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# On the Initial Subalgebra of a Graded Lie Algebra

**Thomas B. Gregory**

Department of Mathematics, The Ohio State University at Mansfield, Mansfield, Ohio, USA
Email: gregory.6@osu.edu

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## Abstract

**We show that each irreducible, transitive finite-dimensional graded Lie algebra over a field of prime characteristic *p* contains an initial subalgebra in which the $p^{th}$ power of the adjoint transformation associated with any element in the lowest gradation space is zero.**

## Keywords

**Prime-Characteristic Lie Algebras**

## 1. Introduction

In the classification of the simple finite-dimensional Lie algebras over fields of prime characteristic, irreducible transitive finite dimensional graded Lie algebras play a fundamental role [1]. The simple finite dimensional Lie algebras over algebraically closed fields of characteristic greater than three have been classified [2]. Work is being done in characteristic three [3]-[7]. It is well known that in Lie algebras of Cartan type, there is a (not necessarily proper) subalgebra, the "initial piece," which contains the sum of the negative gradations spaces of the Lie algebra, and in which the $p^{th}$ power of the adjoint representation associated with any element of the lowest gradation space is zero. In this paper, we prove that any irreducible, transitive finite-dimensional graded Lie algebra contains such an initial subalgebra. Indeed, we prove the following theorem.

## 2. Main Theorem

Let

$$G = G_{-q} \oplus \cdots \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus \cdots \oplus G_r, \quad r \geq 1, \quad q \geq 1$$

be an irreducible, transitive, finite-dimensional graded Lie algebra over a field of characteristic $p$ such that

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$M(G) = 0$ [8]. Then $G$ contains an irreducible, transitive depth-$q$ graded subalgebra

$$R = \prod_{0 \le i \le I} (\operatorname{ad} v_i)^{(p-1)p^{j_i}} G + G_0,$$

where $v_i \in G_{-q}$, and where $I$ is a non-negative whole number. We have $G_{-q} \subseteq R$, $G_{-1} \subseteq R$, and $(\operatorname{ad} v)^p R = 0$ for all $v \in G_{-q}$.

If $q \ge r$, then the conclusion of the theorem obviously holds. In what follows, therefore, we will assume that $r > q$.

## 2. Intermediate Results

To prove the Main Theorem, we will make use of the following series of lemmas, in which we assume the hypotheses and notation of the Main Theorem. We note that by, for example, [9] (Lemma 6), $G$ is transitive in its negative part. (Note that the lemmas we quote from [9] are valid for all prime characteristics.) As usual, we assume throughout that $M(G) = 0$ [8].

**Lemma 1.** *If M is an abelian* $G_0$*-submodule of G, then for any* $m \in M$, $(\operatorname{ad} m)^{p^j}$ *is a* $G_0$*-endomorphism of G for all* $j > 0$.

**Proof.** For any $g \in G_0$, $m \in M$, and $x \in G$, we have

$$[m,[m, g]] \subseteq [m, M] = 0$$

so that modulo $p$

$$\begin{aligned}(\operatorname{ad} m)^{p^j}[g,x] &= \sum_{0 \le k \le p^j} \binom{p^j}{k} \left[(\operatorname{ad} m)^{p^j-k} g, (\operatorname{ad} m)^k x\right] \\ &= p^j \left[[m,g],(\operatorname{ad} m)^{p^j-1} x\right] + \left[g,(\operatorname{ad} m)^{p^j} x\right] \equiv \left[g,(\operatorname{ad} m)^{p^j} x\right].\end{aligned}$$

□

**Lemma 2.** *If* $m \in G$ *is such that* $(\operatorname{ad} m)^{p^j} G_0 = 0$ *for some* $j > 0$*, then* $(\operatorname{ad} m)^{p^j}$ *is a* $G_0$*-endomorphism of G.*

**Proof.** As in the proof of Lemma 1 above, we have, for any $g \in G_0$ and any $x \in G$, that modulo $p$,

$$\begin{aligned}(\operatorname{ad} m)^{p^j}[g,x] &= \sum_{0 \le k \le p^j} \binom{p^j}{k} \left[(\operatorname{ad} m)^{p^j-k} g, (\operatorname{ad} m)^k x\right] \equiv \left[(\operatorname{ad} m)^{p^j} g, x\right] + \left[g,(\operatorname{ad} m)^{p^j} x\right] \\ &= [0,x] + \left[g,(\operatorname{ad} m)^{p^j} x\right] = \left[g,(\operatorname{ad} m)^{p^j} x\right].\end{aligned}$$

