Advances in Pure Mathematics, 2014, 4, 176-188
Published Online May 2014 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2014.45023

Projective Tensor Products of C^*-Algebras*

Ajay Kumar, Vandana Rajpal

Department of Mathematics, University of Delhi, Delhi, India
Email: akumar@maths.du.ac.in, vandanarajpal.math@gmail.com

Received 28 February 2014; revised 28 March 2014; accepted 6 April 2014

Abstract

For C^*-algebras A and B, the constant involved in the canonical embedding of $A^{**} \otimes_\gamma B^{**}$ into $\left(A \otimes_\gamma B\right)^{**}$ is shown to be $\frac{1}{2}$. We also consider the corresponding operator space version of this embedding. Ideal structure of $A \hat{\otimes} B$ is obtained in case A or B has only finitely many closed ideals.

Keywords

Banach Space Projective Tensor Norm, Operator Space Projective Tensor Norm

1. Introduction

The systematic study of various tensor norms on the tensor product of Banach spaces was begun with the work of Schatten [1], which was later studied by Grothendieck in the context of locally convex topological space. One of the most natural and useful tensor norm is the Banach space projective tensor norm. For a pair of arbitrary Banach spaces X and Y and u an element in the algebraic tensor product $X \otimes Y$, the Banach space projective tensor norm is defined to be

$$\|u\|_\gamma = \inf\left\{\sum_{i=1}^{n}\|x_i\|\|y_i\|: u = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

$X \otimes_\gamma Y$ will denote the completion of $X \otimes Y$ with respect to this norm. For operator spaces X and Y, the operator space projective tensor product of X and Y is denoted by $X \hat{\otimes} Y$ and is defined to be the completion of $X \otimes Y$ with respect to the norm:

$$\|u\|_\wedge = \inf\left\{\|\alpha\|\|x\|\|y\|\|\beta\|\right\},$$

the infimum taken over $p, q \in \mathbb{N}$ and all the ways to write $u = \alpha\left(x \otimes y\right)\beta$, where $\alpha \in M_{1,pq}$, $\beta \in M_{pq,1}$,

*2010 Mathematics Subject Classification. Primary 46L06, Secondary 46L07, 47L25.

How to cite this paper: Kumar, A. and Rajpal, V. (2014) Projective Tensor Products of C^*-Algebras. *Advances in Pure Mathematics*, **4**, 176-188. http://dx.doi.org/10.4236/apm.2014.45023

$x \in M_p(X)$ and $y \in M_q(Y)$, and $x \otimes y = (x_{ij} \otimes y_{kl})_{(i,k),(j,l)} \in M_{pq}(X \otimes Y)$.

Kumar and Sinclair defined an embedding μ from $A^{**} \otimes_\gamma B^{**}$ into $(A \otimes_\gamma B)^{**}$, and using the non-commutative version of Grothendieck's theorem to the setting of bounded bilinear forms on C^*-algebras, it was shown that this embedding satisfies $\frac{1}{4}\|u\|_\gamma \le \|\mu(u)\| \le \|u\|_\gamma$ ([2], Theorem 5.1). Recently, analogue of Grothendieck's theorem for jointly completely bounded (jcb) bilinear forms was obtained by Haagerup and Musat [3]. Using this form for jcb, the canonical embedding for the operator space projective tensor product have been studied by Jain and Kumar [4], and they showed that the embedding μ from $A^{**} \hat{\otimes} B^{**}$ into $(A \hat{\otimes} B)^{**}$ satisfies $\frac{1}{2}\|u\|_\wedge \le \|\mu(u)\| \le \|u\|_\wedge$.

In Section 2, an alternate approach for the bi-continuity of the canonical embedding of $A^{**} \otimes_\gamma B^{**}$ into $(A \otimes_\gamma B)^{**}$ has been presented with an improved constant. Our proof essentially uses the fact that the dual of the Banach space projective tensor norm is the Banach space injective tensor norm. We also consider the corresponding operator space version of this embedding and discuss its isomorphism. As a consequence, one can obtain the equivalence between the Haagerup tensor norm and the Banach space projective tensor norm (resp. operator space projective tensor norm).

In the next section, it is shown that if the number of all closed ideals in one of the C^*-algebras is finite then every closed ideal of $A \hat{\otimes} B$ is a finite sum of product ideals. One can obtain all the closed ideals of $B(H) \hat{\otimes} B(H)$ as $B(H) \hat{\otimes} K(H)$, $K(H) \hat{\otimes} B(H)$ and $B(H) \hat{\otimes} K(H) + K(H) \hat{\otimes} B(H)$, and the closed ideals of $B(H) \hat{\otimes} C_0(X)$ as $\sum_{i=1}^{n} A \hat{\otimes} I(E_i)$, where $A = B(H)$ or $K(H)$, $I(E_i) = \{ f \in C_0(X) : f(x) = 0 \text{ for all } x \in E_i \}$ for each i, for an infinite dimensional separable Hilbert space H and locally compact Hausdorff topological space X. Similarly, the closed ideal structure of $(M(A)/A) \hat{\otimes} B$, where B is any C^*-algebra and $M(A)$ is the multiplier algebra of A, A being a nonunital, non-element- ary, separable, simple *AF* C^*-algebra, can be obtained. We may point that such result fails for $A \otimes_{\min} B$, the minimal tensor product of C^*-algebras A and B.

Section 4 is devoted to the inner automorphisms of $A \hat{\otimes} B$ and $A \otimes_h B$ for C^*-algebras as well as for operator algebras. Recall that the Haagerup norm on the algebraic tensor product of two operator spaces X and Y is defined, for $u \in X \otimes Y$, by

$$\|u\|_h = \inf \{ \|x\| \|y\| \}$$

where infimum is taken over all the ways to write

$$u = x \odot y = \sum_{k=1}^{r} x_{1k} \otimes y_{k1}$$

where $x \in M_{1,r}(X), y \in M_{r,1}(Y), r \in \mathbb{N}$. The Haagerup tensor product $X \otimes_h Y$ is defined to be the completion of $X \otimes Y$ in the norm $\|\cdot\|_h$ [5].

2. Isomorphism of Embeddings

For Banach spaces X and Y and $\phi_i \in X^*$, $\psi_i \in Y^*$, define a linear map $J : X^* \otimes Y^* \to B(X \times Y, \mathbb{C})$ as $J\left(\sum_{i=1}^{n} \phi_i \otimes \psi_i\right)(x, y) = \sum_{i=1}^{n} \phi_i(x) \psi_i(y)$, for $x \in X$ and $y \in Y$. Using ([6], Proposition 1.2), it is easy to see that J is well defined. Also, clearly this map is linear and contractive with respect to $\|\cdot\|_\gamma$, and in fact $\|J\| = 1$, and hence can be extended to $X^* \otimes_\gamma Y^*$ with $\|J\| = 1$. A bilinear form T in $B(X \times Y, \mathbb{C})$ is called nuclear if $T \in J(X^* \otimes_\gamma Y^*)$, and the nuclear norm of T is defined to be $\|T\|_N = \inf \left\{ \sum_{n=1}^{\infty} \|\phi_n\| \|\psi_n\| : T = \sum_{n=1}^{\infty} \phi_n \otimes \psi_n \right\}$. The Banach space of nuclear bilinear forms is denoted by $B_N(X \times Y, \mathbb{C})$. For C^*-algebras A and B, consider the canonical map θ from $A \otimes_\gamma B$ into $(A^* \otimes_\lambda B^*)^*$, the dual of the Banach space injective tensor product of

A^* and B^*, defined by

$$\theta = i' \circ J \circ i$$

where i is the natural isometry of $A \otimes_\gamma B$ into $A^{**} \otimes_\gamma B^{**}$, J is as above with $X = A^*$ and $Y = B^*$, i' is the natural inclusion of $B_N\left(A^* \times B^*, \mathbb{C}\right)$ into $B_I\left(A^* \times B^*, \mathbb{C}\right)$, the space of integral bilinear forms.

