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Poisson Vector Fields on Weil Bundles

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Abstract

In this paper, *M* is a smooth manifold of finite dimension *n*, *A* is a local algebra and M^A is the associated Weil bundle. We study Poisson vector fields on M^A and we prove that all globally hamiltonian vector fields on M^A are Poisson vector fields.

Keywords

Weil Algebra, Weil Bundle, Poisson Manifold, Lie Derivative, Poisson 2-Form

1. Introduction

A Weil algebra or local algebra (in the sense of André Weil) [1], is a finite dimensional, associative, commutative and unitary algebra *A* over $\mathbb{R}$ in which there exists a unique maximum ideal $\mathfrak{m}$ of codimension 1. In his case, the factor space $A/\mathfrak{m}$ is one-dimensional and is identified with the algebra of real numbers $\mathbb{R}$. Thus $A=\mathbb{R}\oplus\mathfrak{m}$ and $\mathbb{R}$ is identified with $\mathbb{R}\cdot 1_A$, where 1_A is the unit of *A*.

In what follows we denote by *A* a Weil algebra, *M* a smooth manifold, $C^\infty(M)$ the algebra of smooth functions on *M*.

A near point of $x\in M$ of kind *A* is a homomorphism of algebras

$$\xi : C^\infty(M)\to A$$

such that for any $f\in C^\infty(M)$, $\left[\xi(f)-f(x)\right]\in\mathfrak{m}$.

We denote by M_x^A the set of near points of *x* of kind *A* and $M^A=\bigcup_{x\in M} M_x^A$ the set of near points on *M* of kind *A*. The set M^A is a smooth manifold of dimension $\dim M\times\dim A$ and called manifold of infinitely near points on *M* of kind *A* [1]-[3], or simply the Weil bundle [4] [5].

If $f : M\to\mathbb{R}$ is a smooth function, then the map

$$f^A : M^A\to\mathbb{R}^A = A, \xi\mapsto\left[f^A(\xi)\right](id_{\mathbb{R}})=\xi(id_{\mathbb{R}}\circ f)=\xi(f)$$

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is differentiable of class C^{∞} [4] [6]. The set, $C^{\infty}\left(M^{A},A\right)$ of smooth functions on M^{A} with values on A, is a commutative algebra over A with unit and the map

$$C^{\infty}(M)\to C^{\infty}\left(M^{A},A\right),\ f\mapsto f^{A}$$

is an injective homomorphism of algebras. Then, we have:

$$(f+g)^{A}=f^{A}+g^{A};(\lambda\cdot f)^{A}=\lambda\cdot f^{A};\ (f\cdot g)^{A}=f^{A}\cdot g^{A}.$$

We denote $\mathfrak{X}\left(M^{A}\right)$, the set of vector fields on M^{A} and $Der_{A}\left[C^{\infty}\left(M^{A},A\right)\right]$ the set of A-linear maps

$$X:C^{\infty}\left(M^{A},A\right)\to C^{\infty}\left(M^{A},A\right)$$

such that

$$X(\varphi\cdot\psi)=X(\varphi)\cdot\psi+\varphi\cdot X(\psi),\ \text{for any }\varphi,\psi\in C^{\infty}\left(M^{A},A\right).$$

Thus [4],

$$\mathfrak{X}\left(M^{A}\right)=Der_{A}\left[C^{\infty}\left(M^{A},A\right)\right].$$

If

$$\theta:C^{\infty}(M)\to C^{\infty}(M)$$

is a vector field on M, then there exists one and only one A-linear derivation

$$\theta^{A}:C^{\infty}\left(M^{A},A\right)\to C^{\infty}\left(M^{A},A\right)$$

called prolongation of the vector field θ [4] [6], such that

$$\theta^{A}\left(f^{A}\right)=\left[\theta(f)\right]^{A},\ \text{for any }f\in C^{\infty}(M).$$

Let $\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right]$ be the $C^{\infty}(M)$-module of Kälher differentials of $C^{\infty}(M)$ and

$$\delta_{M}:C^{\infty}(M)\to\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right],\ f\mapsto\overline{f\otimes 1_{C^{\infty}(M)}-1_{C^{\infty}(M)}\otimes f}$$

the canonical derivation which the image of δ_{M} generates the $C^{\infty}(M)$-module $\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right]$ *i.e.* for $x\in\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right]$,

$$x=\sum_{i\in I:\,finite}f_{i}\cdot\delta_{M}(g_{i}),$$

with $f_{i},g_{i}\in C^{\infty}(M)$ for any $i\in I$ [7] et [8].

