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Time Scale Approach to One Parameter Plane Motion by Complex Numbers

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Abstract

This paper presents building one-parameter motion by complex numbers on a time scale. Firstly, we assumed that E and E' were moving in a fixed time scale complex plane and $\{0, e_1, e_2\}$ and $\{0', e_1', e_2'\}$ were their orthonormal frames, respectively. By using complex numbers, we investigated the delta calculus equations of the motion on $\mathbb{T}$. Secondly, we gave the velocities and their union rule on the time scale. Finally, by using the delta-derivative, we got interesting results and theorems for the instantaneous rotation pole and the pole curves (trajectory). In kinematics, investigating one-parameter motion by complex numbers is important for simplifying motion calculation. In this study, our aim is to obtain an equation of motion by using complex numbers on the time scale.

Keywords

Complex Numbers, Kinematic, Time Scales, Pole Curve

1. Introduction

The calculus on time scales was initiated by B. Aulbach and S. Hilger in order to create a theory that can unify discrete and continuous analysis, [1]. Some preliminary definitions and theorems about delta derivative can be found in the references [2]-[4].

In this study, some properties of motion in references [5]-[7] are investigated by using time scale complex planes. We find delta calculus equations of the motion and finally we get some results about the pole curves.

2. Preliminaries

A time scale is an arbitrary nonempty closed subset of the real numbers.

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Definition 2.1. *Let* $\mathbb{T}$ *be any time scale. The forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ *is defined by*

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$$

and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t):=\sup\{s\in\mathbb{T}:s<t\},\quad \text{for } t\in\mathbb{T}$$

In this definition, we put $\inf\Phi=\sup\mathbb{T}$ (*i.e.* $\sigma(t)=t$, if $\mathbb{T}$ has a maximum *t*) and $\sup\Phi=\inf\mathbb{T}$ (*i.e.* $\rho(t)=t$, if $\mathbb{T}$ has a minimum *t*), where Φ denotes the empty set. If $\sigma(t)>t$, we say that *t* is right-scattered, while if $\rho(t)<t$ we say that *t* is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then *t* is called right-dense, and if $t>\inf\mathbb{T}$ and $\rho(t)=t$, then *t* is called left-dense. Points that are right-dense and left-dense at the same time are called dense.

Finally, the graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined by

$$\mu:=\sigma(t)-t.$$

If $f:\mathbb{T}\to\mathbb{R}$ is a function, then we define the function $f^{\sigma}:\mathbb{T}\to\mathbb{R}$ by

$$f^{\sigma}(t)=f(\sigma(t))\quad \text{for all } t\in\mathbb{T},\ \textit{i.e. } f^{\sigma}=f\circ\sigma.$$

Let us define the interior of $\mathbb{T}$ relative to α which is a function that maps $\mathbb{T}$ into $\mathbb{T}$ to be the set

$$\mathbb{T}^{k}=\{t\in\mathbb{T}:\text{either } \alpha(t)\neq t \text{ or } \alpha(t)=t \text{ and } t \text{ is not isolated}\}$$

Definition 2.2. *Assume* $f:\mathbb{T}\to\mathbb{T}$ *is a function and let* $t\in\mathbb{T}^{k}$. *Then we define* $f^{\Delta}(t)$ *to be the number* (*provided it exists*) *with the property that given any* $\varepsilon>0$, *there is a neighborhood U of t* (*i.e.* $U=(t-\delta,t+\delta)\cap\mathbb{T}$ *for some* $\delta>0$) *such that*

$$\left|\left[f(\sigma(t))-f(s)\right]-f^{\Delta}(t)\left[\sigma(t)-s\right]\right|\leq\varepsilon\left|\sigma(t)-s\right|\quad \text{for all } s\in U$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of *f* at *t*. Moreover, we say that *f* is delta (or Hilger) differentiable on $\mathbb{T}^{k}$ provided $f^{\Delta}(t)$ exists for all $t\in\mathbb{T}^{k}$.

