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A Note on the Height of Transitive Depth-One Graded Lie Algebras Generated by Their Local Parts

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Abstract

For a transitive depth-one graded Lie algebra over a field of characteristic greater than two, a limit on the degree of the highest gradation space is determined.

Keywords

Graded Lie Algebras

Let the characteristic p of the base field be greater than two. In [1], V.G. Kac defined the local part of a graded Lie algebra $L = L_{-1} + L_0 + L_1 + \cdots + L_k$ to be $L_{-1} + L_0 + L_1$, which we will assume to be finite dimensional. We will refer to k as the height of L.

Lemma 1. *If* $L = L_{-1} + L_0 + L_1 + \cdots + L_k$ *is a transitive graded Lie algebra which is generated by its finite-dimensional local part, then the height* k *of* L *is less than or equal to* $n(p-1)-1$, *where* $n = \dim L_{-1}$.

Proof. If $c_1, \cdots, c_j$ are any elements of L_1 (or L_0 or L_{-1}, respectively) and v is an element of L_{-1} then we have by the Leibniz rule that

$$(\operatorname{ad} v)^p \left[\cdots[c_1, c_2], \cdots, c_j\right] = \sum_{i=1}^{j} \left[\cdots\left[\cdots[c_1, c_2], \cdots, (\operatorname{ad} v)^p c_i\right], \cdots, c_j\right].$$

Now, $(\operatorname{ad} v)^p c_i$ is an element of L_{1-p}, (or L_{-p} or L_{-1-p}, respectively) which is zero, since p is assumed to be greater than two, and the depth of L is one. Because of the commutativity of L_{-1}, we have that $(\operatorname{ad} L_{-1})^{n(p-1)+1} L = 0$. Consequently, by the transitivity of L, we have $L_m = 0$ for m greater than or equal to $-1 + n(p-1)$. □

Note that the height of the Jacobson-Witt algebra W_n is $n(p-1)-1$.

Because the gradation spaces of positive degree of transitive graded Lie algebras are contained in Cartan

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prolongations, those gradation spaces will be finite dimensional whenever the local part of the transitive Lie algebra is finite dimensional. Thus, when the gradation degrees of a transitive graded Lie algebra with a finite-dimensional local part are bounded, the Lie algebra is itself finite dimensional.

References

[1] Kac, V.G. (1968) Simple Irreducible Graded Lie Algebras of Finite Growth. *Mathematics of the USSR-Izvestiya*, **2**, 1271-1312 (English), *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, **32**, 1323-1367 (Russian).