We denote $\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]$, the $C^{\infty}\left(M^{A},A\right)$-module of Kälher differentials of $C^{\infty}\left(M^{A},A\right)$ which are A-linear. In this case, for $\varphi\in C^{\infty}\left(M^{A},A\right)$, we denote $\overline{\varphi\otimes 1_{C^{\infty}(M^{A},A)}-1_{C^{\infty}(M^{A},A)}\otimes\varphi}$, the class of $\varphi\otimes 1_{C^{\infty}(M^{A},A)}-1_{C^{\infty}(M^{A},A)}\otimes\varphi$ in $C^{\infty}\left(M^{A},A\right)$.

The map

$$C^{\infty}(M)\to\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right],\ f\mapsto\left[\delta_{M}(f)\right]^{A}=\overline{f^{A}\otimes 1_{C^{\infty}(M^{A},A)}-1_{C^{\infty}(M^{A},A)}\otimes f^{A}}$$

is a derivation and there exists a unique A-linear derivation

$$\delta_{M^{A}}^{A}:C^{\infty}\left(M^{A},A\right)\to\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]$$

such that

$$\delta_{M^{A}}^{A}\left(f^{A}\right)=\left[\delta_{M}(f)\right]^{A}$$

for any $f\in C^{\infty}(M)$ [9]. Moreover the map

$$\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right]\to\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right], x\mapsto x^{A}$$

is an injective homomorphism of $\mathbb{R}$-modules. Thus, the pair $\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right],\delta_{M^{A}}^{A}\right)$ satisfies the following universal property: for every $C^{\infty}\left(M^{A},A\right)$-module E and every A-derivation

$$\Phi: C^{\infty}\left(M^{A},A\right)\to E,$$

there exists a unique $C^{\infty}\left(M^{A},A\right)$-linear map

$$\tilde{\Phi}:\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]\to E$$

such that

$$\tilde{\Phi}\circ\delta_{M^{A}}^{A}=\Phi.$$

In other words, there exists a unique $\tilde{\Phi}$ which makes the following diagram commutative

$$\begin{array}{ccc}\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right] & & \\ \delta_{M^{A}}^{A}\uparrow & \overset{\tilde{\Phi}}{\searrow} & \\ C^{\infty}\left(M^{A},A\right) & \underset{\Phi}{\to} & E\end{array}$$

This fact implies the existence of a natural isomorphism of $C^{\infty}\left(M^{A},A\right)$-modules

$$Hom_{C^{\infty}\left(M^{A},A\right)}\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right],E\right)\to Der_{A}\left[C^{\infty}\left(M^{A},A\right),E\right],\psi\mapsto\psi\circ\delta_{M^{A}}^{A}.$$

In particular, if $E=C^{\infty}\left(M^{A},A\right)$, we have

$$\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]^{*}\simeq Der_{A}\left[C^{\infty}\left(M^{A},A\right)\right]=\mathfrak{X}\left(M^{A}\right).$$

For any $p\in\mathbb{N}$, $\Lambda^{p}\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]\right)=\mathfrak{L}_{sks}^{p}\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right],C^{\infty}\left(M^{A},A\right)\right)$ denotes the $C^{\infty}\left(M^{A},A\right)$-module of skew-symmetric multilinear forms of degree p from $\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]$ into $C^{\infty}\left(M^{A},A\right)$ and

$$\Lambda\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]\right)=\bigoplus_{p\in\mathbb{N}}\Lambda^{p}\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]\right)$$

the exterior $C^{\infty}\left(M^{A},A\right)$-algebra of $\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]$ called algebra of Kähler forms on $C^{\infty}\left(M^{A},A\right)$.

$$\Lambda^{0}\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]\right)=C^{\infty}\left(M^{A},A\right),$$

$$\Lambda^{1}\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]\right)=\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]^{*}.$$

