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Upper Bound Estimation of Fractal Dimensions of Fractional Integral of Continuous Functions

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Abstract

Fractional integral of continuous functions has been discussed in the present paper. If the order of Riemann-Liouville fractional integral is *v*, fractal dimension of Riemann-Liouville fractional integral of any continuous functions on a closed interval is no more than 2 − *v*.

Keywords

Box Dimension, Riemann-Liouville Fractional Calculus, Fractal Function

1. Introduction

In [1], fractional integral of a continuous function of bounded variation on a closed interval has been proved to still be a continuous function of bounded variation. The upper bound of Box dimension of the Weyl-Marchaud fractional derivative of self-affine curves has given in [2]. Previous discussion about fractal dimensions of fractional calculus of certain special functions can be found in [3] [4].

In the present paper, we discuss fractional integral of fractal dimension of any continuous functions on a closed interval.

If U is any non-empty subset of n-dimensional Euclidean space, R^n, the diameter of U is defined as $|U| = \sup\{|x-y| : x, y \in U\}$, *i.e.* the greatest distance apart of any pair of points in U. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, *i.e.* $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F.

Suppose that F is a subset of R^n and s is a non-negative number. For any positive number define.

$$\mathcal{H}_\delta^s(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}$$

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Write

$$\mathcal{H}^s(F)=\lim_{\delta\to 0}\mathcal{H}^s_\delta(F)$$

$\mathcal{H}^s_\delta(F)$ is called s-dimensional Hausdorff measure of *F*. Hausdorff dimension is defined as follows:

Definition 1.1 [5] Let *F* be a subset of R^n and *s* is a non-negative number. Hausdorff dimension of *F* is

$$\dim_H(F)=\inf\left\{s:\mathcal{H}^s(F)=0\right\}=\sup\left\{s:\mathcal{H}^s(F)=\infty\right\}$$

If $s=\dim_H(F)$, then $\mathcal{H}^s(F)$ may be zero or infinite, or may satisfy

$$0<\mathcal{H}^s(F)<\infty$$

A Borel set satisfying this last condition is called an *s*-set.

Box dimension is given as follows:

Definition 1.2 [5] Let *F* be any non-empty bounded subset of R^n and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover *F*. Lower and upper Box dimensions of *F* respectively are defined as

$$\underline{\dim}_B(F)=\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \tag{1.1}$$

and

$$\overline{\dim}_B(F)=\overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \tag{1.2}$$

If (1.1) and (1.2) are equal, we refer to the common value as Box dimension of *F*

$$\dim_B(F)=\lim_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \tag{1.3}$$

Definition 1.3 [6] Let $f(x)\in C_{[0,1]}$ and $v>0$. For $t\in[0,1]$ we call

$$D^{-v}f(x)=\frac{1}{\Gamma(v)}\int_0^x(x-t)^{v-1}f(t)\mathrm{d}t$$

Riemann-Liouville integral of $f(x)$ of order *v*.

2. Riemann-Liouville Fractional Integral of 1-Dimensional Fractal Function

Let $f(x)$ be a 1-dimensional fractal function on *I*. We will prove that Riemann-Liouville fractional integral of $f(x)$ is bounded on *I*. Box dimension of Riemann-Liouville fractional integral of $f(x)$ will be estimated.

2.1. Riemann-Liouville Fractional Integral of $f(x)$

Theorem 2.1 Let $D^{-v}f(x)$ be Riemann-Liouville integral of $f(x)$ of order *v*. Then, $D^{-v}f(x)$ is bounded.

Proof. Since $f(x)$ is continuous on a closed interval *I*, there exists a positive constant *M* such that

$$|f(x)|\le M \quad \forall x\in I$$

From **Definition 1.3**, we know

$$D^{-v}f(x)=\frac{1}{\Gamma(v)}\int_0^x(x-t)^{v-1}f(t)\mathrm{d}t \quad 0<v<1$$

For any $x\in I$, it holds

$$\left|D^{-v} f(x)\right| \leq \frac{M}{v\Gamma(v)} x^{v} \leq \frac{M}{\Gamma(v+1)} \qquad 0<v<1$$

$D^{-v} f(x)$ is a bounded function on *I*.