□

**Lemma 3.** *If* $v \in G_{-q}$*, and* $j > 0$ *is maximal such that* $(\operatorname{ad} v)^{p^j} \ne 0$*, then* $(\operatorname{ad} v)^{p^j} G_{p^j q}$ *is a Lie subalgebra.*

**Proof.** Let $x_1$ and $x_2$ be any elements of $G_{p^j q}$, Then for any $v \in G_{-q}$,

$$\left[(\operatorname{ad} v)^{p^j} x_1, (\operatorname{ad} v)^{p^j} x_2\right] = (\operatorname{ad} v)^{p^j}\left[(\operatorname{ad} v)^{p^j} x_1, x_2\right]$$

since, as we have seen in the proofs of the previous lemmas, $(\operatorname{ad} v)^{p^j}$ is a derivation, and $\left((\operatorname{ad} v)^{p^j}\right)^2 x_1 \in G_{-p^j q} = 0$. In addition, since $(\operatorname{ad} v)^{p^j} x_1 \in G_0$ we have $\left[(\operatorname{ad} v)^{p^j} x_1, x_2\right] \in G_{p^j q}$. Hence, $(\operatorname{ad} v)^{p^j}\left[(\operatorname{ad} v)^{p^j} x_1, x_2\right] \in (\operatorname{ad} v)^{p^j} G_{p^i q}$, which, as it is obviously closed under addition, is seen to be a Lie subalgebra, as required. □

**Lemma 4.** *Let I be the minimal (graded) ideal of G* [8]. *If* $v \in G_{-q}$ *is such that* $(\operatorname{ad} v)^{p^j} I_k = 0$ *for some integers j and k, with* $j \ge 0$ *and* $(p^j - 1)q \le k \le r$*, then* $(\operatorname{ad} v)^{p^j} I_m = 0$ *for all m,* $-q \le m \le r$*, i.e.,* $(\operatorname{ad} v)^{p^j} I = 0$.

**Proof.** Suppose $(\operatorname{ad} v)^{p^j} I_k = 0$. Then for all $m$, $k < m \le r$, we have (since for all $m$, $-q \le m < r$, we have

$I_m = [G_{-1}, I_{m+1}]$)

$$(\operatorname{ad} G_{-1})^{m-k} (\operatorname{ad} v)^{p^j} I_m = (\operatorname{ad} v)^{p^j} (\operatorname{ad} G_{-1})^{m-k} I_m = (\operatorname{ad} v)^{p^j} I_k = 0$$

so $(\operatorname{ad} v)^{p^j} I_m = 0$ by transitivity. If $m < k$, then

$$(\operatorname{ad} v)^{p^j} I_m = (\operatorname{ad} v)^{p^j} (\operatorname{ad} G_{-1})^{k-m} I_k = (\operatorname{ad} G_{-1})^{k-m} (\operatorname{ad} v)^{p^j} I_k = 0 \quad \square$$

**Lemma 5.** *If* $(\operatorname{ad} v)^{p^j} G_k = 0$ *for some* $k$ *such that* $p^j q - q \le k \le r$ *and for some* $j > 0$, *then* $(\operatorname{ad} v)^{p^j} G = 0$.

**Proof.** We will show that $(\operatorname{ad} v)^{p^j} G_m = 0$ for all $m$, $p^j q - q \le m \le r$. (If $m < p^j q - q$, then $(\operatorname{ad} v)^{p^j} G_m \subseteq \sum_{n<-q} G_n = 0$.) First of all, suppose that $p^j q - q < m \le r$. Then, since $(\operatorname{ad} v)^{p^j} I_k \subseteq (\operatorname{ad} v)^{p^j} G_k = 0$, we have, by Lemma 4 that $(\operatorname{ad} v)^{p^j} I = 0$. Consequently, we have

$$0 = (\operatorname{ad} v)^{p^j} I_{m-1} = (\operatorname{ad} v)^{p^j} [G_{-1}, G_m] = \left[G_{-1}, (\operatorname{ad} v)^{p^j} G_m\right]$$

so $(\operatorname{ad} v)^{p^j} G_m = 0$ by the transitivity of $G$, if $m \ge p^j q$, or [9] (Lemma 6) otherwise. Finally, if $m = p^j q - q$ and $(\operatorname{ad} v)^{p^j} G_m \ne 0$, then by Lemma 1 (or Lemma 2), $(\operatorname{ad} v)^{p^j} G_m$ is a non-zero $G_0$-submodule of $G_{-q}$. But by, for example, [9] (Lemma 9), $G_{-q}$ is irreducible as a $G_0$ module; therefore, $(\operatorname{ad} v)^{p^j} G_m = G_{-q}$, and we have

$$\left[G_{-q}, I\right] = \left[(\operatorname{ad} v)^{p^j} G_m, I\right] = (\operatorname{ad} v)^{p^j} [G_m, I] \subseteq (\operatorname{ad} v)^{p^j} I = 0$$

