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On the Norm of Elementary Operator

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Abstract

The norm of an elementary operator has been studied by many mathematicians. Varied results have been established especially on the lower bound of this norm. Here, we attempt the same problem for finite dimensional operators.

Keywords

Bounded Linear Operator, Elementary Operator

1. Introduction

Let H be a complex Hilbert space and $B(H)$ be the set of bounded operators on H. A basic elementary operator, $M_{T,S}: B(H) \to B(H)$, is defined as:

$$M_{T,S}(W) = TWS,$$

for $W \in B(H)$ and T, S fixed.

An elementary operator, $E_n: B(H) \to B(H)$, $n \in \mathbf{N}$, is a finite sum of the basic elementary operators, defined as, $E_n(W) = \sum_{i=1}^{n} M_{T_i,S_i} = \sum_{i=1}^{n} T_i W S_i$, for all $W \in B(H)$, where $T_i, S_i \in B(H)$ are fixed, for $i = 1, 2, \cdots, n$. When $n = 2$, we have $E_2(W) = T_1 W S_1 + T_2 W S_2$, for all $W \in B(H)$ and $T_i, S_i \in B(H)$ fixed, for $i = 1, 2$.

Given the elementary operator E_n on $B(H)$, the question on whether the equation

$\|E_n\| = \sum_{i=1}^{n} \|T_i\| \|S_i\|$, $n \in \mathbf{N}$, holds remains an area of interest to many mathematicians. This paper attempts to answer this question for finite dimensional operators.

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For a complex Hilbert space H, with dual H^*, we define a finite rank operator $(u \otimes x): H \to H$ by, $(u \otimes x) y = u(y) x$ for all $y \in H$, where $u \in H^*$, and $x \in H$ is a unit vector, with:

$$\|u \otimes x\| = \sup\left\{\|(u \otimes x) y\| : y \in H, \|y\| \le 1\right\} = \sup\left\{\|u(y) x\| : y \in H, \|y\| \le 1\right\}$$
$$= \sup\left\{|u(y)|\|x\| : y \in H, \|y\| \le 1\right\} = \sup\left\{|u(y)| : y \in H, \|y\| \le 1\right\} = |u(y)|.$$

In this paper, we use finite rank operators to determine the norm of E_2. We first review some known results on the norm of the Jordan elementary operator $U_{T,S} : B(H) \to B(H)$, $W \mapsto TWS + SWT$, for all $W \in B(H)$ with $T, S \in B(H)$ fixed. We will then proceed to show that for an operator $W \in B(H)$ with $\|W\| = 1$ and $W(x) = x$ for all unit vectors $x \in H$, then:

$$\|E_2\| = \sum_{i=1}^{2} \|T_i\|\|S_i\|.$$

Some mathematicians have attempted to determine the norm of E_n. Timoney, used (matrix) numerical ranges and the tracial geometric mean to obtain an approximation of E_n [1], while Nyamwala and Agure used the spectral resolution theorem to calculate the norm of E_n induced by normal operators in a finite dimensional Hilbert space [2].

The study of the norm of the Jordan elementary operator has also attracted many researchers in operator theory. Mathieu [3], in 1990, proved that in the case of a prime C*-algebra, the lower bound of the norm of $U_{T,S}$ can be estimated by $\|U_{T,S}\| \ge \frac{2}{3}\|T\|\|S\|$. In 1994, Cabrera and Rodriguez [4], showed that $\|U_{T,S}\| \ge \frac{1}{20412}\|T\|\|S\|$, for prime JB*-algebras.

On their part, Stacho and Zalar [5], in 1996 worked on the standard operator algebra which is a sub-algebra of $B(H)$, that contains all finite rank operators. They first showed that the operator $U_{T,S}$ actually represents a Jordan triple structure of a C*-algebra. They also showed that if A is a standard operator algebra acting on a Hilbert space H, and $T, S \in A$, then $\|U_{T,S}\| \ge 2(\sqrt{2} - 1)\|T\|\|S\|$. They later (1998), proved that $\|U_{T,S}\| \ge \|T\|\|S\|$ for the algebra of symmetric operators acting on a Hilbert space. They attached a family of Hilbert spaces to standard operator algebra, using the inner products on them to obtain their results.