If $\eta\in\Lambda^{p}\left(\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right]\right)$, then η is of the form $\delta_{M}\left(f_{1}\right)\wedge\cdots\wedge\delta_{M}\left(f_{p}\right)$ with $f_{1},f_{2},\cdots,f_{p}\in C^{\infty}(M)$. Thus, the $C^{\infty}\left(M^{A},A\right)$-module $\Lambda^{p}\left(\Omega_{A}\left[C^{\infty}\left(M^{A},A\right)\right]\right)$ is generated by elements of the form

$$\eta^{A}=\delta_{M^{A}}^{A}\left(\varphi_{1}\right)\wedge\cdots\wedge\delta_{M^{A}}^{A}\left(\varphi_{p}\right)$$

with $\varphi_{1}=f_{1}^{A},\cdots,\varphi_{p}=f_{p}^{A}\in C^{\infty}\left(M^{A},A\right)$.

Let

$$\sigma_{\theta}:\left[\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right]\right]^{p}\to\Lambda^{p}\left(\Omega_{\mathbb{R}}\left[C^{\infty}(M)\right]\right),$$

be the $C^{\infty}(M)$-skew-symmetric multilinear map such that

$$\sigma_\theta\left(x_1,x_2,\cdots,x_p\right)=\sum_{i=1}^{p}(-1)^{i-1}\tilde{\theta}\left(x_i\right)\cdot x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_p,$$

for any $x_1,x_2,\cdots,x_p\in\Omega_{\mathbb{R}}\left[C^\infty(M)\right]$ and, where

$$\tilde{\theta}:\Omega_{\mathbb{R}}\left[C^\infty(M)\right]\to C^\infty(M)$$

is a unique $C^\infty(M)$-linear map such that $\tilde{\theta}\circ\delta_M=\theta$ [8]. Then,

$$\sigma^A_{\theta^A}:\left[\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right]^p\to\Lambda^p\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right)$$

is a unique $C^\infty(M)$-skew-symmetric multilinear map such that

$$\sigma^A_{\theta^A}\left(x^A_1,x^A_2,\cdots,x^A_p\right)=\left[\sigma_\theta\left(x_1,x_2,\cdots,x_p\right)\right]^A.$$

We denote

$$\widetilde{\sigma^A_{\theta^A}}:\Lambda^p\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right)\to\Lambda^{p-1}\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right),$$

the unique $C^\infty\left(M^A,A\right)$-skew-symmetric multilinear map such that

$$\widetilde{\sigma^A_{\theta^A}}\left(x^A_1\wedge x^A_2\wedge\cdots\wedge x^A_p\right)=\sigma^A_{\theta^A}\left(x^A_1,x^A_2,\cdots,x^A_p\right)$$

i.e. $\sigma^A_{\theta^A}$ induces a derivation

$$i_{\theta^A}=\widetilde{\sigma^A_{\theta^A}}:\Lambda\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right)\to\Lambda\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right)$$

of degree −1 [9].

We recall that a Poisson structure on a smooth manifold *M* is due to the existence of a bracket $\{,\}_M$ on $C^\infty(M)$ such that the pair $\left(C^\infty(M),\{,\}_M\right)$ is a real Lie algebra such that, for any $f\in C^\infty(M)$ the map

$$ad(f):C^\infty(M)\to C^\infty(M),g\mapsto\{f,g\}_M$$

is a derivation of commutative algebra *i.e.*

$$\{f,g\cdot h\}_M=\{f,g\}_M\cdot h+g\cdot\{f,h\}_M$$

for $f,g,h\in C^\infty(M)$. In this case we say that $C^\infty(M)$ is a Poisson algebra and *M* is a Poisson manifold [10] [11].

The manifold *M* is a Poisson manifold if and only if there exists a skew-symmetric 2-form

$$\omega_M:\Omega_{\mathbb{R}}\left[C^\infty(M)\right]\times\Omega_{\mathbb{R}}\left[C^\infty(M)\right]\to C^\infty(M)$$

such that for any *f* and *g* in $C^\infty(M)$,

$$\{f,g\}_M=-\omega_M\left[\delta_M(f),\delta_M(g)\right]$$

defines a structure of Lie algebra over $C^\infty(M)$ [8]. In this case, we say that ω_M is the Poisson 2-form of the Poisson manifold *M* and we denote (M,ω_M) the Poisson manifold of Poisson 2-form ω_M.