Theorem 2.1. *Assume* $f:\mathbb{T}\to\mathbb{R}$ *is a function and let* $t\in\mathbb{T}^{k}$. *Then we have the following*:

1) If *f* is differentiable at *t*, then *f* is continuous at *t*.

2) If *f* is continuous at *t* and *t* is right-scattered, then *f* is differentiable at *t* with

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$$

3) If *t* is right-dense, then *f* is differential at *t* if the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case a given

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

4) If *f* is differentiable at *t* then

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$$

Theorem 2.2. *Assume* $f,g:\mathbb{T}\to\mathbb{R}$ *are differentiable at* $t\in\mathbb{T}^{k}$. *Then*:

1) The sum $f+g:\mathbb{T}\to\mathbb{R}$ is differentiable at *t* with

$$(f+g)^{\Delta}(t)=f^{\Delta}(t)+g^{\Delta}(t)$$

2) For any constant, $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$$

3) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t))$$

4) If $f(t) f(\sigma(t)) \neq 0$ then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t) f(\sigma(t))}$$

5) If $g(t) g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))}$$

In the reference [3], the chain rule on time scales is given for various cases.

Theorem 2.3. *Assume* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^k$*, and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then, there exists* c *in the real interval* $[t, \sigma(t)]$ *with*

$$(f \circ g)^{\Delta}(t) = f'(g(c)) g^{\Delta}(t)$$

Theorem 2.4. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and suppose* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable. Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t)) + h\mu(t) g^{\Delta}(t) \mathrm{d}h \right\} g^{\Delta}(t)$$

holds.

Theorem 2.5. *Assume that* $v : \mathbb{T} \to \mathbb{R}$ *is strictly increasing function and* $\tilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scale. Let* $w : \tilde{\mathbb{T}} \to \mathbb{R}$*. If* $v^{\Delta}(t)$ *and* $w^{\tilde{\Delta}}(v(t))$ *exist for* $t \in \mathbb{T}^k$*, then*

$$(w \circ v)^{\Delta} = \left(w^{\tilde{\Delta}} \circ v\right) v^{\Delta} \tag{2.1}$$

Definition 2.3. *For the given time scales* $\mathbb{T}_1$ *and* $\mathbb{T}_2$*, let us set*

$$\mathbb{T}_1 + \mathrm{i}\mathbb{T}_2 = \{ z = x + \mathrm{i}y : x \in \mathbb{T}_1, y \in \mathbb{T}_2 \} \tag{2.2}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. The set $\mathbb{T}_1 + \mathrm{i}\mathbb{T}_2$ is called the time scale complex plane.

Definition 2.4. *For* $h > 0$*, we define the cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ *by*

$$\xi_h(z) = \frac{1}{h} \mathrm{Log}(1 + zh)$$

and for $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$.

Definition 2.5. *If* $p \in \mathbb{R}$*, then we define the exponential function by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right) \quad \text{for } s, t \in \mathbb{T}$$

where the cylinder transformation $\xi_h(z)$ is introduced in Definition 2.4.

Theorem 2.6. *If* $p, q \in \mathbb{R}_h$ *then*

1) $e_0(t,s)\equiv 1$ and $e_p(t,t)\equiv 1$;

2) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$;

3) $\dfrac{1}{e_p(t,s)}=e_{\ominus p}t,s$;

4) $e_p(t,s)=\dfrac{1}{e_p(s,t)}=e_{\ominus p}(s,t)$;

5) $e_p(t,s)e_p(s,r)=e_p(t,r)$;

6) $e_p(t,s)e_q(t,s)=e_{p\oplus q}(t,s)$;

7) $\dfrac{e_p(t,s)}{e_q(t,s)}=e_{p\ominus q}(t,s)$;

8) $\left(\dfrac{1}{e_p(t,s)}\right)^{\Delta}=-\dfrac{p(t)}{e_p^{\sigma}(t,s)}$;