Lemma 2.1 *For C^*-algebras A and B, the canonical map $\theta : A \otimes_\gamma B \to \left(A^* \otimes_\lambda B^*\right)^* \left(= J\left(A^*, B^{**}\right)\right.$, the space of integral operators from A^* to B^{**}) satisfies $\frac{1}{2}\|u\|_\gamma \le \|\theta(u)\| \le \|u\|_\gamma$ for all $u \in A \otimes_\gamma B$. In particular, θ is bi-continuous.*

Proof: The inequality of the right hand side follows directly from the definition of θ. Let $0 \ne u \in A \otimes_\gamma B$ and $\epsilon > 0$. By the Hahn Banach Theorem, there exists $T \in \left(A \otimes_\gamma B\right)^*$ with $\|T\| \le 1$ such that $|T(u)| > \|u\|_\gamma - \epsilon$. Since $\left(A \otimes_\gamma B\right)^* = B\left(A, B^*\right)$, so $T(a \otimes b) = \tilde{T}(a)(b)$, for some $\tilde{T} \in B\left(A, B^*\right)$, for all $a \in A$ and $b \in B$ with $\|\tilde{T}\| = \|T\| \le 1$. By ([7], Proposition 2.1(2)), there is a net $\left(\widetilde{T_\alpha}\right)$ of finite rank operators from A to B^* such that $\left\|\widetilde{T_\alpha}\right\| \le 2\|\tilde{T}\|$ and $\lim_\alpha \left\|\widetilde{T_\alpha}(x) - \tilde{T}(x)\right\| = 0$ for any $x \in A$.

Now, for each α, corresponding to $\widetilde{T_\alpha}$ we can associate $T_\alpha \in \left(A \otimes_\gamma B\right)^*$. For $u \in A \otimes_\gamma B$, there is α_0 such that $\left|T(u) - T_\alpha(u)\right| < \epsilon$ for all $\alpha \ge \alpha_0$. Thus $\left|T_{\alpha_0}(u)\right| > \|u\|_\gamma - 2\epsilon$. Since $\widetilde{T_{\alpha_0}}$ is a finite rank operator, so let $\dim\left(\mathrm{Range}\left(\widetilde{T_{\alpha_0}}\right)\right) = m < \infty$. Choose an Auerbach basis $\{\phi_1, \phi_2, \cdots, \phi_m\}$ for $\mathrm{Range}\left(\widetilde{T_{\alpha_0}}\right)$ with associated coordinate functionals $F_1, F_2, \cdots, F_m$ in B^{**}. Thus, for any $x \in A$, $\widetilde{T_{\alpha_0}}(x) = \sum_{i=1}^m c_i \phi_i$, $c_i \in \mathbb{C}$ for $i = 1, 2, \cdots, m$. By using $F_i\left(\phi_j\right) = \delta_{ij}$, it follows that $\widetilde{T_{\alpha_0}}(x) = \sum_{i=1}^m \psi_i(x)\phi_i$, $\psi_i := F_i \circ \widetilde{T_{\alpha_0}} \in A^*$ for $i = 1, 2, \cdots, m$. Therefore, for any $x \in A$ and $y \in B$,

$$\widetilde{T_{\alpha_0}}(x)(y) = \sum_{i=1}^m \psi_i(x)\phi_i(y) = S\left(\sum_{i=1}^m \psi_i \otimes \phi_i\right)(x)(y)$$

where S is the canonical isometric map from $A^* \otimes_\lambda B^*$ to $B\left(A, B^*\right)$. Thus $\left\|\widetilde{T_{\alpha_0}}\right\| = \left\|\sum_{i=1}^m \psi_i \otimes \phi_i\right\|_\lambda$ and so $\left\|\frac{1}{2}\sum_{i=1}^m \psi_i \otimes \phi_i\right\|_\lambda \le 1$. Moreover, for $u = \sum_{n=1}^\infty a_n \otimes b_n$, we have

$$\|\theta(u)\| \ge \left|\theta(u)\left(\frac{1}{2}\sum_{i=1}^m \psi_i \otimes \phi_i\right)\right| = \left|\frac{1}{2}\sum_{n=1}^\infty \widetilde{T_{\alpha_0}}(a_n)(b_n)\right| = \left|\frac{1}{2}T_{\alpha_0}(u)\right| > \frac{1}{2}\|u\|_\gamma - \epsilon$$

Since $\epsilon > 0$ is arbitrary, so $\|\theta(u)\| \ge \frac{1}{2}\|u\|_\gamma$. □

Next, we consider the map $\phi : A^{**} \otimes_\gamma B^{**} \to \left(A^* \otimes_\lambda B^*\right)^*$ defined by $\phi = i' \circ J$.

Proposition 2.2 *For C^*-algebras A and B, the natural map $\phi : A^{**} \otimes_\gamma B^{**} \to \left(A^* \otimes_\lambda B^*\right)^*$ is bi-continuous and $\frac{1}{2}\|u\|_\gamma \le \|\phi(u)\| \le \|u\|_\gamma$, for all $u \in A^{**} \otimes_\gamma B^{**}$.*

Proof: By the above lemma, we have a map $\theta : A^{**} \otimes_\gamma B^{**} \to J\left(A^{***}, B^{****}\right)$ with $\frac{1}{2}\|u\|_\gamma \le \|\theta(u)\| \le \|u\|_\gamma$ for all $u \in A^{**} \otimes_\gamma B^{**}$. Also, ([6], Proposition 3.21) shows that the natural inclusion map $j' : J\left(A^*, B^{**}\right) \to J\left(A^{***}, B^{****}\right)\left(T \to T^{**}\right)$ is isometric. We will show that $j' \circ \phi = \theta$. For $F \in A^{**}$, $G \in B^{**}$, $\tilde{F} \in A^{***}$ and $\tilde{G} \in B^{***}$,

$$j' \circ \phi(F \otimes G)(\tilde{F})(\tilde{G}) = j'(\phi(F \otimes G))(\tilde{F})(\tilde{G}) = \left[\phi(F \otimes G)\right]^{**}(\tilde{F})(\tilde{G}) = \tilde{F}\left(\left[\phi(F \otimes G)\right]^*(\tilde{G})\right) = \tilde{F}(F)\tilde{G}(G),$$

since $\left[\phi(F \otimes G)\right]^*(\tilde{G})(f) = \tilde{G}(\phi(F \otimes G)(f)) = \tilde{G}(G)F(f)$ for $f \in A^*$. Thus

$j' \circ \phi(F \otimes G)(\tilde{F})(\tilde{G}) = \theta(F \otimes G)(\tilde{G})(\tilde{F})$. Therefore, by linearity and continuity, $j' \circ \phi = \theta$, and hence the map ϕ satisfies $\frac{1}{2}\|u\|_{\gamma} \le \|\phi(u)\| \le \|u\|_{\gamma}$ for all $u \in A^{**} \otimes_{\gamma} B^{**}$. □

Haagerup proved that every bounded bilinear form on $A \times B$ can be extended uniquely to a separately normal norm preserving bounded bilinear form on $A^{**} \times B^{**}$ ([7], Corollary 2.4), so we have a continuous isometric map $\chi : (A \otimes_{\gamma} B)^{*} \to (A^{**} \otimes_{\gamma} B^{**})^{*}$. Set

$$\mu = \chi^{*} \circ i : A^{**} \otimes_{\gamma} B^{**} \to (A \otimes_{\gamma} B)^{**}$$

where i is the natural embedding of $A^{**} \otimes_{\gamma} B^{**}$ into $(A^{**} \otimes_{\gamma} B^{**})^{**}$. Kumar and Sinclair proved that this embedding is a bi-continuous map with lower bound $\frac{1}{4}$ ([2], Theorem 5.1). We re-establish its bi-continuity with an alternate proof and an improved lower bound $\frac{1}{2}$.

Theorem 2.3 *For C^{*}-algebras A and B, the natural embedding μ satisfies $\frac{1}{2}\|u\|_{\gamma} \le \|\mu(u)\| \le \|u\|_{\gamma}$ for all $u \in A^{**} \otimes_{\gamma} B^{**}$.*

Proof: We know that the natural embedding $j : A^{*} \otimes_{\lambda} B^{*} \to B(A \times B, \mathbb{C}) = (A \otimes_{\gamma} B)^{*}$ is isometric. Thus, by the Hahn Banach theorem, $j^{*} : (A \otimes_{\gamma} B)^{**} \to (A^{*} \otimes_{\lambda} B^{*})^{*}$ is a quotient map. We will show that

$$j^{*} \circ \mu = \phi$$

where ϕ is as in Proposition 2.2. Since $j^{*} \circ \mu$ and ϕ are linear and continuous, it suffices to show that $j^{*} \circ \mu$ and ϕ agree on $A^{**} \otimes B^{**}$. Note that, for $F \in A^{**}$, $G \in B^{**}$, $f \in A^{*}$ and $g \in B^{*}$,

$$\begin{aligned} j^{*} \circ \mu(F \otimes G)(f \otimes g) &= j^{*}\left(\chi^{*}\left(\widehat{F \otimes G}\right)\right)(f \otimes g) = \chi^{*}\left(\widehat{F \otimes G}\right)\left(j(f \otimes g)\right) \\ &= \chi\left(j(f \otimes g)\right)(F \otimes G) = \chi\left(\overline{j(f \otimes g)}\right)(F \times G), \end{aligned}$$

where $\chi\left(\overline{j(f \otimes g)}\right)$ is the bilinear form corresponding to $\chi(j(f \otimes g)) \in \left(A^{**} \hat{\otimes} B^{**}\right)^{*}$.