2. Poisson 2-Form on Weil Bundles

When $\left(M,\{,\}_M\right)$ is a Poisson manifold, the map

$$ad:C^\infty(M)\to Der_{\mathbb{R}}\left[C^\infty(M)\right],f\mapsto ad(f)$$

such that $\left[ad(f)\right](g)=\{f,g\}_M$ for any $g\in C^\infty(M)$, is a derivation. Thus, there exists a derivation

$$ad^A:C^\infty\left(M^A,A\right)\to Der_A\left[C^\infty\left(M^A,A\right)\right]$$

such that

$$ad^A\left(f^A\right)=\left[ad\left(f\right)\right]^A.$$

Let

$$\widetilde{ad^A}:\Omega_A\left[C^\infty\left(M^A,A\right)\right]\to Der_A\left[C^\infty\left(M^A,A\right)\right]$$

be a unique $C^\infty\left(M^A,A\right)$-linear map such that

$$\widetilde{ad^A}\circ\delta^A_{M^A}=ad^A.$$

Let us consider the canonical isomorphism

$$\sigma_{M^A}:\Omega_A\left[C^\infty\left(M^A,A\right)\right]^*\to Der_A\left[C^\infty\left(M^A,A\right)\right],\ \Psi\mapsto\Psi\circ\delta^A_{M^A}$$

and let

$$\sigma^{-1}_{M^A}\circ\widetilde{ad^A}:\Omega_A\left[C^\infty\left(M^A,A\right)\right]\xrightarrow{\widetilde{ad^A}}Der_A\left[C^\infty\left(M^A,A\right)\right]\xrightarrow{\sigma^{-1}_{M^A}}\Omega_A\left[C^\infty\left(M^A,A\right)\right]^*$$

be the map.

Proposition 1. [9] *If* $\left(M,\omega_M\right)$ *is a Poisson manifold, then the map,*

$$\omega^A_{M^A}:\Omega_A\left[C^\infty\left(M^A,A\right)\right]\times\Omega_A\left[C^\infty\left(M^A,A\right)\right]\to C^\infty\left(M^A,A\right)$$

such that for any $X,Y\in\Omega_A\left[C^\infty\left(M^A,A\right)\right]$

$$\omega^A_{M^A}\left(X,Y\right)=-\left[\sigma^{-1}_{M^A}\circ\widetilde{ad^A}\left(X\right)\right]\left(Y\right)$$

is a skew-symmetric 2*-form on* $\Omega_A\left[C^\infty\left(M^A,A\right)\right]$ *such that*

$$\omega^A_{M^A}\left(x^A,y^A\right)=\left[\omega_M\left(x,y\right)\right]^A,$$

for any x and y in $\Omega_{\mathbb{R}}\left[C^\infty\left(M\right)\right]$. *Moreover,* $\left(M^A,\omega^A_{M^A}\right)$ *is a Poisson manifold.*

Theorem 2. [9] *The manifold* M^A *is a Poisson manifold if and only if there exists a skew-symmetric* 2*-form*

$$\omega^A_{M^A}:\Omega_A\left[C^\infty\left(M^A,A\right)\right]\times\Omega_A\left[C^\infty\left(M^A,A\right)\right]\to C^\infty\left(M^A,A\right)$$

such that for any φ *and* ψ *in* $C^\infty\left(M^A,A\right)$,

$$\left\{\varphi,\psi\right\}_{M^A}=-\omega^A_{M^A}\left(\delta^A_{M^A}\left(\varphi\right),\delta^A_{M^A}\left(\psi\right)\right)$$

defines a structure of A-Lie algebra over $C^\infty\left(M^A,A\right)$. *Moreover, for any f and g in* $C^\infty\left(M\right)$,

$$\left\{f^A,g^A\right\}_{M^A}=\left\{f,g\right\}^A_M.$$

In this case, we will say that $\omega^A_{M^A}$ is the Poisson 2-form of the *A*-Poisson manifold M^A and we denote $\left(M^A,\omega^A_{M^A}\right)$ the *A*-Poisson manifold of Poisson 2-form $\omega^A_{M^A}$ [9].