In 2001, Barraa and Boumazguor [6], used the concept of the maximal numerical range and finite rank operators to show that if $T, S \in B(H)$ with $S \ne 0$, then:

$$\|U_{T,S}\| \ge \sup_{\lambda \in W_S(T^*S)} \left\{\left\|\|S\|T + \frac{\bar{\lambda}}{\|S\|} S\right\|\right\},$$

where,

$$W_S(T^*S) = \left\{\lambda \in \mathbf{C} : \exists x_n \in H, \|x_n\| = 1, \lim_{n\to\infty} \langle T^*Sx_n, x_n\rangle = \lambda, \lim_{n\to\infty} \|Sx_n\| = \|S\|\right\},$$

is the maximal numerical range of T^*S relative to S, and T^* is the Hilbert adjoint of T.

Okelo and Agure [7] used the finite rank operators to determine the norm of the basic elementary operator. Their work forms the basis of the results in this paper.

2. The Norm of Elementary Operator

In this section, we present some of the known results on elementary operators and proceed to determine norm of the elementary operator E_2.

In the following theorem Okelo and Agure [7], determined the norm of the basic elementary operator.

***Theorem* 2.1** [5]: Let H be a complex Hilbert space and $B(H)$ the algebra of bounded linear operators on H. Let $M_{T,S} : B(H) \to B(H)$ be defined by $M_{T,S}(W) = TWS$ for all $W \in B(H)$ with T, S as fixed elements in $B(H)$. If for all $W \in B(H)$ with $\|W\| = 1$, we have $W(x) = x$ for all unit vectors $x \in H$, then;

$$\|M_{T,S}\| = \|T\|\|S\|.$$

Proof: Since $\left\|M_{T,S}|B(H)\right\| = \sup\left\{\left\|M_{T,S}(W)\right\| : W \in B(H), \|W\| = 1\right\}$, we have, $\forall \varepsilon > 0$;

$$\left\|M_{T,S}|B(H)\right\| - \varepsilon < \left\|M_{T,S}(W)\right\|, \ \forall W \in B(H), \ \|W\| = 1$$

Therefore:

$$\left\|M_{T,S}|B(H)\right\| - \varepsilon < \|TWS\| \le \|T\|\|S\|.$$

Letting $\varepsilon \to 0$, we obtain:

$$\left\|M_{T,S}|B(H)\right\| \le \|T\|\|S\|. \tag{1}$$

On the other hand, we have:

$$\left\|M_{T,S}|B(H)\right\| \ge \left\|M_{T,S}(W)\right\|, \ \forall W \in B(H), \ \|W\| = 1,$$

with:

$$\left\|M_{T,S}(W)\right\| = \sup\left\{\left\|\left(M_{T,S}(W)\right)x\right\| : \forall x \in H, \|x\| = 1\right\}.$$

So, setting $T = u \otimes x_1$, $\forall x_1 \in H$, $\|x_1\| = 1$, and $S = v \otimes x_2$, $\forall x_2 \in H$, $\|x_2\| = 1$, we have: $\left\|M_{T,S}|B(H)\right\| \ge \left\|M_{T,S}(W)\right\|$, $\forall W \in B(H)$, $\|W\| = 1$, with T, S fixed in $B(H)$.

$$\begin{aligned} &= \left\|\left((u \otimes x_1) W (v \otimes x_2)\right) y\right\| = \left\|(u \otimes x_1) W \left(v(y) x_2\right)\right\| \\ &= \left\|(u \otimes x_1) v(y) W(x_2)\right\| = |v(y)| \left\|(u \otimes x_1) W(x_2)\right\| \\ &= |v(y)| \left\|u\left(W(x_2)\right) x_1\right\| = |v(y)| \left|u\left(W(x_2)\right)\right| \|x_1\| = \|T\|\|S\|, \end{aligned}$$

obtaining;

$$\left\|M_{T,S}|B(H)\right\| \ge \|T\|\|S\|. \tag{2}$$

Hence, from (1) and (2), we obtain

$$\left\|M_{T,S}|B(H)\right\| = \|T\|\|S\|.$$

□

For any vectors $y, z \in H$, the rank one operator, $y \otimes z \in B(H)$, is defined by $(y \otimes z)x = \langle x, z\rangle y$, for all $x \in H$.