Since $F \in A^{**}$ and $G \in B^{**}$ so, by Goldstine's Lemma, there are nets $x_{\lambda} \in A$ and $y_{\mu} \in B$ such that $\widehat{x_{\lambda}}$ converges to F in $\sigma(A^{**}, A^{*})$ and $\widehat{y_{\mu}}$ converges to G in $\sigma(B^{**}, B^{*})$. The separate w^{*}-continuity of the bilinear form $\chi\left(\overline{j(f \otimes g)}\right)$ and the equality $\chi\left(\overline{j(f \otimes g)}\right)\left(\widehat{x_{\lambda}}, \widehat{y_{\mu}}\right) = \widehat{x_{\lambda}}(f)\widehat{y_{\mu}}(g)$ shows that $j^{*} \circ \mu(F \otimes G)(f \otimes g) = \phi(F \otimes G)(f \otimes g)$. Thus, $j^{*} \circ \mu = \phi$. Hence, by Proposition 2.2, we deduce that $\frac{1}{2}\|u\|_{\gamma} \le \|\mu(u)\| \le \|u\|_{\gamma}$. □

Remark 2.4 (i) *Note that, for a C^{*}-algebra A having Completely positive approximation property, the canonical embedding of $A^{**} \otimes_{\min} B^{**}$ into $(A \otimes_{\min} B)^{**}$ is isometric by* ([8], *Theorem* 3.6) *and* ([9], *Theorem* 3.6). *However, for the largest Banach space tensor norm, the embedding μ is isometric if one of the C^{*}-algebra has the metric approximation property, which follows directly by using* ([6], *Theorem* 4.14) *in the above theorem.*

(ii) For a locally compact Hausdorff topological group G, let $C^{*}(G)$ and $C_{r}^{*}(G)$ be the group C^{*}-algebra and the reduced group C^{*}-algebra of G, respectively. Then, for any C^{*}-algebra A and a discrete amenable group G, the natural embedding of $C_{r}^{*}(G)^{**} \otimes_{\gamma} A^{**}$ into $(C_{r}^{*}(G) \otimes_{\gamma} A)^{**}$ is isometric by ([8], *Theorem* 4.2); and for any amenable group G, the natural embedding of $C^{*}(G)^{**} \otimes_{\gamma} A^{**}$ into $(C^{*}(G) \otimes_{\gamma} A)^{**}$ is isometric by ([8], Proposition 4.1).

(*iii*) The natural embedding μ is isomorphism if A^{*} has the approximation property, A^{**} has the Radon Nikodym property and every bilinear form on $A \times B$ is nuclear. This follows directly by observing that if A^{**} has the Radon Nikodym property then ([6], Theorem 5.32) gives us

$$N\left(B^*, A^{**}\right) = PJ\left(B^*, A^{**}\right) = J\left(B^*, A^{**}\right)$$

where $PJ\left(B^*, A^{**}\right)$ and $N\left(B^*, A^{**}\right)$ denote the Pietsch integral and nuclear operators from B^* to A^{**}, respectively [6]. Clearly, bijectivity follows if we show that j is an onto map. For this, let $T \in B\left(A \times B, \mathbb{C}\right)$ so it is nuclear. Since A^* has the approximation property, so there exists an element $u \in A^* \otimes_\gamma B^*$ such that

$$J\left(u\right) = T$$

where J is an isometric isomorphism from $A^* \otimes_\gamma B^*$ to $B\left(A \times B, \mathbb{C}\right)$ ([6], *Corollary* 4.8). Consider the canonical map $i : A^* \otimes_\gamma B^* \to A^* \otimes_\lambda B^*$. Of course $j \circ i = J$ on $A^* \otimes B^*$, and hence by linearity and continuity $j \circ i = J$.

We now discuss the operator space version of the above embedding, which is already discussed in [4]. Note that in this case the embedding is positive, and becomes an isomorphism under the conditions weaker than that required in case of the Banach space projective tensor product. For operator spaces X and Y, an operator from X into Y is called completely nuclear if it lies in the image of the map $J : X^* \hat{\otimes} Y \to X^* \check{\otimes} Y$ [10]. The space of completely nuclear operators will be denoted by $CN\left(X, Y\right)$. This space has the natural operator space structure determined by the identification $CN\left(X, Y\right) \cong \dfrac{X^* \hat{\otimes} Y}{\ker\left(J\right)}$.

For C^*-algebras A and B, consider the map θ from $A \hat{\otimes} B$ into the dual of operator space injective tensor product $\left(A^* \check{\otimes} B^*\right)^*$ given by

$$\theta = S \circ J \circ i$$

where $i : A \hat{\otimes} B \to A^{**} \hat{\otimes} B^{**}$ is the natural completely isometric map, $J : A^{**} \hat{\otimes} B^{**} \to CN\left(A^*, B^{**}\right)$ and $S : CN\left(A^*, B^{**}\right) \to \left(A^* \check{\otimes} B^*\right)^*$ [10]. Making use of the fact that the dual of the operator space projective tensor norm is the operator space injective ([10], Proposition 8.1.2) and an application of Grothendieck's theorem for jcb ([11], Proposition 1) and the techniques of Lemma 2.1, we obtain the following:

Lemma 2.5 *For C^*-algebras A and B, the canonical map $\theta : A \hat{\otimes} B \to \left(A^* \check{\otimes} B^*\right)^*$ satisfies $\frac{1}{2}\|u\|_\wedge \le \|\theta(u)\| \le \|u\|_\wedge$ for all $u \in A \hat{\otimes} B$. In particular, θ is bi-continuous.*

Proposition 2.6 *For C^*-algebras A and B, the natural map $\phi : A^{**} \hat{\otimes} B^{**} \to \left(A^* \check{\otimes} B^*\right)^*$, defined by $\phi = S \circ J$, is bi-continuous satisfying $\frac{1}{2}\|u\|_\wedge \le \|\phi(u)\| \le \|u\|_\wedge$ for all $u \in A^{**} \hat{\otimes} B^{**}$.*

Proof. By ([10], Theorem 15.3.1) we have A^* is locally reflexive operator space. Therefore, ([10], Theorem 14.3.1) implies that $\left(A^* \check{\otimes} B^*\right)^*$ can be identified with $I\left(A^*, B^{**}\right)$, where $I\left(A^*, B^{**}\right)$ denotes the space of completely integral operators from A^* to B^{**}. Now, the result follows by using the techniques of Proposition 2.2 and ([10], Proposition 15.4.4).

By ([4], Proposition 2.5), we have a continuous completely isometric map $\chi : \left(A \hat{\otimes} B\right)^* \to \left(A^{**} \hat{\otimes} B^{**}\right)^*$. Let

$$\mu = \chi^* \circ i : A^{**} \hat{\otimes} B^{**} \to \left(A \hat{\otimes} B\right)^{**}$$

where i is the natural embedding of $A^{**} \hat{\otimes} B^{**}$ into $\left(A^{**} \hat{\otimes} B^{**}\right)^{**}$. Then clearly $\|\mu\| \le 1$.

For a matrix ordered space A and its dual space A^*, we define *-operation on A^* by $f^*(x) = \overline{f\left(x^*\right)}$, $x \in A$ and $M_n\left(A^*\right)^+ = \left\{\phi \in CB\left(A, M_n\right) : \phi \text{ is completely positive}\right\}$. Note that, for C^*-algebras A and B, $A \hat{\otimes} B$ is a Banach *-algebra ([12], Proposition 3).

Theorem 2.7 *For C^*-algebras A and B, the natural embedding μ is *-preserving positive bounded map which satisfies $\frac{1}{2}\|u\|_\wedge \le \|\mu(u)\| \le \|u\|_\wedge$ for all $u \in A^{**} \hat{\otimes} B^{**}$.*

Proof: Given $\alpha \in M_{1,\infty^2}$, $\beta \in M_{\infty^2,1}$, $m \in M_\infty\left(A^{**}\right)$, $n \in M_\infty\left(A^{**}\right)$ and $f \in \left(A \hat{\otimes} B\right)^*$,

$$\mu\left(\alpha\left(m \otimes n\right)\beta\right)^*\left(f\right) = \overline{\mu\left(\alpha\left(m \otimes n\right)\beta\right)\left(f^*\right)} = \overline{\chi\left(f^*\right)\left(\alpha\left(m \otimes n\right)\beta\right)} = \chi\left(f^*\right)^*\left(\beta^*\left(m^* \otimes n^*\right)\alpha^*\right)$$

On the other hand, $\mu\left(\beta^*\left(m^* \otimes n^*\right)\alpha^*\right)\left(f\right) = \chi\left(f\right)\left(\beta^*\left(m^* \otimes n^*\right)\alpha^*\right)$. So in order to prove that μ is *-

preserving, we have to show that $\chi(f)^* = \chi(f^*)$.