3. Poisson Vector Field on Weil Bundles

Proposition 3. *For any* $\theta\in Der_{\mathbb{R}}\left[C^\infty\left(M\right)\right]$ *and for any* $\eta\in\Lambda^p\left(\Omega_{\mathbb{R}}\left[C^\infty\left(M\right)\right]\right)$, *we have*

$$i_{\theta^A}\left(\eta^A\right)=\left[i_\theta\left(\eta\right)\right]^A.$$

Proof. If $\eta\in\Lambda^p\left(\Omega_{\mathbb{R}}\left[C^\infty\left(M\right)\right]\right)$, then there exists $f_1,f_2,\cdots,f_p\in C^\infty\left(M\right)$, such that $\eta=\delta_M\left(f_1\right)\wedge\cdots\wedge\delta_M\left(f_p\right)$. Thus,

$$\begin{aligned}
i_{\theta^A}\left(\eta^A\right) &= i_{\theta^A}\left(\left[\delta_M\left(f_1\right)\wedge\cdots\wedge\delta_M\left(f_p\right)\right]^A\right) \\
&= i_{\theta^A}\left(\left[\delta_M\left(f_1\right)\right]^A\wedge\cdots\wedge\left[\delta_M\left(f_p\right)\right]^A\right) \\
&= \sigma^A_{\theta^A}\left(\left[\delta_M\left(f_1\right)\right]^A,\cdots,\left[\delta_M\left(f_p\right)\right]^A\right) \\
&= \left[\sigma_\theta\left(\delta_M\left(f_1\right),\cdots,\delta_M\left(f_p\right)\right)\right]^A \\
&= \left[i_\theta\left(\delta_M\left(f_1\right)\wedge\cdots\wedge\delta_M\left(f_p\right)\right)\right]^A \\
&= \left[i_\theta\left(\eta\right)\right]^A.
\end{aligned}$$

3.1. Lie Derivative

The Lie derivative with respect to $D\in Der_A\left[C^\infty\left(M^A,A\right)\right]$ is the derivation of degree 0

$$\mathfrak{L}_D = i_D\circ\delta^A_{M^A}+\delta^A_{M^A}\circ i_D : \Lambda\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right)\to\Lambda\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right).$$

Proposition 4. *For any* $\theta\in\mathfrak{X}(M)$, *lthe map*

$$\mathfrak{L}_{\theta^A} : \Lambda\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right)\to\Lambda\left(\Omega_A\left[C^\infty\left(M^A,A\right)\right]\right)$$

is a unique A-linear derivation such that

$$\mathfrak{L}_{\theta^A}\left(\eta^A\right)=\left[\mathfrak{L}_\theta\left(\eta\right)\right]^A,$$

for any $\eta\in\Lambda\left(\Omega_{\mathbb{R}}\left[C^\infty\left(M\right)\right]\right)$.

Proof. For any $\eta\in\Lambda\left(\Omega_{\mathbb{R}}\left[C^\infty\left(M\right)\right]\right)$, we have

$$\begin{aligned}
\mathfrak{L}_{\theta^A}\left(\eta^A\right) &= i_{\theta^A}\left[\delta^A_{M^A}\left(\eta^A\right)\right]+\delta^A_{M^A}\left[i_{\theta^A}\left(\eta^A\right)\right] \\
&= i_{\theta^A}\left(\left[\delta_M\left(\eta\right)\right]^A\right)+\delta^A_{M^A}\left(\left[i_\theta\left(\eta\right)\right]^A\right) \\
&= \left(i_\theta\left[\delta_M\left(\eta\right)\right]\right)^A+\left(\delta_M\left[i_\theta\left(\eta\right)\right]\right)^A \\
&= \left(i_\theta\left[\delta_M\left(\eta\right)\right]+\delta_M\left[i_\theta\left(\eta\right)\right]\right)^A \\
&= \left[\mathfrak{L}_\theta\left(\eta\right)\right]^A.
\end{aligned}$$

A vector field θ on a Poisson manifold (M,ω_M) is called Poisson vector field if the Lie derivative of ω_M with respect to θ vanishes *i.e.* $\mathfrak{L}_\theta\omega_M=0$. A vector field

$$X : C^\infty\left(M^A,A\right)\to C^\infty\left(M^A,A\right)$$

on a *A*-Poisson manifold of Poisson 2-form $\omega^A_{M^A}$ will be said Poisson vector field if $\mathfrak{L}_X\omega_{M^A}=0$.