(by Lemma 4, as we noted earlier in the proof). But then, since $G_{-1} \subset I$, we would have

$$0 = \left[G_{-q}, I_1\right] \supseteq \left[G_{-q}, [G_{-1}, G_2]\right] = \left[G_{-1}, \left[G_{-q}, G_2\right]\right]$$

so $\left[G_{-q}, G_2\right] = 0$ by, for example, [9] (Lemma 6), to contradict, for example, [9] (Lemma 8). Thus, we must have $(\operatorname{ad} v)^{p^j} G_m = 0$ in this case, also, so $(\operatorname{ad} v)^{p^j} G = 0$ as required. $\square$

**Lemma 6.** *If* $(\operatorname{ad} v)^{p^j} G \ne 0$ *for some* $v \in G_{-q}$ *and* $j > 0$, *then both* $G_{p^j q}$ *and* $(\operatorname{ad} v)^{p^j} G_{p^j q}$ *are non-zero, and* $G_{-q} \subseteq (\operatorname{ad} v)^{p^j} G$.

**Proof.** If $(\operatorname{ad} v)^{p^j} G \ne 0$, then $r \ge p^j q - q$, since otherwise we would have $(\operatorname{ad} v)^{p^j} G \subseteq \sum_{n<-q} G_n = 0$, contrary to hypothesis. By Lemma 5, $(\operatorname{ad} v)^{p^j} G_{p^j q - q}$ is not zero, and by Lemma 1 (or Lemma 2), $(\operatorname{ad} v)^{p^j} G_{p^j q - q}$ is a $G_0$-submodule of $G_{-q}$; hence, by, for example, [9] (Lemma 9),

$$(\operatorname{ad} v)^{p^j} G_{p^j q - q} = G_{-q}$$

Since $(\operatorname{ad} v)^{p^j}$ is a derivation of $G$ which annihilates $G_q$, we have, by, for example, [9] (Lemma 8) that

$$0 \ne \left[G_{-q}, G_q\right] = \left[(\operatorname{ad} v)^{p^j} G_{p^j q - q}, G_q\right] = (\operatorname{ad} v)^{p^j} \left[G_{p^j q - q}, G_q\right] \subseteq (\operatorname{ad} v)^{p^j} G_{p^j q}.$$

Thus, both $G_{p^j q}$ and $(\operatorname{ad} v)^{p^j} G_{p^j q}$ are non-zero, and Lemma 6 is proved. $\square$

**Lemma 7.** *Let* $v$ *be a non-zero element of* $G_{-q}$. *If* $j > 0$ *is maximal such that* $(\operatorname{ad} v)^{p^j} G \ne 0$, *then* $(\operatorname{ad} v)^{(p-1)p^j} G \ne 0$.

**Proof.** Suppose $(\operatorname{ad} v)^{(p-1)p^j} G = 0$. Then for any $x \in G_{p^j q}$, which is non-zero by Lemma 6, we have that

$$0 = \left((\operatorname{ad} v)^{p^j}\right)^{p-1} (\operatorname{ad} x)^{p-1} G_{-1} = (p-1)! \left(\operatorname{ad} \, (\operatorname{ad} v)^{p^j} x\right)^{p-1} G_{-1}.$$

Thus $\left(\operatorname{ad} \, (\operatorname{ad} v)^{p^j} x\right)^{p-1} G_{-1} = 0$, so $\operatorname{ad}_{G_{-1}} (\operatorname{ad} v)^{p^j} G_{p^j q}$ is a nil set of endomorphisms of $G_{-1}$. By Lemma 3, this nil set of endomorphisms is weakly closed, so by Jacobson's theorem on nil weakly closed sets [10], $\mathrm{ad}_{G_{-1}}(\mathrm{ad}\,v)^{p^j} G_{p^j q}$ acts nilpotently on $G_{-1}$ and therefore annihilates some non-zero element of $G_{-1}$ By Lemma 1 (or Lemma 2), $(\mathrm{ad}\,v)^{p^j} G_{p^j q}$ is a $G_0$-submodule of $G_0$ (*i.e.*, an ideal of $G_0$). Hence, the annihilator of $(\mathrm{ad}\,v)^{p^j} G_{p^j q}$ in $G_{-1}$ must be a $G_0$-submodule of $G_{-1}$. By the assumed irreducibility of $G$, $G_{-1}$ is irreducible as a $G_0$-module. Consequently,