Theorem 2.7. *Assume* $\sigma(t)>t$ *for* $t\in\mathbb{T}^{\kappa}$

$$e_p^{\Delta}(t,t_0)=p(t)e_p(t,t_0)$$

Theorem 2.8. *If* $p,\ q\in\mathbb{R}_h$ *then*

$$e_{p\ominus q}^{\Delta}(t,t_0)=(p-q)\frac{e_p(t,t_0)}{e_q^{\sigma}(t,t_0)}$$

Theorem 2.9. *If* $p\in\mathbb{R}_h$ *and* $a,\ b,\ c\in\mathbb{T}$ *then*

$$\left[e_p(c,s)\right]^{\Delta}=-p\left[e_p(c,s)\right]^{\sigma}$$

3. One Parameter Motion and Hilger Complex Numbers on a Time Scale

Assume that $\mathbb{T}$ is a time scale. Let us set the time scale complex plane for as

$$\mathbb{T}+\mathrm{i}\mathbb{R}=\{z=x+\mathrm{i}y:x\in\mathbb{R},y\in\mathbb{R}\} \tag{3.1}$$

Here, let E and E' be moving in a fixed time scale complex plane. The motion is called as one-parameter planar motion by the complex numbers on the time scale and denoted as $H_1=E/E'$ for a planar motion of E relative to E'. $\{0,\boldsymbol{e}_1,\boldsymbol{e}_2\}$ and $\{0',\boldsymbol{e}_1',\boldsymbol{e}_2'\}$ be their orthonormal frames, respectively. We suppose that $\{0',\boldsymbol{e}_1',\boldsymbol{e}_2'\}$ is fixed, then we say that $\{0,\boldsymbol{e}_1,\boldsymbol{e}_2\}$ moves with respect to $\{0',\boldsymbol{e}_1',\boldsymbol{e}_2'\}$, $\boldsymbol{e}_i$, $(i=1,2)$ are the functions of a time scale parameter t. Let $\boldsymbol{x}=(x_1,x_2)$ and $\boldsymbol{x}'=(x_1',x_2')$ be the position vectors of a point X in the plane, as following we can write the coordinates of the point X by using complex numbers on the time scale with respect to a fixed or moving plane E and E', respectively. So:

$$\boldsymbol{x}=x_1+\mathrm{i}x_2 \quad \text{and} \quad \boldsymbol{x}'=x_1'+\mathrm{i}x_2'$$

The translation vector $\overrightarrow{O'O}$ can be written as the following equation on a fixed plane E':

$$\boldsymbol{u}=u_1'+\mathrm{i}u_2'$$

by using the definition of the time scale complex plane. The translation vector is more suitable as

$$\boldsymbol{u}=\overrightarrow{OO'}=-u_1-\mathrm{i}u_2$$

for doing the formulas symmetric on the moving plane.

Thus, $\boldsymbol{u}$ is equivalent to the vector $\overrightarrow{OO'}$. Let φ be a rotation angle between the vectors $\boldsymbol{e}_1'$ and $\boldsymbol{e}_1$ (or the time scale complex planes E and E'), in **Figure 1**. So we can find the equation