Proposition 5. *If* (M,ω_M) *is a Poisson manifold, then a vector field*

$$\theta : C^\infty(M)\to C^\infty(M)$$

is a Poisson vector field if and only if

$$\theta^A : C^\infty\left(M^A,A\right)\to C^\infty\left(M^A,A\right)$$

is a Poisson vector field.

Proof. indeed, for any $\theta\in\mathfrak{X}(M)$,

$$\mathfrak{L}_{\theta^A}\omega_{M^A}=\left[\mathfrak{L}_\theta\omega_M\right]^A.$$

Thus, $\mathfrak{L}_\theta \omega_M = 0$ if and only if $\mathfrak{L}_{\theta^A} \omega_{M^A} = 0$.

Proposition 6. *Let* (M^A, ω_{M^A}) *be a A-Poisson manifold. Then, all globally hamiltonian vector fields are Poisson vector fields.*

Proof. Let X be a globally hamiltonian vector field, then there exists $\varphi \in C^\infty(M^A, A)$ such that $X = ad(\varphi)$ *i.e.* X is the interior derivation of the Poisson A-algebra $C^\infty(M^A, A)$ [6]. For any ψ and $\phi \in C^\infty(M^A, A)$,

$$\mathfrak{L}_X \omega_{M^A}\left(\left[\delta^A_{M^A}(\psi), \delta^A_{M^A}(\phi)\right]\right) = \left(\mathfrak{L}_{ad(\varphi)}\omega_{M^A}\right)\left(\left[\delta^A_{M^A}(\psi), \delta^A_{M^A}(\phi)\right]\right)$$
$$= ad(\varphi)\left(\omega_{M^A}\left[\delta^A_{M^A}(\psi), \delta^A_{M^A}(\phi)\right]\right) - \left(\omega_{M^A}\left[\mathfrak{L}_{ad(\varphi)}\delta^A_{M^A}(\psi), \delta^A_{M^A}(\phi)\right]\right)$$
$$-\left(\omega_{M^A}\left[\delta^A_{M^A}(\psi), \mathfrak{L}_{ad(\varphi)}\delta^A_{M^A}(\phi)\right]\right)$$
$$= -ad(\varphi)(\{\psi, \phi\}) - \omega_{M^A}\left[\delta^A_{M^A}\{\varphi, \psi\}, \delta_M(\phi)\right] - \omega_{M^A}\left[\delta^A_{M^A}(\psi), \delta^A_{M^A}\{\varphi, \phi\}\right]$$
$$= -\{\varphi, \{\psi, \phi\}\} + \{\{\varphi, \psi\}, \phi\} + \{\psi, \{\varphi, \phi\}\} = -\left(\{\varphi, \{\psi, \phi\}\} + \{\phi, \{\varphi, \psi\}\} + \{\psi, \{\phi, \varphi\}\}\right) = 0.$$

Thus, all globally hamiltonian vector fields are Poisson vector fields.

When (M, Ω) is a symplectic manifold, then (M^A, Ω^A) is a symplectic A-manifold [6] [12]. For $\varphi \in C^\infty(M^A, A)$, we denote X_φ the unique vector field on M^A, considered as a derivation of $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$, such that

$$i_{X_\varphi}\Omega^A = d^A(\varphi),$$

where

$$d^A : \Lambda(M^A, A) \to \Lambda(M^A, A)$$

denotes the operator of cohomology associated with the representation

$$\mathfrak{X}(M^A) \to Der_A\left[C^\infty(M^A, A)\right], X \mapsto X.$$

When (M^A, Ω^A) is a symplectic A-manifold, then for any $X \in \mathfrak{X}(M^A)$,

$$\mathfrak{L}_X \Omega^A = d^A(i_X \Omega^A).$$

Therefore, all globally hamiltonian vector fields are Poisson vector fields.