In the following three results Baraa and Boumazgour give three estimations to the lower bound of the norm of the Jordan elementary operator. See [6]. Recall that the Jordan elementary operator is the operator $U_{T,S} : B(H) \to B(H)$, $W \mapsto TWS + SWT$, for all $W \in B(H)$ with $T, S \in B(H)$ fixed.

***Theorem* 2.2.** Let $U_{T,S}$ be the Jordan elementary operator with $T, S \in B(H)$ fixed, and with $S \ne 0$. Then

$$\left\|U_{T,S}\right\| \ge \sup_{\lambda \in W_S(T^*S)} \left\{\left\|\|S\|T + \frac{\bar{\lambda}}{\|S\|} S\right\|\right\},$$

where, $W_S(T^*S)$ is the maximal numerical range of T^*S relative to S, as defined earlier.

Proof: Let $\lambda \in W_S(T^*S)$. Then there exists a sequence $\{x_n\}_{n\ge1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle T^*Sx_n, x_n\rangle = \lambda$ and $\lim_{n\to\infty}\|Sx_n\| = \|S\|$. Consider unit vectors $y, z \in H$, and recall the rank one operator, $y \otimes z \in B(H)$, defined as $(y \otimes z)x = \langle x, z\rangle y$, for all unit vectors $x \in H$. For fixed operators $T, S \in B(H)$, we have;

$$\left\|\left(U_{T,S}(y \otimes Sx_n)\right)x_n\right\| \le \left\|U_{T,S}(y \otimes Sx_n)\right\| \le \left\|U_{T,S}\right\|\|y\|\|Sx_n\| \le \left\|U_{T,S}\right\|\|S\|.$$

That is $\|U_{T,S}\|\|S\| \geq \|(U_{T,S}(y \otimes Sx_n))x_n\|$.

Thus we have:

$$\begin{aligned}\|U_{T,S}\|\|S\| &\geq \|(U_{T,S}(y \otimes Sx_n))x_n\| = \|(T(y \otimes Sx_n)S + S(y \otimes Sx_n)T)x_n\| \\ &= \|T(y \otimes Sx_n)Sx_n + S(y \otimes Sx_n)Tx_n\| = \|T\langle Sx_n, Sx_n\rangle y + S\langle Tx_n, Sx_n\rangle y\| \\ &= \left\| \|Sx_n\|^2 Ty + \langle x_n, T^*Sx_n\rangle Sy \right\|.\end{aligned}$$

Hence

$$\|U_{T,S}\| \geq \frac{1}{\|S\|}\left\|\|Sx_n\|^2 Ty + \langle x_n, T^*Sx_n\rangle Sy\right\|. \tag{3}$$

Letting $n \to \infty$, we obtain:

$$\|U_{T,S}\| \geq \left\|\|S\|Ty + \frac{\bar{\lambda}}{\|S\|}Sy\right\|,$$

and this is true for any $\lambda \in W_S(T^*S)$, and for any unit vector $y \in H$.

Now, consider the set $\left\{\left\|\|S\|Ty + \frac{\bar{\lambda}}{\|S\|}Sy\right\| : \lambda \in W_S(T^*S), y \in H, \|y\| = 1\right\}$.

We have:

$$\|U_{T,S}\| \geq \sup\left\{\left\|\|S\|Ty + \frac{\bar{\lambda}}{\|S\|}Sy\right\| : \lambda \in W_S(T^*S), y \in H, \|y\| = 1\right\}$$

But $\sup\left\{\left\|\|S\|Ty + \frac{\bar{\lambda}}{\|S\|}Sy\right\| : \lambda \in W_S(T^*S), y \in H, \|y\| = 1\right\} = \sup\left\{\left\|\|S\|T + \frac{\bar{\lambda}}{\|S\|}S\right\| : \lambda \in W_S(T^*S)\right\}$.