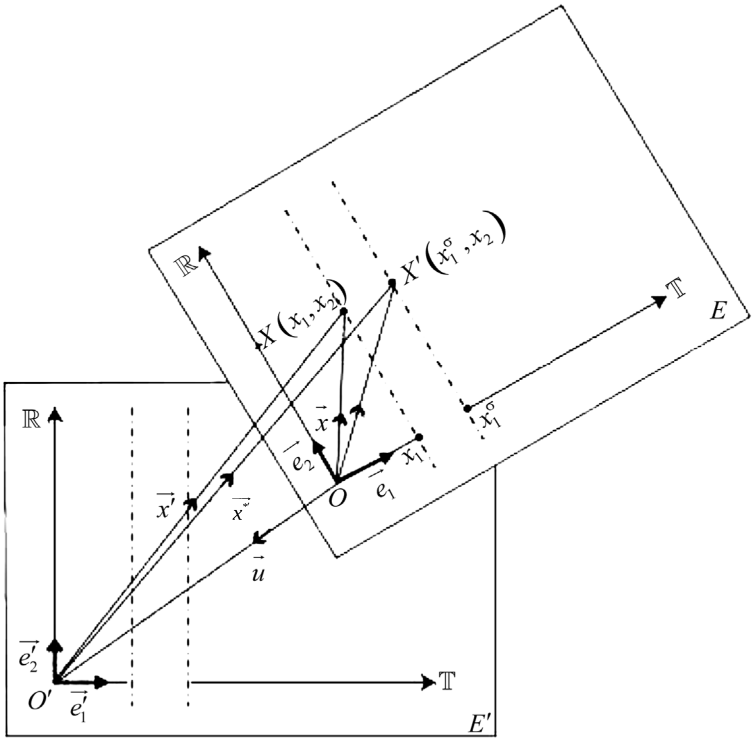

Figure 1. One parameter planar motion on time scale.

$$\boldsymbol{u} = -\boldsymbol{u}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right] \tag{3.2}$$

For any point X, the vector $\boldsymbol{x}'$ is

$$\boldsymbol{x}' = \boldsymbol{u}' + \boldsymbol{x}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right] \tag{3.3}$$

By substituting $\boldsymbol{u}'$ in the Equation (3.3)

$$\boldsymbol{x}' = -\boldsymbol{u}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right] + \boldsymbol{x}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]$$

$$\boldsymbol{x}' = \left(-\boldsymbol{u}+\boldsymbol{x}\right)\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right] \tag{3.4}$$

Then, we can obtain the vector $\boldsymbol{x}$ as follows:

$$\boldsymbol{x} = \boldsymbol{u} + \frac{\boldsymbol{x}'}{\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]} = \boldsymbol{u} + \boldsymbol{x}'\left[e_{\ominus p}\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]$$

Here, assume the functions

$$\boldsymbol{u} = \boldsymbol{u}\left(t\right),\quad \boldsymbol{u}' = \boldsymbol{u}\left(t\right),\quad \varphi = \varphi\left(t\right)$$

are Δ-differentiable functions and the parameter t is defined as $t_0 \le t \le t_1$ on the $\mathbb{T}$ time scale. We will calculate the formulas for a fixed or moving plane.

Definition 3.1. *A velocity vector of the point X with respect to E is called Δ-relative velocity vector of the point X on the time scale. The equation of relative Δ-velocity vector is*

$$\boldsymbol{x}_{\Delta r} = \frac{\mathrm{d}\boldsymbol{x}}{\Delta t} = \boldsymbol{x}^{\Delta} \tag{3.5}$$

for the moving time scale complex plane.

Definition 3.2. *A velocity vector of the point X with respect to E is called* Δ*-relative velocity vector of the point X on the time scale. The equation of the relative* Δ*-velocity vector is*

$$\boldsymbol{x}'_{\Delta r} = \boldsymbol{x}_{\Delta r}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right] \tag{3.6}$$

$$= \boldsymbol{x}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right] \tag{3.7}$$

for the fixed time scale complex plane.

Definition 3.3. *A velocity vector of the point X with respect to the time scale complex plane* E' *on the planar motion* $H_I = E/E'$ *which belongs to a curve* $\left(X'\right)$ *of the point* X' *on* E' *is called the* Δ*-absolute velocity vector of the point X on the time scale and is denoted by* $\boldsymbol{x}_{\Delta a}$.

Definition 3.4. *On the planar motion* $H_I = E/E'$*, while the point X is fixed on the moving time scale complex plane* E (*i.e.* $\boldsymbol{x}_{\Delta r} = \mathbf{0}$), *a velocity vector of the point X is called the* Δ*-dragging velocity vector of this point on the time scale and is denoted by* $\boldsymbol{x}_{\Delta f}$.