Proposition 7. *For any* $\varphi \in C^\infty(M^A, A)$ *and for any Poisson vector field Y, we have*

$$[Y, X_\varphi] = X_{Y(\varphi)}.$$

Proof.

$$i_{[Y, X_\varphi]}\Omega^A = \left[\mathfrak{L}_Y, i_{X_\varphi}\right]\Omega^A = \mathfrak{L}_Y\left(i_{X_\varphi}\Omega^A\right) - i_{X_\varphi}\left(\mathfrak{L}_Y \Omega^A\right)$$
$$= \mathfrak{L}_Y\left(i_{X_\varphi}\Omega^A\right) = i_Y\left[d^A\left(d^A\varphi\right)\right] + d^A\left[i_Y\left(d^A\varphi\right)\right]$$
$$= d^A\left[i_Y\left(d^A\varphi\right)\right] = d^A\left[Y(\varphi)\right] = i_{X_{Y(\varphi)}}\Omega^A.$$

Thus,

$$[Y, X_\varphi] = X_{Y(\varphi)}.$$

3.2. Example

When $\alpha = \sum_{i=1}^n p_i dx_i$ is a Liouville form, where $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ is a local system of coordinates in the cotangent bundle T^*M of M, then (T^*M, $\Omega = d\alpha = \sum_{i=1}^n dp_i \wedge dx_i$) is a symplectic manifold on T^*M [7]. Let

α^A be the unique differential A-form of degree −1 on T^*M^A such that

$$d^A\left(\alpha^A\right)=\left[d\left(\alpha\right)\right]^A.$$

Thus,

$$\Omega^A=\left[d\left(\alpha\right)\right]^A=d^A\left(\alpha^A\right)=\sum_{i=1}^{n}d^A\left(p_i^A\right)\Lambda d^A\left(x_i^A\right).$$

Therefore, (T^*M^A, $\Omega^A=d^A\left(\alpha^A\right)$) is a symplectic A-manifold.

For $H\in C^\infty\left(M^A,A\right)$, let X_H be the globally hamiltonian vector field

$$i_{X_H}\Omega^A=-d^A\left(H\right).$$

As [13]

$$\frac{\partial}{\partial x_i^A}=\left(\frac{\partial}{\partial x_i}\right)^A,$$

we have

$$X_H=\sum_{i=1}^{n}f_i\cdot\frac{\partial}{\partial x_i^A}+\sum_{i=1}^{n}g_i\cdot\frac{\partial}{\partial p_i^A},$$

$$\begin{aligned}i_{X_H}\Omega^A&=\sum_{i=1}^{n}\left(i_{X_H}\Omega^A\right)\left(\frac{\partial}{\partial x_i^A}\right)d^A\left(x_i^A\right)+\sum_{i=1}^{n}\left(i_{X_H}\Omega^A\right)\left(\frac{\partial}{\partial p_i^A}\right)d^A\left(p_i^A\right)\\&=\sum_{i=1}^{n}\Omega^A\left(X_H,\frac{\partial}{\partial x_i^A}\right)d^A\left(x_i^A\right)+\sum_{i=1}^{n}\Omega^A\left(X_H,\frac{\partial}{\partial p_i^A}\right)d^A\left(p_i^A\right).\end{aligned}$$

As

$$\begin{aligned}\Omega^A\left(X_H,\frac{\partial}{\partial x_i^A}\right)&=\sum_{j=1}^{n}f_i\cdot\Omega^A\left(\frac{\partial}{\partial x_j^A},\frac{\partial}{\partial x_i^A}\right)+\sum_{j=1}^{n}g_j\cdot\Omega^A\left(\frac{\partial}{\partial p_j^A},\frac{\partial}{\partial x_i^A}\right)\\&=\sum_{j,k}^{n}f_j\cdot\left(d^A\left(p_k^A\right)\Lambda d^A\left(x_k^A\right)\right)\left(\frac{\partial}{\partial x_j^A},\frac{\partial}{\partial x_i^A}\right)+\sum_{j,k}^{n}g_j\cdot\left(d^A\left(p_k^A\right)\Lambda d^A\left(x_k^A\right)\right)\left(\frac{\partial}{\partial p_j^A},\frac{\partial}{\partial x_i^A}\right)\\&=\sum_{j,k}^{n}g_j\cdot\left[d^A\left(p_k^A\right)\left(\frac{\partial}{\partial p_j^A}\right)d^A\left(x_k^A\right)\left(\frac{\partial}{\partial x_i^A}\right)-d^A\left(p_k^A\right)\left(\frac{\partial}{\partial x_i^A}\right)d^A\left(x_k^A\right)\left(\frac{\partial}{\partial p_i^A}\right)\right]\\&=\sum_{j,k}^{n}g_j\cdot\left(\delta_{kj}\cdot\delta_{ki}\right)=g_j,\end{aligned}$$