Note that, for $a \in A$ and $b \in B$, $\chi(f)^*(\hat{a}\otimes\hat{b}) = \overline{\chi(f)(\hat{a}^*\otimes\hat{b}^*)} = \overline{f(a^*\otimes b^*)} = f^*(a\otimes b) = \chi(f^*)(\hat{a}\otimes\hat{b})$, and hence the result follows from the separate w^*-continuity of the bilinear forms corresponding to $\chi(f^*)$ and $\chi(f)^*$.

Now given an algebraic element $\alpha(m\otimes n)\alpha^* \in C_1$, where C_n is defined as in [13]. For the positivity of μ, we have to show that $\mu(\alpha(m\otimes n)\alpha^*)(f)\geq 0$ for $f \in \left(\left(A\hat{\otimes}B\right)^*\right)^+$. By ([13], Theorem 1.9), it suffices to show that if $\tilde{f}\in CP(A,B^*)$ then

$$\widetilde{\chi(f)}\in CP\left(A^{**},B^{***}\right)$$

where $\tilde{f}(a)(b) = f(a\otimes b)$ for all $a\in A$, $b\in B$ and $\widetilde{\chi(f)}(m)(n) = \chi(f)(m\otimes n)$ for all $m\in A^{**}$, $n\in B^{**}$. Since $M_n(A)^+$ is w^*-dense in $M_n(A^{**})^+$, so given $[F_{ij}]\in M_n(A^{**})$ we obtain a net $[a_{ij}^\lambda]\in M_n(A)^{++}$ which is w^*-convergent to $[F_{ij}]$. Now note that $\widetilde{\chi(f)}_n[F_{ij}] = w^*-\lim_\lambda \tilde{f}_n[a_{ij}^\lambda]$. Hence the result follows.

The bi-continuity of the map μ follows as in Theorem 2.3. □

Remark 2.8 *By* ([14], *Theorem* 2.2), *the natural embedding* μ *is completely isometric if one of the* C^**-algebras has the* W^* *MAP.*

We now discuss the isomorphism of this embedding. For $x\in A$, the map $f\otimes g\to\hat{x}(f)g = f(x)g$, for $f\in A^*$ and $g\in B^*$, has a unique continuous extension to a map $R_{\hat{x}}: A^*\hat{\otimes}B^*\to B^*$, with $\|R_{\hat{x}}\|\leq\|x\|$. The next proposition does not have counterpart in the Banach space context.

Proposition 2.9 *For* C^**-algebras* A *and* B, *the family* $\{R_{\hat{x}}: x\in A\}$ *is total on* $A^*\hat{\otimes}B^*$.

Proof: Suppose that $u\in A^*\hat{\otimes}B^*$ such that $R_{\hat{x}}(u)=0$ for all $x\in A$. Let $T\in\left(A^*\hat{\otimes}B^*\right)^*$ with $\|T\|\leq 1$. Since $\left(A^*\hat{\otimes}B^*\right)^* = CB(B^*,A^{**})$, so $T(f\otimes g) = \tilde{T}(g)(f)$ for some $\tilde{T}\in CB(B^*,A^{**})$, for all $f\in A^*$ and $g\in B^*$, with $\|\tilde{T}\|_{cb} = \|T\|_{cb}\leq 1$. If A^{**} is taken in the universal representation of A then $\tilde{T}$ satisfies the W^*AP by ([10], Theorem 15.1) and ([5], § 1.4.10). So there exists a net $\tilde{T}_\alpha$ of finite rank w^*-continuous mapping from B^* to A^{**} such that $\|\tilde{T}_\alpha\|_{cb}\leq\|\tilde{T}\|_{cb}$, and $\tilde{T}_\alpha(g)\to\tilde{T}(g)$ for all $g\in B^*$. Thus for $u\in A^*\hat{\otimes}B^*$ and $\epsilon>0$, there exists α_0 such that $|T(u)-T_\alpha(u)|<\epsilon$ for all $\alpha\geq\alpha_0$. Since $\tilde{T}_\alpha\in CB(B^*,A^{**})$, we have $T_\alpha\in\left(A^*\hat{\otimes}B^*\right)^*$ such that $T_\alpha(f\otimes g) = \tilde{T}_\alpha(g)(f)$. Since $\tilde{T}_\alpha$ is a finite rank operator so, as in Lemma 2.1, $\tilde{T}_\alpha(g) = \sum_{j=1}^{l}\Phi_j(g)\Psi_j$ for $\Phi_j\in B^{**}$ and $\Psi_j\in A^{**}$. Thus, for

$$u = \sum_{k=1}^{\infty}\alpha_k\left(f_k\otimes g_k\right)\beta_k$$

where $\alpha_k\in M_{1,p_k\times q_k}$, $\beta_k\in M_{p_k\times q_k,1}$, $f_k\in M_{p_k}(A^*)$, and $g_k\in M_{q_k}(B^*)$, a norm convergent representation in $A^*\hat{\otimes}B^*$ [10], $T_\alpha(u) = \sum_{k=1}^{\infty}\sum_{m,n,p,q}\alpha_{1,mn}^k\tilde{T}_\alpha(g_{nq}^k)(f_{mp}^k)\beta_{pq,1}^k = \sum_{j=1}^{l}\Psi_j\left(\sum_{k=1}^{\infty}\sum_{m,n,p,q}\alpha_{1,mn}^k\Phi_j(g_{nq}^k)f_{mp}^k\beta_{pq,1}^k\right)$. Given $\sum_{k=1}^{\infty}\sum_{m,n,p,q}\alpha_{1,mn}^k\hat{x}(f_{mp}^k)g_{nq}^k\beta_{pq,1}^k = 0$ for all $x\in A$. Therefore, $\sum_{k=1}^{\infty}\sum_{m,n,p,q}\alpha_{1,mn}^k f_{mp}^k G(g_{nq}^k)\beta_{pq,1}^k = 0$ for any $G\in B^{**}$. Thus $T_\alpha(u)=0$, giving that $|T(u)|\leq|T(u)-T_\alpha(u)|+|T_\alpha(u)|<\epsilon$ for all $\alpha\geq\alpha_0$, and hence $u=0$. □

In particular, the map $J: A^*\hat{\otimes}B^*\to CN(A,B^*)$ defined above is 1-1. Thus $A^*\hat{\otimes}B^*\cong CN(A,B^*)$.

Now, as in Remark 2.4(iii), we have the following:

Corollary 2.10 *Let* A *and* B *be* C^**-algebras such that every completely bounded operator from* A *to* B^* *is completely nuclear and the map* ϕ *defined in the Proposition* 2.6 *is onto. Then the natural embedding*

$\mu : A^{**} \hat{\otimes} B^{**} \to \left(A \hat{\otimes} B\right)^{**}$ *is an isomorphism map.*

Remark 2.11 *The embedding in the case of the Haagerup tensor product turns out to be completely isometric, which can be seen as below. For operator spaces* X, Y, *using the fact that* $T_n^* = M_n$ *and* ([5], § 1.6.7), *the map* $\chi : \left(X \otimes_h Y\right)^* \to \left(X^{**} \otimes_h Y^{**}\right)^*$ *is completely isometric. Set*

$$\tau := \chi^* \circ i$$

where $i : X^{**} \otimes_h Y^{**} \to \left(X^{**} \otimes_h Y^{**}\right)^{**}$. *Then, clearly* $\|\tau\|_{cb} \le 1$. *By the self-duality of the Haagerup norm, the map* $\phi : X^{**} \otimes_h Y^{**} \to \left(X^* \otimes_h Y^*\right)^*$ *is completely isometric. As in Theorem 2.3,*

$$j^* \circ \tau = \phi,$$

where j *is the completely isometric map from* $X^* \otimes_h Y^*$ *to* $\left(X \otimes_h Y\right)^*$, *which further gives* $j_n^* \circ \tau_n = \phi_n$ *for any* $n \in \mathbb{N}$. *Thus* τ *is completely isometric.*

3. Closed Ideals in $A \hat{\otimes} B$

It was shown in ([4], Theorem 3.8) that if A or B is a simple C^*-algebra then every closed ideal of $A \hat{\otimes} B$ is the product ideal, *i.e.* of the form $A \hat{\otimes} J$ or $I \hat{\otimes} B$ for closed ideal I of A and J of B. In the following, we generalize this result to the C^*-algebra which has only a finite number of closed ideals. More precisely, it is shown that if one of the C^*-algebras A and B has only finitely many closed ideals, then every closed ideal in $A \hat{\otimes} B$ is precisely of the form $\sum_{j=1}^{n} I_j \hat{\otimes} J_j$, for some $n \in N$ and closed ideals I_j in A, J_j in B, $j = 1, 2, \cdots, n$. Thus obtaining the complete lattice of closed ideals of $B(H) \hat{\otimes} B(H)$, $B(H) \hat{\otimes} C_0(X)$, $\left(M(A)/A\right) \hat{\otimes} B$, where H is an infinite dimensional separable Hilbert space, X is a locally compact Hausdorff space, B is any C^*-algebra and $M(A)$ is the multiplier algebra of A, A being a nonunital, non-elementary, separable, simple *AF* C^*-algebra ([15], Theorem 2). We would like to remark that in [4] the lattice of closed ideals of $B(H) \hat{\otimes} B(H)$ has already been explored.