Therefore:

$$\|U_{T,S}\| \geq \sup\left\{\left\|\|S\|T + \frac{\bar{\lambda}}{\|S\|}S\right\| : \lambda \in W_S(T^*S)\right\},$$

and this completes the proof. □

***Corollary* 2.3:** Let H be a complex Hilbert space and T, S be bounded linear operators on H. Let $0 \in W_S(T^*S) \cup W_T(S^*T)$. Then we have $\|U_{T,S}\| \geq \|T\|\|S\|$.

***Proof*:** Let $0 \in W_S(T^*S) \cup W_T(S^*T)$. Then, $0 \in W_S(T^*S)$ or $0 \in W_T(S^*T)$, and therefore, either there is a sequence $\{x_n\}_{n\geq 1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle T^*Sx_n, x_n\rangle = 0$ and $\lim_{n\to\infty}\|Sx_n\| = \|S\|$ or, there is a sequence $\{y_n\}_{n\geq 1}$ of unit vectors in H such that $\lim_{n\to\infty}\langle S^*Ty_n, y_n\rangle = 0$ and $\lim_{n\to\infty}\|Ty_n\| = \|T\|$.

Recall that in the previous theorem (Inequality (3)), we obtained:

$$\|U_{T,S}\| \geq \frac{1}{\|S\|}\left\|\|Sx_n\|^2 Ty + \langle x_n, T^*Sx_n\rangle Sy\right\|$$

This is equivalent to:

$$\|U_{T,S}\| \geq \frac{1}{\|S\|}\left\|\|Sy_n\|^2 Ty + \langle S^*Ty_n, y_n\rangle Sy\right\|, \tag{4}$$

considering the sequence $\{y_n\}_{n\geq 1}$,

Taking limits in either (3) or (4), we obtain

$$\|U_{T,S}\| \geq \|\|S\|Ty\|,$$

and this is true for any unit vector $y \in H$.

Now, consider the set $\left\{ \left\| \|S\| Ty \right\| : y \in H, \|y\| = 1 \right\}$.

We have:

$$\|U_{T,S}\| \ge \sup\left\{ \left\| \|S\| Ty \right\| : y \in H, \|y\| = 1 \right\}.$$

But $\sup\left\{ \left\| \|S\| Ty \right\| : y \in H, \|y\| = 1 \right\} = \|T\|\|S\|$.

Therefore:

$$\|U_{T,S}\| \ge \|T\|\|S\|,$$

and this completes the proof. □

***Proposition* 2.4:** Let H be a complex Hilbert space and T, S be bounded linear operators on H. If $\|T\|\|S\| \in W_T\left(S^*T\right) \cap W_{T^*}\left(ST^*\right)$ then:

$$\|U_{T,S}\| = 2\|T\|\|S\|.$$

Proof: Suppose $\|T\|\|S\| \in W_T\left(S^*T\right) \cap W_{T^*}\left(ST^*\right)$. Then $\|T\|\|S\| \in W_T\left(S^*T\right)$ and $\|T\|\|S\| \in W_{T^*}\left(ST^*\right)$, and therefore we can find two sequences $\{x_n\}_{n\ge1}$ and $\{y_n\}_{n\ge1}$ of unit vectors in H such that:

$$\lim_{n\to\infty}\left\langle S^*Tx_n, x_n\right\rangle = \|T\|\|S\|, \quad \lim_{n\to\infty}\|Tx_n\| = \|T\| \quad \text{and} \quad \lim_{n\to\infty}\left\langle ST^*y_n, y_n\right\rangle = \|T\|\|S\|, \quad \lim_{n\to\infty}\|T^*y_n\| = \|T\|.$$