and

$$\begin{aligned}\Omega^A\left(X_H,\frac{\partial}{\partial p_i^A}\right)&=\sum_{j=1}^{n}f_i\cdot\Omega^A\left(\frac{\partial}{\partial x_j^A},\frac{\partial}{\partial p_i^A}\right)+\sum_{j=1}^{n}g_j\cdot\Omega^A\left(\frac{\partial}{\partial p_j^A},\frac{\partial}{\partial p_i^A}\right)\\&=\sum_{j=1}^{n}f_i\cdot\Omega^A\left(\frac{\partial}{\partial x_j^A},\frac{\partial}{\partial p_i^A}\right)=\sum_{j,k}^{n}f_j\cdot\left(d^A\left(p_k^A\right)\Lambda d^A\left(x_k^A\right)\right)\left(\frac{\partial}{\partial x_j^A},\frac{\partial}{\partial p_i^A}\right)\\&=\sum_{j,k}^{n}f_j\cdot\left[d^A\left(p_k^A\right)\left(\frac{\partial}{\partial x_j^A}\right)d^A\left(x_k^A\right)\left(\frac{\partial}{\partial p_i^A}\right)-d^A\left(p_k^A\right)\left(\frac{\partial}{\partial p_i^A}\right)d^A\left(x_k^A\right)\left(\frac{\partial}{\partial x_j^A}\right)\right]\\&=\sum_{j,k}^{n}f_j\cdot\left(-\delta_{kj}\cdot\delta_{ki}\right)=-f_j.\end{aligned}$$

As,

$$i_{X_H}\Omega^A = \sum_{i=1}^{n} g_i \cdot d^A\left(x_i^A\right) + \sum_{i=1}^{n} f_i \cdot d^A\left(p_i^A\right) = -dH$$
$$= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i^A} d^A\left(x_i^A\right) - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i^A} d^A\left(p_i^A\right).$$

Thus,

$$\begin{cases} f_i = \dfrac{\partial H}{\partial p_i^A} \\ g_i = -\dfrac{\partial H}{\partial x_i^A} \end{cases}$$

where $f_i, g_i \in C^\infty\left(U^A, A\right)$. An integral curve of X_H is a solution the following system of ordinary equation

$$\begin{cases} \dfrac{\mathrm{d}^A\left(x_i^A\right)}{\mathrm{d}^A t} = \dfrac{\partial H}{\partial p_i^A} \\ \dfrac{\mathrm{d}^A\left(p_i^A\right)}{\mathrm{d}^A t} = -\dfrac{\partial H}{\partial x_i^A} \end{cases} \quad \forall i = 1, 2, \cdots, n.$$

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i^A} \frac{\partial}{\partial x_i^A} - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i^A} \frac{\partial}{\partial p_i^A}$$

When $\left(x_1, x_2, \cdots, x_{2n}\right)$ is a local system of coordinates corresponding at a chart U of M,

$$\Omega\big|_U = \sum_{i=1}^{n} dx_i \Lambda dx_{i+n}.$$

Thus,

$$\Omega^A\big|_{U^A} = \sum_{i=1}^{n} d^A\left(x_i^A\right) \Lambda d^A\left(x_{i+n}^A\right),$$

$$X\big|_{U^A} = \sum_{i=1}^{n} f_i \left(\frac{\partial}{\partial x_i}\right)^A + \sum_{i=1}^{n} f_{i+n} \left(\frac{\partial}{\partial x_{i+n}}\right)^A$$

where $f_i, f_{i+n} \in C^\infty\left(U^A, A\right)$ for $i = 1, 2, \cdots, n$. For $\varphi \in C^\infty\left(M^A, A\right)$,

$$x_\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_{n+i}^A} \frac{\partial}{\partial x_i^A} - \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i^A} \frac{\partial}{\partial x_{n+i}^A}.$$

$$i_{X_\varphi}\Omega^A = -\left[\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i^A} d^A\left(x_i^A\right) + \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_{n+i}} d^A\left(x_{n+i}^A\right)\right].$$

As

$$\{\varphi, \psi\}_{\Omega^A} = X_\varphi(\psi),$$

we have

$$\{\varphi, \psi\}_{\Omega^A} = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_{n+i}^A} \frac{\partial \psi}{\partial x_i^A} - \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i^A} \frac{\partial \psi}{\partial x_{n+i}^A}.$$

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