So, we obtain the Δ-absolute velocity $\boldsymbol{x}_{\Delta a}$, *i.e.* the velocity of *X* with respect to the plane E', from the Equation (3.4) using Equation (3.2).

$$\begin{aligned}
\boldsymbol{x}'_{\Delta a} = \frac{\mathrm{d}\boldsymbol{x}'}{\Delta t} = \boldsymbol{x}^{\Delta} &= \left\{\left(-\boldsymbol{u}+\boldsymbol{x}\right)\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\right\}^{\Delta} \\
&= \left\{-\boldsymbol{u}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]+\boldsymbol{x}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\right\}^{\Delta} \\
&= \left\{\boldsymbol{u}'+\boldsymbol{x}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\right\}^{\Delta} \\
&= \boldsymbol{u}'^{\Delta}+\boldsymbol{x}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]+\boldsymbol{x}^{\sigma}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]^{\Delta} \\
&= \boldsymbol{u}'^{\Delta}+\boldsymbol{x}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]+\boldsymbol{x}^{\sigma}\left[e_p^{\tilde{\Delta}}\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\mathrm{i}\varphi^{\Delta}\left(t\right).
\end{aligned}$$

by Theorem 2.5. Also

$$\boldsymbol{x}'_{\Delta a} = \boldsymbol{x}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]+\boldsymbol{u}'^{\Delta}+\boldsymbol{x}^{\sigma}\left[e_p^{\tilde{\Delta}}\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\mathrm{i}\varphi^{\Delta}\left(t\right)$$

and using Theorem 2.7, we have

$$\boldsymbol{x}'_{\Delta a} = \boldsymbol{x}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]+\boldsymbol{u}'^{\Delta}+\boldsymbol{x}^{\sigma}\left[pe_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\mathrm{i}\varphi^{\Delta}\left(t\right)$$

Here, φ^{Δ} is called a delta-angular velocity of the motion H_I on a time scale, and remembering Equations (3.3) and (3.7), we can find the dragging velocity vector $\boldsymbol{x}_{\Delta f}$ of the point *X*

$$\boldsymbol{x}_{\Delta f} = \boldsymbol{u}'^{\Delta}+\mathrm{i}\varphi^{\Delta}\left(t\right)\boldsymbol{x}^{\sigma}\left[p\cdot e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right] \tag{3.8}$$

$$= \boldsymbol{u}'^{\Delta}+\mathrm{i}p\varphi^{\Delta}\left(t\right)\left(\boldsymbol{x}'^{\sigma}-\boldsymbol{u}'^{\sigma}\right) \tag{3.9}$$

with the restriction $\varphi^{\Delta}\neq 0$, from Equation (3.2) by taking the Δ-derivative with respect to the parameter *t*, we get the following equation.

$$\begin{aligned}
u'^{\Delta} &= \left[-\boldsymbol{u}e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]^{\Delta} \\
&= -\boldsymbol{u}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]-\boldsymbol{u}^{\sigma}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]^{\Delta} \\
&= -\boldsymbol{u}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]-\boldsymbol{u}^{\sigma}\left[e_p^{\tilde{\Delta}}\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\mathrm{i}\varphi^{\Delta}\left(t\right) \\
&= -\boldsymbol{u}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]-\boldsymbol{u}^{\sigma}\left[pe_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]\mathrm{i}\varphi^{\Delta}\left(t\right) \\
&= -\boldsymbol{u}^{\Delta}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right]-\mathrm{i}p\varphi^{\Delta}\left(t\right)\boldsymbol{u}^{\sigma}\left[e_p\left(t,t_0\right)\circ\left(\mathrm{i}\varphi\left(t\right)\right)\right].
\end{aligned}$$

and using Equation (3.2), we get

$$u'^{\Delta} = -\boldsymbol{u}^{\Delta}\left[e_p(t,t_0)\circ(\mathrm{i}\varphi(t))\right] + \mathrm{i}p\varphi^{\Delta}(t)\boldsymbol{u}'^{\sigma} \tag{3.10}$$