Since $\left|\left\langle S^*Tx_n, x_n\right\rangle\right| \le \|Tx_n\|\|Sx_n\|$ and $\left|\left\langle ST^*y_n, y_n\right\rangle\right| \le \|T^*y_n\|\|S^*y_n\|$, then $\lim_{n\to\infty}\|Sx_n\| = \|S\|$ and

$$\lim_{n\to\infty}\|S^*y_n\| = \|S\|.$$

For each $n \ge 1$, we have:

$$\begin{aligned}
&\left\|\left(U_{T,S}\left(x_n \otimes y_n\right)\right)S^*y_n\right\|^2 = \left\|\left(T\left(x_n \otimes y_n\right)S + S\left(x_n \otimes y_n\right)T\right)S^*y_n\right\|^2 \\
&= \left\|T\left(x_n \otimes y_n\right)SS^*y_n + S\left(x_n \otimes y_n\right)TS^*y_n\right\|^2 = \left\|\left\langle SS^*y_n, y_n\right\rangle Tx_n + \left\langle TS^*y_n, y_n\right\rangle Sx_n\right\|^2 \\
&= \left\|\left\langle S^*y_n, S^*y_n\right\rangle Tx_n + \left\langle TS^*y_n, y_n\right\rangle Sx_n\right\|^2 = \left\|\left\|S^*y_n\right\|^2 Tx_n + \left\langle TS^*y_n, y_n\right\rangle Sx_n\right\|^2 \\
&= \left\langle \left\|S^*y_n\right\|^2 Tx_n + \left\langle TS^*y_n, y_n\right\rangle Sx_n, \left\|S^*y_n\right\|^2 Tx_n + \left\langle TS^*y_n, y_n\right\rangle Sx_n \right\rangle \\
&= \left\|\left\|S^*y_n\right\|^2 Tx_n\right\|^2 + 2\operatorname{Re}\left\langle \left\langle TS^*y_n, y_n\right\rangle Sx_n, \left\|S^*y_n\right\|^2 Tx_n \right\rangle + \left|\left\langle TS^*y_n, y_n\right\rangle\right|^2 \|Sx_n\|^2 \\
&= \left\|S^*y_n\right\|^4 \|Tx_n\|^2 + \left\|S^*y_n\right\|^2 2\operatorname{Re}\left\langle TS^*y_n, y_n\right\rangle\left\langle Sx_n, Tx_n\right\rangle + \left|\left\langle TS^*y_n, y_n\right\rangle\right|^2 \|Sx_n\|^2 \\
&= \left\|S^*y_n\right\|^4 \|Tx_n\|^2 + \left\|S^*y_n\right\|^2 2\operatorname{Re}\left\langle TS^*y_n, y_n\right\rangle\left\langle T^*Sx_n, x_n\right\rangle + \left|\left\langle TS^*y_n, y_n\right\rangle\right|^2 \|Sx_n\|^2
\end{aligned}$$

Now, we have:

$$\begin{aligned}
\left\|\left(U_{T,S}\left(x_n \otimes y_n\right)\right)S^*y_n\right\| &\le \left\|U_{T,S}\left(x_n \otimes y_n\right)\right\|\left\|S^*y_n\right\| \\
&\le \left\|U_{T,S}\right\|\left\|x_n \otimes y_n\right\|\left\|S^*y_n\right\| \le \left\|U_{T,S}\right\| \left\|S^*y_n\right\|.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\left\|U_{T,S}\right\|^2 \left\|S^*y_n\right\|^2 \ge{}& \left\|S^*y_n\right\|^4 \|Tx_n\|^2 + \left\|S^*y_n\right\|^2 2\operatorname{Re}\left\langle TS^*y_n, y_n\right\rangle\left\langle T^*Sx_n, x_n\right\rangle \\
&+ \left|\left\langle TS^*y_n, y_n\right\rangle\right|^2 \|Sx_n\|^2.
\end{aligned}$$

Letting $n \to \infty$ we obtain:

$$\|U_{T,S}\|^2 \|S\|^2 \ge \|S\|^4 \|T\|^2 + 2\|S\|^4 \|T\|^2 + \|S\|^4 \|T\|^2 = 4\|S\|^4 \|T\|^2 .$$

That is $\|U_{T,S}\|^2 \ge 4\|S\|^2 \|T\|^2$ and this implies that $\|U_{T,S}\| \ge 2\|S\|\|T\|$.

