h^{-}(\cdot,x)\in\kappa BV_{p(\cdot)}^{W}([a,b])$, for each $x\in\mathbb{R}$, we obtain that $\alpha,\beta\in\kappa BV_{p(\cdot)}^{W}([a,b])$. □

J. Matkowski [42] introduced the notion of a uniformly bounded operator and proved that any uniformly bounded composition operator acting between general Lipschitz function normed spaces must be of the form (21).

Definition 5.6. ([42], *Def.* 1) *Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric* (*or normed*) *spaces. We say that a mapping $H:\mathcal{X}\to\mathcal{Y}$ is uniformly bounded if, for any $t>0$ there exists a nonnegative real number $\gamma(t)$ such that for any nonempty set $B\subset\mathcal{X}$ we have*

$$diamB\leq t\Rightarrow diamH(B)\leq\gamma(t).$$

Remark 5.7. *Every uniformly continuous operator or Lipschitzian operator is uniformly bounded.*

Theorem 5.8. *Let $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ and H be the composition operator associated with h. Suppose that H maps $\kappa BV_{p(\cdot)}^{W}([a,b])$ into itself and is uniformly continuous, then, there exist functions $\alpha,\beta\in\kappa BV_{p(\cdot)}^{W}([a,b])$ such that*

$$h^{-}(t,x)=\alpha(t)x+\beta(t),\ t\in[a,b],x\in\mathbb{R}.$$

where $h^{-}(\cdot,x):(a,b]\to\mathbb{R}$ is the left regularization of $h(\cdot,x)$ for all $x\in\mathbb{R}$.

Proof. Take any $t\geq 0$ and $f,g\in\kappa BV_{p(\cdot)}^{W}([a,b])$ such that

$$\|f-g\|_{\kappa p(\cdot)}^{W}\leq diamH(\{f,g\})$$

Since $diam\{f,g\}\leq t$ by the uniform boundedness of *H*, we have

$$diamH(\{f,g\})\leq\gamma(t),$$

that is,

$$\left\|H(f)-H(g)\right\|_{\kappa p(\cdot)}^{W} = diamH\left(\{f,g\}\right) \le \gamma\left(\left\|f-g\right\|_{\kappa p(\cdot)}^{W}\right),$$

and therefore, by the Theorem 5.5 we get

$$h^{-}(t,x)=\alpha(t)x+\beta(t),\ \ t\in[a,b], x\in\mathbb{R}.$$

Acknowledgements

This research has been partially supported by the Central Bank of Venezuela. We want to give thanks to the library staff of B.C.V for compiling the references.

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