Theorem 3.1. *A Δ-absolute velocity vector is equal to adding a Δ-relative velocity vector and Δ-dragging velocity vector on the motion $H_I = E/E'$, i.e.*

$$x_{\Delta a} = x_{\Delta f} + x_{\Delta r} \tag{3.11}$$

Proof. By using Equation (3.10) and Equation (3.5), we can get the following equations:

$$\begin{aligned} \boldsymbol{x}_{\Delta a} &= \boldsymbol{x}'_{\Delta a}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] \\ &= \left\{\boldsymbol{x}^{\Delta}\left[e_p(t,t_0)\circ(\mathrm{i}\varphi(t))\right] + \boldsymbol{u}'^{\Delta} + \boldsymbol{x}^{\sigma}\left[pe_p(t,t_0)\circ(\mathrm{i}\varphi(t))\right]\mathrm{i}\varphi^{\Delta}(t)\right\}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] \\ &= \boldsymbol{x}^{\Delta} + \boldsymbol{u}'^{\Delta}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] + p\boldsymbol{x}^{\sigma}\mathrm{i}\varphi(t) \\ &= \boldsymbol{x}_{\Delta r} + \boldsymbol{x}_{\Delta f}. \end{aligned}$$

and thus, we get the relation of the velocities:

$$\boldsymbol{x}_{\Delta a} = \boldsymbol{x}_{\Delta r} + \boldsymbol{x}_{\Delta r}$$

We have

$$\boldsymbol{x}_{\Delta f} = \boldsymbol{x}'_{\Delta f}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right]$$

We will calculate $\boldsymbol{x}_{\Delta f}$ here using Equation (3.9) and Equation (3.10);

$$\begin{aligned} \boldsymbol{x}_{\Delta f} &= \left\{u'^{\Delta} + \mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{x}'^{\sigma} - u'^{\sigma}\right)\right\}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] \\ &= u'^{\Delta}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] + \mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{x}'^{\sigma} - u'^{\sigma}\right)\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] \\ &= \left\{-\boldsymbol{u}^{\Delta}\left[e_p(t,t_0)\circ(\mathrm{i}\varphi(t))\right] + \mathrm{i}p\varphi^{\Delta}(t)\boldsymbol{u}'^{\sigma}\right\}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] \\ &\quad + \mathrm{i}p\varphi^{\Delta}(t)\boldsymbol{x}'^{\sigma}\left\{\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] - \mathrm{i}p\varphi^{\Delta}(t)\boldsymbol{u}'^{\sigma}\right\}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right] \\ &= -\boldsymbol{u}^{\Delta} + \mathrm{i}p\varphi^{\Delta}(t)\boldsymbol{x}'^{\sigma}\left[e_{\ominus p}(t,t_0)\circ(\mathrm{i}\varphi(t))\right]. \end{aligned}$$

and

$$\boldsymbol{x}_{\Delta f} = -\boldsymbol{u}^{\Delta} + \mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{x}^{\sigma} - \boldsymbol{u}^{\sigma}\right) \tag{3.12}$$

Theorem 3.2. *There is only one point at which the Δ-dragging velocity is zero for any instant $t \in \mathbb{T}$, i.e. which is fixed on the both of the planes E and E', with the restriction $\varphi^{\Delta} \neq 0$ on the motion $H_I = E/E'$.*