Clearly, $\|U_{T,S}\| \le 2\|S\|\|T\|$ and therefore we obtain $\|U_{T,S}\| = 2\|S\|\|T\|$. □

We recall that an elementary operator, $E_n : B(H) \to B(H)$, $n \in \mathbf{N}$, is defined as $E_n(W) = \sum_{i=1}^{n} T_i W S_i$, for all $W \in B(H)$ where $T_i, S_i \in B(H)$ are fixed, for $i = 1, 2, \cdots, n$. When $n = 2$, we have $E_2(W) = T_1 W S_1 + T_2 W S_2$, for all $W \in B(H)$ and $T_i, S_i \in B(H)$ fixed, for $i = 1, 2$.

The following result gives the norm of E_2.

***Theorem* 2.5:** Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H: Let E_2 be the elementary operator on $B(H)$ defined above. If for an operator $W \in B(H)$ with $\|W\| = 1$, we have $W(x) = x$ for all unit vectors $x \in H$, then:

$$\|E_2\| = \sum_{i=1}^{2} \|T_i\|\|S_i\| .$$

***Proof*:** Recall that $E_2 : B(H) \to B(H)$ is defined as $E_2(W) = T_1 W S_1 + T_2 W S_2$, for all $W \in B(H)$ and $T_i, S_i \in B(H)$ fixed, for $i = 1, 2$.

We have:

$$\|E_2 | B(H)\| = \sup\{\|E_2(W)\| : W \in B(H), \|W\| = 1\} .$$

Therefore, $\|E_2 | B(H)\| \ge \|E_2(W)\|$ for all $W \in B(H)$ with $\|W\| = 1$.

So, for all $\varepsilon > 0$, $\|E_2 | B(H)\| - \varepsilon < \|E_2(W)\|$ for all $W \in B(H)$ with $\|W\| = 1$.

Therefore, $\|E_2 | B(H)\| - \varepsilon < \sum_{i=1}^{2} \|T_i\|\|S_i\|$.

Letting $\varepsilon \to 0$, we obtain:

$$\|E_2 | B(H)\| \le \sum_{i=1}^{2} \|T_i\|\|S_i\| . \quad (5)$$

Next, we show that $\|E_2 | B(H)\| \ge \sum_{i=1}^{2} \|T_i\|\|S_i\|$.

Since $\|E_2(W)\| = \sup\{\|(E_2(W))x\| : x \in H, \|x\| = 1\}$, then we have $\|E_2(W)\| \ge \|(E_2(W))x\|$ for all $x \in H, \|x\| = 1$. But $E_2(W)x = (T_1 W S_1 + T_2 W S_2)x$.

Now, let $u_i, v_i : H \to \mathbf{R}^+$ be functionals for $i = 1, 2$.

Choose unit vectors $y, z \in H$ and define finite rank operators $T_i = u_i \otimes y$ and $S_i = v_i \otimes z$ on H, for $i = 1, 2$ by $T_i x = (u_i \otimes y)x = u_i(x) y$ for all $x \in H$ with $\|x\| = 1$, for $i = 1, 2$, and $S_i x = (v_i \otimes z)x = v_i(x) z$, for $x \in H$ with $\|x\| = 1$, for $i = 1, 2$.

Observe that the norm of T_i for $i = 1, 2$ is,

$$\begin{aligned}
\|T_i\| &= \sup\{\|(u_i \otimes y)x\| : x \in H, \|x\| \le 1\} \\
&= \sup\{\|u_i(x) y\| : x \in H, \|x\| \le 1\} \\
&= \sup\{|u_i(x)|\|y\| : x \in H, \|x\| \le 1\} \\
&= \sup\{|u_i(x)| : x \in H, \|x\| \le 1\} = |u_i(x)|
\end{aligned}$$

That is $\|T_i\| = |u_i(x)|$ for any unit vector $x \in H$ with $\|x\| = 1$, for $i = 1, 2$.