Proposition 3.1 *Let* A *and* B *be* C^**-algebras and* I *a closed ideal in* $A \hat{\otimes} B$. *If* $a \otimes b \in I_h$, *the closure of* $i(I)$ *in* $\|\cdot\|_h$, *then* $a \otimes b \in I$, *where* i *is the natural map from* $A \hat{\otimes} B$ *into* $A \otimes_h B$.

Proof: Since $a \otimes b \in I_h$ so there exists a sequence $i_n \in I$ such that $\left\|a \otimes b - i(i_n)\right\|_h \to 0$ as n tends to infinity. Consider the identity map $\epsilon : A \otimes_h B \to A \otimes_{\min} B$ and $i' : A \hat{\otimes} B \to A \otimes_{\min} B$. Of course, $i' = \varepsilon \circ i$ on $A \otimes B$, and hence by continuity $i' = \varepsilon \circ i$. Thus $a \otimes b \in I_{\min}$ and so $a \otimes b \in I$ by ([12], Theorem 6). □

The following lemma can be proved as a routine modification to the arguments of ([16], Lemma 1.1).

Lemma 3.2 *For closed ideals* M *of* A *and* N *of* B, $A \hat{\otimes} N + M \hat{\otimes} B = \left(A \otimes_h N + M \otimes_h B\right) \cap \left(A \hat{\otimes} B\right)$.

In order to prove our main result. We first investigate the inverse image of product ideals of $A_2 \hat{\otimes} B_2$ for C^*-algebras A_2 and B_2, which is largely based on the ideas of ([10], Proposition 7.1.7)

Proposition 3.3 *For* C^**-algebras* A_1, A_2, B_1, *and* B_2 *and the complete quotient maps* $\phi : A_1 \to A_2$, $\psi : B_1 \to B_2$. *Let* I_2 *and* J_2 *be closed ideals in* A_2 *and* B_2, *respectively. Then*

$$\left(\phi \hat{\otimes} \psi\right)^{-1}\left(I_2 \hat{\otimes} J_2\right) = \phi^{-1}(0) \hat{\otimes} B_1 + \phi^{-1}(I_2) \hat{\otimes} \psi^{-1}(J_2) + A_1 \hat{\otimes} \psi^{-1}(0).$$

Proof: By ([4], Proposition 3.2) and the Bipolar theorem, it suffices to show that

$$\left(\phi \hat{\otimes} \psi\right)^{-1}\left(I_2 \hat{\otimes} J_2\right)^{\perp} = \left[\phi^{-1}(0) \hat{\otimes} A_2 + \phi^{-1}(I_2) \hat{\otimes} \psi^{-1}(J_2) + A_1 \hat{\otimes} \psi^{-1}(0)\right]^{\perp}.$$

Let $F \in \left[\phi^{-1}(0) \hat{\otimes} A_2 + \phi^{-1}(I_2) \hat{\otimes} \psi^{-1}(J_2) + A_1 \hat{\otimes} \psi^{-1}(0)\right]^{\perp}$ then $F \in \left(A_1 \hat{\otimes} B_1\right)^*$ and $F\left(\phi^{-1}(0) \hat{\otimes} A_2 + \phi^{-1}(I_2) \hat{\otimes} \psi^{-1}(J_2) + A_1 \hat{\otimes} \psi^{-1}(0)\right) = 0$. Since $JCB\left(A_1 \times B_1, \mathbb{C}\right) = \left(A_1 \hat{\otimes} B_1\right)^*$, so $F\left(v \otimes w\right) = F_1\left(v, w\right)$ for some $F_1 \in JCB\left(A_1 \times B_1, \mathbb{C}\right)$, for all $v \in A_1$ and $w \in B_1$. Define a bilinear map $F_2 : A_2 \times B_2 \to \mathbb{C}$ as

$$F_2(v_1, w_1) = F_1(v, w)$$

where $\phi(v) = v_1$ and $\psi(w) = w_1$. Clearly, F_2 is well defined. Note that, for $p \in \mathbb{N}$, $[v_{ij}] \in M_p(A_1)$ and $[w_{kl}] \in M_p(B_1)$, we have $(F_2)_p(\phi_p[v_{ij}], \psi_p[w_{kl}]) = (F_1)_p([v_{ij}],[w_{kl}])$. For any $\epsilon > 0$, there are $[v_{ij}^1] \in M_p(A_2)$ and $[w_{kl}^1] \in M_p(B_2)$ with $\|[v_{ij}^1]\| \le 1$, $\|[w_{kl}^1]\| \le 1$ such that $\|(F_2)_p\| - \epsilon < \|(F_2)_p([v_{ij}^1],[w_{kl}^1])\|$. We can find $r, s \in \mathbb{R}$ such that $\|[v_{ij}^1]\| < r \le 1$, $\|[w_{kl}^1]\| < s \le 1$. By definition, we may write

$$\phi_p[v_{ij}] = \frac{[v_{ij}^1]}{r} \text{ and } \psi_p[w_{kl}] = \frac{[w_{kl}^1]}{s}$$

where $[v_{ij}] \in M_p(A_1)$, $[w_{kl}] \in M_p(B_1)$ both have norm $<$ 1. Thus

$$\|(F_1)_p\| > \|(F_1)_p([v_{ij}],[w_{kl}])\| = \|(F_2)_p(\phi_p[v_{ij}], \psi_p[w_{kl}])\| = \left\|(F_2)_p\left(\frac{[v_{ij}^1]}{r}, \frac{[w_{kl}^1]}{s}\right)\right\|, \text{ and so } \|(F_1)_p\| \ge \|(F_2)_p\|.$$

This shows that $F_2 : A_2 \times B_2 \to \mathbb{C}$ is jcb bilinear form. Thus it will determine a $\tilde{F}_2 \in (A_2 \hat{\otimes} B_2)^*$. We have $F(v \otimes w) = F_1(v, w) = F_2(\phi(v), \psi(w)) = \tilde{F}_2(\phi(v) \otimes \psi(w)) = \tilde{F}_2 \circ (\phi \hat{\otimes} \psi)(v \otimes w)$ for all $v \in A_1$ and $w \in B_1$. This implies that $F = \tilde{F}_2 \circ (\phi \hat{\otimes} \psi)$ on $A_1 \otimes B_1$, and so by continuity $F = \tilde{F}_2 \circ (\phi \hat{\otimes} \psi)$. Now let $z \in (\phi \hat{\otimes} \psi)^{-1}(I_2 \hat{\otimes} J_2)$. We may assume that $\|z\|_\wedge < 1$. Then $\phi \hat{\otimes} \psi(z) \in I_2 \hat{\otimes} J_2$ and $\|\phi \hat{\otimes} \psi(z)\|_\wedge < 1$. So $\phi \hat{\otimes} \psi(z) = \sum_{k=1}^{\infty} \alpha_k (i_k \otimes j_k) \beta_k = \sum_{k=1}^{\infty} \sum_{m,n,p,q} \alpha_{1,mn}^k (i_{mp}^k \otimes j_{nq}^k) \beta_{pq,1}^k$ with $i_{mp}^k \in I_2$, $j_{nq}^k \in J_2$ and $\|i_{mp}^k\| < 1$, $\|j_{nq}^k\| < 1$ [10]. Since ϕ and ψ are complete quotient maps and $F(\phi^{-1}(0) \hat{\otimes} B_1 + \phi^{-1}(I_2) \hat{\otimes} \psi^{-1}(J_2) + A_1 \hat{\otimes} \psi^{-1}(0)) = 0$, so it follows that $F(z) = 0$. Hence

$$[\phi^{-1}(0) \hat{\otimes} A_2 + \phi^{-1}(I_2) \hat{\otimes} \psi^{-1}(J_2) + A_1 \hat{\otimes} \psi^{-1}(0)]^\perp \subseteq (\phi \hat{\otimes} \psi)^{-1}(I_2 \hat{\otimes} J_2)^\perp.$$

Since the annihilator is reverse ordering, so converse is trivial. □

Now we are ready to prove the main result.