Proof. The points at which the Δ-dragging velocity vector is zero for any instant $t \in \mathbb{T}$ have to stay fixed for not only the plane E, but also for the plane E' on the motion $H_I = E/E'$. By taking $x_{\Delta f} = x'_{\Delta f} = 0$ for fixed and moving planes, from (3.15) and (3.8):

$$\boldsymbol{x}_{\Delta f} = -\boldsymbol{u}^{\Delta} + \mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{x}^{\sigma} - \boldsymbol{u}^{\sigma}\right) = 0 \tag{3.13}$$

$$\boldsymbol{x}'_{\Delta f} = \boldsymbol{u}'^{\Delta} + \mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{x}'^{\sigma} - \boldsymbol{u}'^{\sigma}\right) = 0 \tag{3.14}$$

we can obtain the following complex vectors;

$$\boldsymbol{x}^{\sigma} = P^{\sigma} = \boldsymbol{u}^{\sigma} + \mathrm{i}\frac{\boldsymbol{u}^{\Delta}}{p\varphi^{\Delta}(t)} \tag{3.15}$$

$$\boldsymbol{x}'^{\sigma} = P'^{\sigma} = \boldsymbol{u}'^{\sigma} - \mathrm{i}\frac{\boldsymbol{u}'^{\Delta}}{p\varphi^{\Delta}(t)} \tag{3.16}$$

which are given σ-instantaneous rotation pole *P* on both coordinate systems. Because, the affine axioms P^{σ}, P'^{σ} are the end-points of X^{σ}, X'^{σ}, respectively.

Definition 3.5. *The point* P^{σ} *which corresponds to the position vector* $P^{\sigma}=\left(p_1^{\sigma},p_2^{\sigma}\right)$ *is called the forward pole or the instantaneous rotation pole or the instantaneous rotation center for the moving plane on the time scale motion* H_I, *in* **Figure 2**.

Definition 3.6. *The point* P'^{σ} *which corresponds to the position vector* $P'\sigma=\left(p_1'^{\sigma},p_2'^{\sigma}\right)$ *is called the forward pole or the instantaneous rotation pole or the instantaneous rotation center for the fixed plane on the time scale motion* H_I, *in* **Figure 2**.

We can get the following equations from Equation (3.15) and Equation (3.16):

$$\boldsymbol{u}^{\Delta}=\mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{P}^{\sigma}-\boldsymbol{u}^{\sigma}\right) \tag{3.17}$$

$$\boldsymbol{u}'^{\Delta}=-\mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{P}'^{\sigma}-\boldsymbol{u}'^{\sigma}\right) \tag{3.18}$$

By eliminating $\boldsymbol{u}^{\Delta}$ and $\boldsymbol{u}'^{\Delta}$ from Equation (3.13) and Equation (3.14), the dragging velocity becomes as following:

$$\boldsymbol{x}_{\Delta f}=\mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{x}^{\sigma}-\boldsymbol{P}^{\sigma}\right) \tag{3.19}$$

$$=\mathrm{i}p\varphi^{\Delta}(t)\boldsymbol{P}^{\sigma}\boldsymbol{x}^{\sigma} \tag{3.20}$$

and;

$$\begin{aligned}\boldsymbol{x}'_{\Delta f}&=\mathrm{i}p\varphi^{\Delta}(t)\left(\boldsymbol{x}'^{\sigma}-\boldsymbol{P}'^{\sigma}\right)\\&=\mathrm{i}p\varphi^{\Delta}(t)\boldsymbol{P}'^{\sigma}\boldsymbol{x}'^{\sigma}.\end{aligned}$$

4. Conclusions

Result 4.1. *Two results for the* Δ*-dragging velocity of the point* X *on the moving plane can be obtained as follows*:

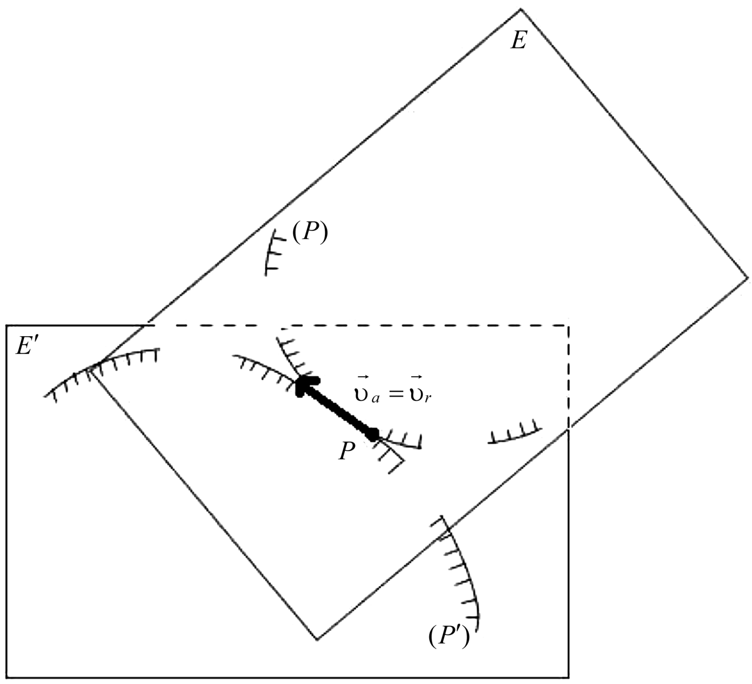

Figure 2. The pole curve on time scale.

1) Since scalar product of the vector is

$$\boldsymbol{P}^{\sigma}\boldsymbol{X}^{\sigma}=\left(x_1^{\sigma}-p_1^{\sigma}\right)+\mathrm{i}\left(x_2^{\sigma}-p_2^{\sigma}\right)$$

and the vector $\boldsymbol{x}_{\Delta f}$ is zero, these vectors are perpendicular.

2) The length of the vector $\boldsymbol{x}_f$ can be calculated as follows:

$$\left|\boldsymbol{x}_{\Delta f}\right|=\sqrt{\left(x_1^{\sigma}-p_1^{\sigma}\right)^2+\left(x_2^{\sigma}-p_2^{\sigma}\right)^2}\,p\varphi^{\Delta}=\left|\boldsymbol{P}^{\sigma}\boldsymbol{X}^{\sigma}\right|p\varphi^{\Delta}$$

here $\left|\boldsymbol{P}^{\sigma}\boldsymbol{X}^{\sigma}\right|$ denotes for the length of $\boldsymbol{P}^{\sigma}\boldsymbol{X}^{\sigma}$. From this result, we get the following theorem:

Theorem 4.1. *On the motion* H_I, *the points X of the moving plane E draw trajectories on the fixed time scale complex plane* E' *which their normals* (*trajectory normals*) *pass from the instantaneous rotation pole* P^{σ}.

Theorem 4.2. *Every point of X of the moving plane E is doing rotational movement* (*instantaneous rotation movement*) *with a* P^{σ}*-centered,* φ^{Δ}*-angular velocity and p factor on instant t.*

Since *X* is an arbitrary point of the time scale complex plane *E*, we can give the following theorem:

Theorem 4.3. *A one-parameter motion consists of rotation with* φ^{Δ} *angular velocity and p factor around the instantaneous rotation pole* P^{σ} *of the moving plane E on t instant, i.e. the plane E rotates with the angle* φ^{Δ} *and the factor p around the point* P^{σ} *on the time element* Δt.

Theorem 4.4. *The velocity vectors of the instantaneous rotation pole* P^{σ} *which draws the forward pole curves on the moving and fixed planes is the same vector at each instant t.*

Theorem 4.5. *On one-parameter planar motion* H_I *the moving pole curve* $\left(P^{\sigma}\right)$ *of the plane E rolls onto the fixed pole curve* $\left(P'^{\sigma}\right)$ *of the plane* E' *without sliding.*

Result 4.2. *Without being depended on time, a motion* H_I *occurs by rolling, without sliding, the curve* (P) *of E onto the curve* (P') *of* E'.

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