$$(0 \neq)\,\mathrm{Ann}_{G_{-1}}(\mathrm{ad}\,v)^{p^j} G_{p^j q} = G_{-1}$$

*i.e.*, $\left[G_{-1},(\mathrm{ad}\,v)^{p^j} G_{p^j q}\right]=0$, But then, we have by transitivity that $(\mathrm{ad}\,v)^{p^j} G_{p^j q}=0$, so that, by Lemma 6 again $(\mathrm{ad}\,v)^{p^j} G=0$ contrary to the choice of $j$. Thus, $(\mathrm{ad}\,v)^{(p-1)p^j} G$ must be non-zero, as asserted. □

**Lemma 8.** *Let* $v$ *be a non-zero element of* $G_{-q}$*, and let* $j>0$ *be maximal such that* $(\mathrm{ad}\,v)^{p^j} G \neq 0$*. Then* $(\mathrm{ad}\,v)^{(p-1)p^j} G$ *is a Lie algebra, and we have that both* $G_{-q} \subseteq (\mathrm{ad}\,v)^{(p-1)p^j} G$ *and* $G_{-1} \subseteq (\mathrm{ad}\,v)^{(p-1)p^j} G$*. Consequently,* $(\mathrm{ad}\,v)^{(p-1)p^j} G + G_0$ *is an irreducible, transitive, depth-*$q$ *graded Lie algebra which is annihilated by* $(\mathrm{ad}\,v)^{p^j}$.

**Proof.** For $j \geq 1$ (since $(\mathrm{ad}\,v)^{p^j}(\mathrm{ad}\,v)^{(p-1)p^j} G = (\mathrm{ad}\,v)^{p^{j+1}} G = 0$, by the definition of $j$), we have

$$\begin{aligned}(\mathrm{ad}\,v)^{(p-1)p^j} G &\supseteq (\mathrm{ad}\,v)^{(p-1)p^j}\left[(\mathrm{ad}\,v)^{(p-1)p^j} G, G\right] = \left((\mathrm{ad}\,v)^{p^j}\right)^{(p-1)}\left[(\mathrm{ad}\,v)^{(p-1)p^j} G, G\right] \\ &= \left[(\mathrm{ad}\,v)^{(p-1)p^j} G, (\mathrm{ad}\,v)^{(p-1)p^j} G\right].\end{aligned}$$

so $\left[(\mathrm{ad}\,v)^{(p-1)p^j} G,(\mathrm{ad}\,v)^{(p-1)p^j} G\right] \subseteq (\mathrm{ad}\,v)^{(p-1)p^j} G$; *i.e.* $(\mathrm{ad}\,v)^{(p-1)p^j} G$ is a Lie algebra whenever $j \geq 1$, its closure under addition being obvious. Note that we must have $r \geq q(p-1)p^j - q$, since otherwise we would have $(\mathrm{ad}\,v)^{(p-1)p^j} G \subseteq \sum_{n<-q} G_n = 0$, to contradict Lemma 7.

By Lemma 6, $(\mathrm{ad}\,v)^{p^j} G_{p^j q} \neq 0$. By Lemma 1 (or Lemma 2), $(\mathrm{ad}\,v)^{p^j} G_{p^j q}$ is a non-zero ideal of $G_0$. Thus, by transitivity and irreducibility, $\left[G_{-1},(\mathrm{ad}\,v)^{p^j} G_{p^j q}\right] = G_{-1}$. Thus, we have

$$G_{-1} = \left[G_{-1},(\mathrm{ad}\,v)^{p^j} G_{p^j q}\right] = (\mathrm{ad}\,v)^{p^j}\left[G_{-1}, G_{p^j q}\right] \subseteq (\mathrm{ad}\,v)^{p^j} G_{p^j q-1} \subseteq G_{-1}$$

Consequently, we conclude that $G_{-1} = (\mathrm{ad}\,v)^{p^j} G_{p^j q-1}$, so $G_{-1} \subseteq (\mathrm{ad}\,v)^{p^j} G$. By Lemma 6, $G_{-q} \subseteq (\mathrm{ad}\,v)^{p^j} G$, also. Thus, $(\mathrm{ad}\,v)^{p^j} G + G_0$ is an irreducible, transitive depth-$q$ graded Lie algebra. Since by Lemma 7, $(\mathrm{ad}\,v)^{p^j(p-1)} G \neq 0$, it follows that $(\mathrm{ad}\,v)^{p^j}\left((\mathrm{ad}\,v)^{p^j} G\right) \neq 0$, so we may repeat the argument just made to conclude that $\left((\mathrm{ad}\,v)^{2p^j} G\right)$ is an irreducible, transitive depth-$q$ graded Lie algebra. Repeating the argument $p-3$ more times, we conclude the proof of Lemma 8. □