Theorem 3.4 *If A and B are C^*-algebras such that number of closed ideals in A is finite. Then every closed ideal in $A \hat{\otimes} B$ is a finite sum of product ideals.*

Proof. Proof is by induction on $n(A)$, the number of closed ideals in A counting both $\{0\}$ and A. If $n(A) = 2$ then the result follows directly by ([4], Theorem 3.8). Suppose that the result is true for all C^*-algebras with $n(A) \le n-1$. Let A be a C^*-algebra with $n(A) = n$.

Since there are only finitely many closed ideals in A so there exists a minimal non-zero closed ideal, say I, which is simple by definition. Let K be a closed ideal in $A \hat{\otimes} B$ then $K \cap (I \hat{\otimes} B)$ is a closed ideal in $I \hat{\otimes} B$. So it is equal to $I \hat{\otimes} J$ for some closed ideal J in B by ([4], Theorem 3.8). Consider the closed ideal K_h, the closure of $i(K)$ in $\|\cdot\|_h$, where $i : A \hat{\otimes} B \to A \otimes_h B$ is an injective map ([11], Theorem 1). Then $K_h \cap (I \otimes_h B) = I \otimes_h \tilde{J}$ for some closed ideal $\tilde{J}$ in B by ([17], Proposition 5.2). We first show that $\tilde{J} = J$. Since the map $i : A \hat{\otimes} B \to A \otimes_h B$ is injective so $K_h \cap (A \hat{\otimes} B) \supseteq K$. Thus $I \otimes_h \tilde{J} \cap (A \hat{\otimes} B) \supseteq K \cap (I \otimes_h B)$, which by using ([18], Corollary 4.6), ([19], Proposition 4), and Lemma 3.2, gives that $I \hat{\otimes} \tilde{J} \supseteq I \hat{\otimes} J$ and so $\tilde{J} \supseteq J$. To see the equality, let $\tilde{j} \in \tilde{J}$. Take any $0 \ne i \in I$ then $i \otimes \tilde{j} \in K_h$ so it belongs to K by Proposition 3.1. Thus $i \otimes \tilde{j} \in I \hat{\otimes} J$. Hence $\tilde{j} \in J$.

As in ([17], Theorem 5.3), $K_h \subseteq A \otimes_h J + M \otimes_h B$ for $M = ann(I)$. Thus $K \subseteq A \hat{\otimes} J + M \hat{\otimes} B$ by Lemma 3.2. Since M cannot contain I, so $n(M) \le n(A) - 1 = n - 1$. Thus $K \cap (M \hat{\otimes} B)$, which is a closed ideal in $M\hat{\otimes}B$, is a finite sum of product ideals by induction hypothesis. Let $T=K\cap\left(A\hat{\otimes}J\right)$ then clearly T contains $I\hat{\otimes}J$. Corresponding to the complete quotient map $\pi:A\to A/I$, we have a quotient map $\pi\hat{\otimes}id:A\hat{\otimes}J\to A/I\hat{\otimes}J$ with kernel $I\hat{\otimes}J$ and $\pi\hat{\otimes}id(T)$ is a closed ideal of $A/I\hat{\otimes}J$ ([19], Lemma 2). Also $n(A/I)\le n(A)-1=n-1$ and so by the induction hypothesis

$$\pi\hat{\otimes}id(T)=\sum_{r=1}^{t}I_r\hat{\otimes}J_r$$

where I_r and J_r are closed ideals in A/I and J, for $r=1,\cdots,t$, respectively. Thus, by ([19], Lemma 2) and Theorem 3.3, $T=\sum_{r=1}^{t}\pi^{-1}(I_r)\hat{\otimes}J_r+I\hat{\otimes}J$. So $K\cap\left(A\hat{\otimes}J\right)+K\cap\left(M\hat{\otimes}B\right)$ is a finite sum of product ideal and hence closed by ([4], Proposition 3.2).

We now claim that $K\cap\left(A\hat{\otimes}J+M\hat{\otimes}B\right)=K\cap\left(A\hat{\otimes}J\right)+K\cap\left(M\hat{\otimes}B\right)$.

Let $z\in K\cap\left(A\hat{\otimes}J+M\hat{\otimes}B\right)$. Since the closed ideal $A\hat{\otimes}J+M\hat{\otimes}B$ has a bounded approximate identity so there exist $x,y\in A\hat{\otimes}J+M\hat{\otimes}B$ such that $z=xy$ and y belongs to the least closed ideal of $A\hat{\otimes}J+M\hat{\otimes}B$ containing z ([20], § 11, Corollary 11). This implies that $y\in K$ so $z\in K\cap\left(A\hat{\otimes}J\right)+K\cap\left(M\hat{\otimes}B\right)$. Hence $K\cap\left(A\hat{\otimes}J+M\hat{\otimes}B\right)=K\cap\left(A\hat{\otimes}J\right)+K\cap\left(M\hat{\otimes}B\right)$. Therefore K is a finite sum of product ideals.

4. Inner Automorphisms of $A\hat{\otimes}B$

For unital C^*-algebras A and B, isometric automorphism of $A\hat{\otimes}B$ is either of the form $\phi\hat{\otimes}\psi$ or $\nu\hat{\otimes}\rho\circ\tau$, where $\phi:A\to A$, $\psi:B\to B$, $\nu:B\to A$ and $\rho:A\to B$ are isometric isomorphisms ([11], Theorem 4). In the following, we characterize the isometric inner *-automorphisms of $A\hat{\otimes}B$ completely.

Proposition 4.1 *For unital C^*-algebras A and B, the map $\phi\hat{\otimes}\psi$ $(\phi\otimes_h\psi)$ is inner automorphism of $A\hat{\otimes}B$ (resp. $A\otimes_h B$) if and only if ϕ is inner automorphism of A and ψ is inner automorphism of B.*

Proof: Suppose that $\phi\hat{\otimes}\psi$ is implemented by $u\in A\hat{\otimes}B$. We will show that $\phi\otimes_{\min}\psi$ is implemented by $i(u)$, where i is *-homomorphism from $A\hat{\otimes}B$ into $A\otimes_{\min}B$ [11]. It is easy to see that $i\circ\phi\hat{\otimes}\psi=\phi\otimes_{\min}\psi\circ i$. So, for $x\in A\hat{\otimes}B$, $\phi\otimes_{\min}\psi\left(i(x)\right)=i\left(uxu^{-1}\right)=i(u)i(x)i\left(u^{-1}\right)=i(u)i(x)i(u)^{-1}$. As $i\left(A\hat{\otimes}B\right)$ is $\|\cdot\|_{\min}$-dense in $A\otimes_{\min}B$, so $\phi\otimes_{\min}\psi$ is implemented by $i(u)$. Hence the result follows from ([21], Theorem 1). Converse is trivial. □

We now characterize the isometric inner automorphism of $A\hat{\otimes}B$ for C^*-algebras A and B other than M_n.

Theorem 4.2 *For unital C^*-algebras A and B other than M_n for some $n\in\mathbb{N}$, the isometric inner *-automorphism of $A\hat{\otimes}B$ is of the form $\phi\hat{\otimes}\psi$, where ϕ and ψ are inner *-automorphisms of A and B, respectively.*

Proof: Suppose that θ is the isometric inner *-automorphism of $A\hat{\otimes}B$. So

$$\theta=\phi\hat{\otimes}\psi$$

where ϕ and ψ are *-automorphisms of A and B, respectively or

$$\theta=\nu\hat{\otimes}\rho\circ\tau$$

where $\nu:B\to A$ and $\rho:A\to B$ are *-isomorphisms, $\tau:A\hat{\otimes}B\to B\hat{\otimes}A$ is a flip map [11]. In view of Proposition 4.1, it suffices to show that the second case will never arise for C^*-algebras A and B other than M_n. Let $J\neq\{0\}$ be a proper closed ideal in B and $I_1=A\hat{\otimes}J$, which is a closed ideal in $A\hat{\otimes}B$ by ([12], Theorem 5). Since $\theta=\nu\hat{\otimes}\rho\circ\tau$ is inner so it preserves $A\hat{\otimes}J$. Thus, for any $x\in J$, $\theta(1\otimes x)\in I_1$. Therefore $\nu(x)\otimes(1+J)=0$ [19], which further gives that $x=0$. Hence $J=\{0\}$ and so B is simple. Similarly, one can show that A is simple. By hypothesis there exists $d\in A\hat{\otimes}B$ which implements θ so that $\nu(b)\otimes\rho(a)=d(a\otimes b)d^{-1}$ for all $a\in A$ and $b\in B$. Choose $z,w\in A\otimes B$ such that $\|z-d\|_\wedge<\frac{1}{4}\|d^{-1}\|_\wedge^{-1}$,

$\|z\|_\wedge < \|d\|_\wedge + 1$, and $\|w - d^{-1}\|_\wedge < \frac{1}{4}(\|d\|_\wedge + 1)^{-1}$. Thus, for $z = \sum_{i=1}^{r} x_i \otimes y_i$ and $w = \sum_{j=1}^{s} u_j \otimes v_j$, we have $\|v(b) \otimes \rho(a) - z(a \otimes b)w\|_\wedge \le \|d(a \otimes b)d^{-1} - z(a \otimes b)d^{-1}\| + \|z(a \otimes b)d^{-1} - z(a \otimes b)w\|_\wedge \le \frac{1}{2}\|a\|\|b\|$, hence $\left\|1 \otimes \rho(a) - \sum_{i=1, j=1}^{r,s} x_i a u_j \otimes y_i v_j\right\|_\wedge \le \frac{1}{2}\|a\|$.