Likewise, the norm of S_i is $\|S_i\| = |v_i(x)|$ for any unit vector $x \in H$ with $\|x\| = 1$, for $i = 1, 2$. Therefore, for all $x \in H$ with $\|x\| = 1$, we have

$$\begin{aligned} E_2(W)x &= (T_1WS_1 + T_2WS_2)x = (T_1WS_1)x + (T_2WS_2)x \\ &= (u_1 \otimes y)W(v_1 \otimes z)x + (u_2 \otimes y)W(v_2 \otimes z)x \\ &= (u_1 \otimes y)Wv_1(x)z + (u_2 \otimes y)Wv_2(x)z \\ &= v_1(x)(u_1 \otimes y)W(z) + v_2(x)(u_2 \otimes y)W(z) \\ &= v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y. \end{aligned}$$

Since $\|E_2|B(H)\| = \sup\{\|E_2(W)\| : W \in B(H), \|W\| = 1\}$, we have:

$$\begin{aligned} \|E_2|B(H)\|^2 &\geq \|v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y\|^2 \\ &= \langle v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y, v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y \rangle \\ &= \langle v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y, v_1(x)u_1(W(z))y \rangle \\ &\quad + \langle v_1(x)u_1(W(z))y + v_2(x)u_2(W(z))y, v_2(x)u_2(W(z))y \rangle \\ &= \langle v_1(x)u_1(W(z))y, v_1(x)u_1(W(z))y \rangle + \langle v_2(x)u_2(W(z))y, v_1(x)u_1(W(z))y \rangle \\ &\quad + \langle v_1(x)u_1(W(z))y, v_2(x)u_2(W(z))y \rangle + \langle v_2(x)u_2(W(z))y, v_2(x)u_2(W(z))y \rangle \\ &= \|v_1(x)u_1(W(z))y\|^2 + v_2(x)u_2(W(z))v_1(x)u_1(W(z))\langle y, y \rangle \\ &\quad + v_1(x)u_1(W(z))v_2(x)u_2(W(z))\langle y, y \rangle + \|v_2(x)u_2(W(z))y\|^2 \\ &= |v_1(x)|^2 |u_1(W(z))|^2 \|y\|^2 + v_2(x)u_2(W(z))v_1(x)u_1(W(z)) \\ &\quad + v_1(x)u_1(W(z))v_2(x)u_2(W(z)) + |v_2(x)|^2 |u_2(W(z))|^2 \|y\|^2 \\ &= \{|v_1(x)||u_1(W(z))|\}^2 + v_2(x)u_2(W(z))v_1(x)u_1(W(z)) \\ &\quad + v_1(x)u_1(W(z))v_2(x)u_2(W(z)) + \{|v_2(x)||u_2(W(z))|\}^2 \\ &= \{|v_1(x)||u_1(W(z))|\}^2 + 2v_1(x)u_1(W(z))v_2(x)u_2(W(z)) + \{|v_2(x)||u_2(W(z))|\}^2 \end{aligned}$$

Now, since $v_1(x), u_1(W(z)), v_2(x)$ and $u_2(W(z))$ are all positive real numbers, we have $v_1(x) = |v_1(x)| = |S_1|$, $u_1(W(z)) = |u_1(W(z))| = |T_1|$, $v_2(x) = |v_2(x)| = |S_2|$, and $u_2(W(z)) = |u_2(W(z))| = |T_2|$.

Thus $\|E_2|B(H)\|^2 \geq \{|T_1||S_1|\}^2 + 2|T_1||S_1||T_2||S_2| + \{|T_2||S_2|\}^2 = \{|T_1||S_1| + |T_2||S_2|\}^2$ and hence we have $\|E_2|B(H)\| \geq \{|T_1||S_1| + |T_2||S_2|\} = \sum_{i=1}^{2} \|T_i\|\|S_i\|$.

That is,

$$\|E_2|B(H)\| \geq \sum_{i=1}^{2} \|T_i\|\|S_i\|. \quad (6)$$

Now, (5) and (6) implies that:

$$\|E_2|B(H)\| = \sum_{i=1}^{2} \|T_i\|\|S_i\|,$$

and this completes the proof. □

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