2.2. Fractal Dimensions of Riemann-Liouville Fractional Integral of $f(x)$

Theorem 2.2 Let $D^{-v} f(x)$ be Riemann-Liouville integral of $f(x)$ of order *v*. Then,

$$1 \leq \dim_H \Gamma\left(D^{-v} f, I\right) \leq \overline{\dim}_B \Gamma\left(D^{-v} f, I\right) \leq 2-v, \qquad 0<v<1$$

Proof. Let $0<\delta<1/2$, and *m* is the least integer greater than or equal to $1/\delta$. If $0 \leq a_1 < b_1 \leq \delta$, we have

$$\Gamma(v)\left[D^{-v} f(b_1)-D^{-v} f(a_1)\right]=\left(\int_0^{b_1}(b_1-t)^{v-1} f(t)\mathrm{d}t-\int_0^{a_1}(a_1-t)^{v-1} f(t)\mathrm{d}t\right)$$
$$=\int_0^{a_1}\left[(b_1-t)^{v-1}-(a_1-t)^{v-1}\right] f(t)\mathrm{d}t+\int_{a_1}^{b_1}(b_1-t)^{v-1} f(t)\mathrm{d}t.$$

For $1 \leq i \leq m$, let $M_i=\max_{x\in[(i-1)\delta, i\delta]} f(x)$, $m_i=\min_{x\in[(i-1)\delta, i\delta]} f(x)$ $M=\max_{x\in I} f(x)$ If $D^{-v} f(b_1)-D^{-v} f(a_1) \geq 0$, it holds

$$\Gamma(v+1)\left[D^{-v} f(b_1)-D^{-v} f(a_1)\right] \leq (b_1-a_1)^{v}(M_1-m_1)+\left(b_1^{v}-a_1^{v}\right) m_1$$

If $D^{-v} f(b_1)-D^{-v} f(a_1)<0$, it holds

$$\Gamma(v+1)\left|D^{-v} f(b_1)-D^{-v} f(a_1)\right| \leq (b_1-a_1)^{v}(M_1-m_1)$$

We have

$$\left|D^{-v} f(b_1)-D^{-v} f(a_1)\right| \leq \frac{1}{\Gamma(v+1)}(b_1-a_1)^{v}(M_1-m_1)+M\delta^{v}$$

Let $1 \leq n \leq m-1$. If $n\delta \leq a_{n+1} \leq b_{n+1} \leq (n+1)\delta$, we have

$$\Gamma(v)\left[D^{-v} f(b_{n+1})-D^{-v} f(a_{n+1})\right]=\int_0^{n\delta+b_{n+1}}(n\delta+b_{n+1}-t)^{v-1} f(t)\mathrm{d}t-\int_0^{n\delta+a_{n+1}}(n\delta+a_{n+1}-t)^{v-1} f(t)\mathrm{d}t$$
$$=\int_0^{n\delta}\left[(n\delta+b_{n+1}-t)^{v-1}-(n\delta+a_{n+1}-t)^{v-1}\right] f(t)\mathrm{d}t$$
$$+\int_{n\delta}^{n\delta+a_{n+1}}\left[(n\delta+b_{n+1}-t)^{v-1}-(n\delta+a_{n+1}-t)^{v-1}\right] f(t)\mathrm{d}t$$
$$+\int_{n\delta+a_{n+1}}^{n\delta+b_{n+1}}(n\delta+b_{n+1}-t)^{v-1} f(t)\mathrm{d}t.$$

If $D^{-v} f(b_{n+1})-D^{-v} f(a_{n+1}) \geq 0$, it holds

$$\Gamma(v+1)\left[D^{-v} f(b_{n+1})-D^{-v} f(a_{n+1})\right] \leq (b_{n+1}-a_{n+1})^{v}(M_{n+1}-m_{n+1})+\left(b_{n+1}^{v}-a_{n+1}^{v}\right) m_{n+1}$$

If $D^{-v} f(b_{n+1})-D^{-v} f(a_{n+1})<0$, it holds

$$\Gamma(v+1)\left|D^{-v} f(b_{n+1})-D^{-v} f(a_{n+1})\right| \leq (b_{n+1}-a_{n+1})^{v}(M_{n+1}-m_{n+1})+(2\delta)^{v} M$$

We get

$$\left|D^{-v} f(b_{n+1})-D^{-v} f(a_{n+1})\right| \leq \frac{1}{\Gamma(v+1)}(b_{n+1}-a_{n+1})^{v}(M_{n+1}-m_{n+1})+2M\delta^{v}$$

There exists a positive constant *C*, such that

$$R_{D^{-v}f}\left[i\delta,(i+1)\delta\right]\le C\delta^{v},\quad 1\le i\le m-1$$

If $N_\delta\left(D^{-v}f\right)$ is the number of squares of the δ mesh that intersects $\Gamma\left(D^{-v}f,I\right)$, by Proposition 11.1 of [1], we have

$$N_\delta\left(D^{-v}f\right)\le 2m+\delta^{-1}\sum_{i=0}^{m-1}R_{D^{-v}f}\left[i\delta,(i+1)\delta\right]\le C\delta^{v-2}$$

From (1.2) of **Definition 1.2**, we know

$$\overline{\dim}_B\Gamma\left(D^{-v}f,I\right)=\overline{\lim}_{\delta\to 0}\frac{\log N_\delta\left(D^{-v}f\right)}{-\log\delta}\le 2-v,\quad 0<v<1$$

With **Definition 1.1**, we get the conclusion of **Theorem 2.2**.

This is the first time to give estimation of fractal dimensions of fractional integral of any continuous function on a closed interval.

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