Now choose $f \in A^*$ such that $f(1) = \|f\| = 1$. Therefore, $\left\|\rho(a) - \sum_{i=1, j=1}^{r,s} f(x_i a u_j) y_i v_j\right\| \le \frac{1}{2}\|a\|$ for all $a \in A$. Take any $b \in B$, ρ being an isomorphism, there exists a unique $a \in A$ such that $b = \rho(a)$. Thus $\left\|b - \sum_{i=1, j=1}^{r,s} f(x_i \rho^{-1}(b) u_j) y_i v_j\right\| \le \frac{1}{2}\|b\|$ for any $b \in B$. Now define a finite dimensional subspace D of B by

$$D = \operatorname{span}\{y_i v_j : i = 1, 2, \cdots, r, j = 1, 2, \cdots, s\}.$$

The above inequality implies that $D \cap B\left[b, \frac{\|b\|}{2}\right] \ne \varnothing$, where $B\left[b, \frac{\|b\|}{2}\right]$ is the closed ball center at b and radius $\frac{\|b\|}{2}$. If D is proper then Riesz Lemma implies that for $r > \frac{1}{2}$ there exists $x_r \in B$ such that $\|x_r\| = 1$ and $\operatorname{dist}(x_r, D) \ge r$. Since $D \cap B\left[b, \frac{\|b\|}{2}\right] \ne \varnothing$ for any $b \in B$, so we can choose $d \in D$ such that $\|d - x_r\| \le \frac{1}{2}$, and, because $r > \frac{1}{2}$, a contradiction arises. Therefore, $D = B$. Thus, by the classical Wedderburn-Artin Theorem, $B = M_n$ for some $n \in \mathbb{N}$. Similarly, $A = M_n$ for some $n \in \mathbb{N}$. □

However, by ([11], Theorem 5), for unital C^*-algebras A and B with at least one being non-commutative, isometric inner automorphism of $A \otimes_h B$ is of the form $\phi \otimes_h \psi$, where ϕ and ψ are inner automorphisms of A and of B, respec- tively.

Corollary 4.3 *For an infinite dimensional separable Hilbert space H, every inner automorphism of $B(H) \hat{\otimes} B(H)$ is of the form $\phi \hat{\otimes} \psi$, where ϕ and ψ are inner automorphisms of $B(H)$.*

We now give an equivalent form of Proposition 4.1 in case of operator algebras. For operator algebras V and W, we do not know if $\phi \hat{\otimes} \psi$ (or $\phi \otimes_h \psi$) is inner then ϕ and ψ are inner or not. However, if one of the automorphism is an identity map then we have an affirmative answer for the Haagerup tensor product. In order to prove this, we need the following results.

Proposition 4.4 *For operator spaces V and W, the family $\{R_\phi : \phi \in V^*\}$ $(\{L_\psi : \psi \in W^*\})$ is total on $V \otimes_h W$.*

Proof: For $u \in V \otimes_h W$, assume that $R_\phi(u) = 0$ for all $\phi \in V^*$. We can assume that $\|u\|_h < 1$. Therefore, for $u = \sum_{i=1}^{\infty} a_i \otimes b_i$ a norm convergent representation in $V \otimes_h W$, where $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are strongly independent with $\left\|\sum_{i=1}^{\infty} a_i a_i^*\right\|^{1/2} < 1$ and $\left\|\sum_{i=1}^{\infty} b_i^* b_i\right\|^{1/2} < 1$. Then we have $\sum_{i=1}^{\infty} \phi(a_i) b_i = 0$ for all $\phi \in V^*$. From the strongly independence of $\{a_i\}_{i=1}^{\infty}$, choose linear functionals $\phi_j \in V^*$ such that

$$\|(\phi_j(a_1), \phi_j(a_2), \cdots, \cdots) - e_j\| < \epsilon$$

where $\{e_j\}$ are the standard basis for l_2 by the equivalent form of ([17], Lemma 2.2). Thus $\left\|b_j - \sum_{i=1}^{\infty} \phi_j(a_i) b_i\right\| < \epsilon$ and so $\|b_j\| < \epsilon$. Because ϵ was arbitrary, we conclude that $b_j = 0$ for each j, hence

$u=0$. □

Corollary 4.5 *For operator algebras V and W, if ϕ and ψ are completely contractive automorphisms of V and W, respectively. Then $\phi\otimes_h\psi$ is a completely contractive automorphism of $V\otimes_h W$.*

Proof: By the functoriality of the Haagerup tensor product, the map $\phi\otimes_h\psi : V\otimes_h W\to V\otimes_h W$ is completely contractive. One can see that $\phi\otimes_h\psi$ is an algebra homomorphism. Let $u=\sum_{i=1}^{\infty}a_i\otimes b_i$ be a norm convergent representation in $V\otimes_h W$. Since ϕ and ψ are bijective maps, so there exist unique $\tilde{a}_i\in V$ and $\tilde{b}_i\in W$, for each i, such that $u=\sum_{i=1}^{\infty}\phi(\tilde{a}_i)\otimes\psi(\tilde{b}_i)$. By [22], there is a new norm on V and W with respect to that V and W become a new operator algebras, say $\tilde{V}$ and $\tilde{W}$, and the natural maps θ_1 from V to $\tilde{V}$, θ_2 from W to $\tilde{W}$ and their inverses are completely bounded, and the maps $\theta_1\circ\phi$ and $\theta_2\circ\psi$ are completely isometric. Therefore, $(\theta_1\circ\phi)\otimes_h(\theta_2\circ\psi)$ is completely isometric, so for all positive integers $k\le l$

$$\left\|\sum_{i=k}^{l}\tilde{a}_i\otimes\tilde{b}_i\right\|_{V\otimes_h W}=\left\|\sum_{i=k}^{l}(\theta_1\circ\phi)(\tilde{a}_i)\otimes(\theta_2\circ\psi)(\tilde{b}_i)\right\|_{\tilde{V}\otimes_h\tilde{W}}\le\|\theta_1\|_{cb}\|\theta_2\|_{cb}\left\|\sum_{i=k}^{l}\phi(\tilde{a}_i)\otimes\psi(\tilde{b}_i)\right\|_{V\otimes_h W}.$$

This shows that the partial sums of $\sum_{i=1}^{\infty}\tilde{a}_i\otimes\tilde{b}_i$ form a Cauchy sequence in $V\otimes_h W$, and so we may define an element $z=\sum_{i=1}^{\infty}\tilde{a}_i\otimes\tilde{b}_i\in V\otimes_h W$. Then, clearly $\phi\otimes_h\psi(z)=u$. Thus the map $\phi\otimes_h\psi$ is onto. To prove the injectivity of the map $\phi\otimes_h\psi$, let $\phi\otimes_h\psi(u)=0$ for $u\in V\otimes_h W$. Then, for $u=\sum_{i=1}^{\infty}a_i\otimes b_i$ a norm convergent representation in $V\otimes_h W$, we have $\sum_{i=1}^{\infty}\phi(a_i)\otimes\psi(b_i)=0$. Thus, for any $\Phi\in V^*$, $\sum_{i=1}^{\infty}\Phi(\phi(a_i))\psi(b_i)=0$. But ψ is one-to-one, so $\sum_{i=1}^{\infty}\Phi(\phi(a_i))b_i=0$. Now Proposition 4.4 yields that $\sum_{i=1}^{\infty}\phi(a_i)\otimes b_i=0$. Again by applying the same technique we obtain $u=0$. □

By the above corollary, for operator algebras V and W and automorphisms ϕ of V and ψ of W, it is clear that if ϕ and ψ are inner then $\phi\otimes_h\psi$ is.

In the following, by a *-reduced operator algebra we mean an operator algebra having isometric involution with respect to which it is *-reduced, and for any *-reduced operator algebra V having approximate identity, we denote by $P(V)$ the set of all pure states of V.