## 3. Proof of Main Theorem

Let $j_1$ be the maximum whole number such that $(\mathrm{ad}\,v)^{p^{j_1}} G \neq 0$ for some $v \in G_{-q}$. Such a maximal $j_1$ must exist, since the height $r$ of the finite-dimensional graded Lie algebra $G$ is finite. If $j_1 = 0$, then we are done. Suppose then that $j_1 \geq 1$. Let $v_1$ be an element of $G_{-q}$ such that $(\mathrm{ad}\,v_1)^{j_1} G \neq 0$. Then by Lemma 8,

$$G^{\{1\}} \stackrel{\mathrm{def}}{=} (\mathrm{ad}\,v_1)^{(p-1)p^{j_1}} G + G_0$$

is an irreducible, transitive, finite-dimensional depth-$q$ graded Lie algebra to which we may apply Lemma 8 to obtain a $j_2$ and $v_2$ such that

$$G^{\{2\}} \overset{\text{def}}{=} \left(\operatorname{ad} v_2\right)^{(p-1)p^{j_2}} \left(\operatorname{ad} v_1\right)^{(p-1)p^{j_1}} G + G_0$$

is an irreducible, transitive, finite-dimensional depth-$q$ graded Lie algebra to which we may apply Lemma 8 again. Since $G_{-q}$ is abelian, it follows that

$$\left(\operatorname{ad} v_1 + \operatorname{ad} v_2\right)^{p^{j_1}} G^{\{1\}} = \left(\operatorname{ad} v_1\right)^{p^{j_1}} G^{\{1\}} + \left(\operatorname{ad} v_2\right)^{p^{j_1}} G^{\{1\}} = \left(\operatorname{ad} v_2\right)^{p^{j_1}} G^{\{1\}}$$

Consequently, if $j_1 = j_2$, then $\left(\operatorname{ad} v_2\right)^{p^{j_1}} G^{\{1\}} \neq 0$, so $v_2$ is linearly independent of $v_1$. Since $G_{-q}$, like $G$ is finite-dimensional, we can, by repeating this process, arrive at an integer $t_1 \leq \dim G_{-q}$ such that $j_{t_1} = j_1$, but $j_{t_1+m} < j_{t_1}$ for any $m > 0$ for which $j_{t_1+m}$ is ultimately defined through the repetitive process we just described. Then

$$\left(\operatorname{ad} v_k\right)^{p^{j_k}} \coprod_{i=1}^{t_1} \left( \left(\operatorname{ad} v_i\right)^{(p-1)p^{j_i}} \right) G = 0, \quad 1 \leq k \leq t_1$$

Since the sequence $j_1, j_2, \cdots, j_{t_1}$ is non-increasing, the aforementioned commutativity of $G_{-q}$ entails that

$$\left(\operatorname{ad} G_{-1}\right)^{p^{j_1}} \coprod_{i=1}^{t_1} \left( \left(\operatorname{ad} v_i\right)^{(p-1)p^{j_i}} \right) G = 0 .$$

If, in the above argument, we replace $v_1$ and $j_1$ with $v_{t_1+1}$ and $j_{t_1+1}$, we eventually, by the finite dimensionality of $G_{-q}$, obtain a $t_2$ such that $j_{t_2} = j_{t_1+1}$, but $j_{t_2+1} < j_{t_2}$. Continuing in this way, using $v_{t_i+1}$ and $j_{t_i+1}$, $i \geq 1$, in the above argument, we see that the series $j_{t_i}$, $i = 1, 2, \cdots$ must eventually decrease to zero; *i.e.*, we obtain a Lie algebra $G^{\{n\}}$ such that $\left(\operatorname{ad} v\right)^p G^{\{n\}} = 0$ for all $v \in G_{-q}$, as required. □

**Remark.** Note that if we define $t_0 \overset{\text{def}}{=} 0$, and $V_{j_i} = \left\langle v_{t_i+1}, \cdots, v_{t_{i+1}} \right\rangle$ for $i > 0$, and $V_0 \overset{\text{def}}{=} G_{-q}$, then we get, for depth $q$, something analogous to a flag in the sense of [1].

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