Corollary 4.6 *For *-reduced operator algebra V having approximate identity and any operator algebra W, the family $\{R_\phi:\phi\in P(V)\}$ is total on $V\otimes_h W$.*

Proof: Using ([23], Proposition 2.5.5), we have $\tilde{V}=\overline{co}(\{0\}\cup P(V))$, where $\tilde{V}$ is the set of continuous positive forms on V of norm less than equal to 1. Therefore, if $R_\phi(u)=0$ for all $\phi\in P(V)$ then $R_\phi(u)=0$ for all $\phi\in\tilde{V}$. Thus $\sum_{i=1}^{\infty}\phi(a_i)\psi(b_i)=0$ for any $\phi\in\tilde{V}$ and $\psi\in W^*$. Since the algebra V is *-reduced, so it admits a faithful *-representation, say π_1, on some Hilbert space, say H_1. For a fix ζ in the closed unit ball of H_1, define $\phi\in V^*$ as $\phi(a)=\langle\pi_1(a)\zeta,\zeta\rangle$ for $a\in V$. One can easily verify that $\phi\in\tilde{V}$. As π_1 is faithful so ϕ is one-to-one. Therefore, $\sum_{i=1}^{\infty}a_i\psi(b_i)=0$ for any $\psi\in W^*$ and hence the result follows from Proposition 4.4. □

Corollary 4.7 *For any operator algebra V and *-reduced operator algebra W having approximate identity, the family $\{L_\psi:\psi\in P(W)\}$ is total on $V\otimes_h W$.*

The following can be proved on the similar lines as those in ([21], Lemma 2) by using ([23], Proposition 2.5.4), so we skip the proof.

Lemma 4.8 *For unital Banach *-algebra V and any Banach algebra W and a pure state ϕ of V, we have $R_\phi(cxd)=R_\phi(c)R_\phi(x)R_\phi(d)$ for $x\in V\otimes_h W$ and $c,d\in Z(V)\otimes_h W$ (Similarly, for any Banach algebra*

V and unital Banach *-algebra W, $L_\psi(cxd) = L_\psi(c)L_\psi(x)L_\psi(d)$ for $x \in V \otimes_h W$ and $c, d \in V \otimes_h Z(W)$, $\psi \in P(W)$).

Theorem 4.9 *Let V and W be unital operator algebras. Suppose that W is *-reduced and V has a completely contractive outer automorphism. Then $V \otimes_h W$ has a completely contractive outer automorphism.*

Proof: Let Φ be a completely contractive outer automorphism. Define a map μ from $V \otimes_h W$ into $V \otimes_h W$ as $\mu\left(\sum_{i=1}^{t} a_i \otimes b_i\right) = \sum_{i=1}^{t} \Phi(a_i) \otimes b_i$. By Corollary 4.5, μ is a completely contractive automorphism of $V \otimes_h W$. Assume that μ is a inner automorphism implemented by u. Then $\mu(x) = uxu^{-1}$. As $u \neq 0$ so we can find the pure state ψ on W such that $L_\psi(u) \neq 0$ by Corollary 4.7. Let $u_\psi := L_\psi(u)$. Note that for any $b \in W$ we have $u(1 \otimes b) = (1 \otimes b)u$. This implies that $u \in (\mathbb{C}1 \otimes_h W)^c$, the relative commutant of $\mathbb{C}1 \otimes_h W$ in $V \otimes_h W$, which is $V \otimes_h Z(W)$ by ([18], Corollary 4.7). For $a \in V$,
$u_\psi a = L_\psi(u(a \otimes 1)) = L_\psi((\Phi(a) \otimes 1)u) = \Phi(a)u_\psi$ by the module property of the slice map. Since $u \in V \otimes_h Z(W)$ is invertible, so u_ψ is invertible by Lemma 4.8. Therefore, $\Phi(a) = u_\psi a u_\psi^{-1}$ and hence Φ is inner, a contradiction. Thus μ is an outer automorphism. □

References

[1] Schatten, R. (1943) On the Direct Product of Banach Spaces. *Transactions of the American Mathematical Society*, **53**, 195-217. http://dx.doi.org/10.1090/S0002-9947-1943-0007568-7

[2] Kumar, A. and Sinclair, A.M. (1998) Equivalence of Norms on Operator Space Tensor Products of C^*-Algebras. *Transactions of the American Mathematical Society*, **350**, 2033-2048. http://dx.doi.org/10.1090/S0002-9947-98-02190-4

[3] Haagerup, U. and Musat, M. (2008) The Effros-Ruan Conjecture for Bilinear Forms on C^*-Algebras. *Inventiones Mathematicae*, **174**, 139-163. http://dx.doi.org/10.1007/s00222-008-0137-7

[4] Jain, R. and Kumar, A. (2011) Operator Space Projective Tensor Product: Embedding into Second Dual and Ideal Structure. Available on arXiv:1106.2644v1.

[5] Blecher, D.P. and LeMerdy, C. (2004) Operator Algebras and Their Modules—An Operator Space Approach. London Mathematical Society Monographs, New Series, The Clarendon Press, Oxford University Press, Oxford.

[6] Ryan, R. (2002) Introduction to Tensor Products of Banach Spaces. Springer Monographs in Mathematics, Springer-Verlag, Berlin, Heidelberg. http://dx.doi.org/10.1007/978-1-4471-3903-4

[7] Haagerup, U. (1985) The Grothendieck Inequality for Bilinear Forms on C^*-Algebras. *Advances in Mathematics*, **56**, 93-116. http://dx.doi.org/10.1016/0001-8708(85)90026-X

[8] Lance, C. (1973) On Nuclear C^*-Algebras. *Journal of Functional Analysis*, **12**, 157-176. http://dx.doi.org/10.1016/0022-1236(73)90021-9

[9] Archbold, R.J. and Batty, C.J.K. (1980) C^*-Tensor Norms and Slice Maps. *Journal of the London Mathematical Society*, **22**, 127-138. http://dx.doi.org/10.1112/jlms/s2-22.1.127

[10] Effros, E.G. and Ruan, Z-J. (2000) Operator Spaces. Claredon Press, Oxford.

[11] Jain, R. and Kumar, A. (2008) Operator Space Tensor Products of C^*-Algebras. *Mathematische Zeitschrift*, **260**, 805-811. http://dx.doi.org/10.1007/s00209-008-0301-1

[12] Kumar, A. (2001) Operator Space Projective Tensor Product of C^*-Algebras. *Mathematische Zeitschrift*, **237**, 211-217. http://dx.doi.org/10.1007/PL00004864

[13] Itoh, T. (2000) Completely Positive Decompositions from Duals of C^*-Algebras to Von Neumann Algebras. *Mathematica Japonica*, **51**, 89-98.

[14] Effros, E.G. and Ruan, Z-J. (1992) On Approximation Properties for Opertaor Spaces. *International Journal of Mathematics*, **1**, 163-187. http://dx.doi.org/10.1142/S0129167X90000113

[15] Lin, H. (1988) Ideals of Multiplier Algebras of Simple *AF* C^*-Algebras. *Proceedings of the American Mathematical Society*, **104**, 239-244.

[16] Archbold, R.J., *et al.* (1997) Ideal Space of the Haagerup Tensor Product of C^*-Algebras. *International Journal of Mathematics*, **8**, 1-29. http://dx.doi.org/10.1142/S0129167X97000020

[17] Allen, S.D., Sinclair, A.M. and Smith, R.R. (1993) The Ideal Structure of the Haagerup Tensor Product of C^*-Algebras, *Journal Für die Reine und Angewandte Mathematik*, **442**, 111-148.

[18] Smith, R.R. (1991) Completely Bounded Module Maps and the Haagerup Tensor Product. *Journal of Functional*

Analysis, **102**, 156-175. http://dx.doi.org/10.1016/0022-1236(91)90139-V

[19] Jain, R. and Kumar, A. (2011) Ideals in Operator Space Projective Tensor Products of C^*-Algebras. *Journal of the Australian Mathematical Society*, **91**, 275-288. http://dx.doi.org/10.1017/S1446788711001479

[20] Bonsall, F.F. and Duncan, J. (1973) Complete Normed Algebras. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin, Heidelberg, New York. http://dx.doi.org/10.1007/978-3-642-65669-9

[21] Wassermann, S. (1975) Tensor Products of *-Automorphisms of C^*-Algbras. *Bulletin London Mathematical Society*, **7**, 65-70. http://dx.doi.org/10.1112/blms/7.1.65

[22] Clouâtre, R. (2014) Completely Bounded Isomorphism of Operator Algebras. Available on arXiv:1401.0748v2.

[23] Dixmier, J. (1977) Von Neumann Algebras. North Holland Publishing Company, Amsterdam.