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On Bifurcation from Infinity and Multipoint Boundary Value Problems

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Abstract

We generalize a result on bifurcation from infinity of high order ordinary differential equations with multi-point boundary conditions. Our abstract setting represents a variant of Nonlinear Krein-Ruthman theorems. Furthermore, an analysis of this abstract setting raises an open question motivated by some misunderstanding and inconclusive proofs about the simplicity of principal eigenvalues in some articles in the literature.

Keywords

Krein-Rutman, Positivity, Cone, Homogeneity, Compactness, Eigenvalue, Simplicity, Bifurcation

1. Introduction

In this paper, we generalize and improve a result of Coyle *et al*. [1] about the bifurcation from infinity after stating in the line of Nussbaum [2], Schmitt [3], etc., a type of nonlinear Krein-Rutman theorem for a class of positively 1-homogeneous, compact and continuous operators in Banach spaces leaving invariant cones.

Our method is motivated by the maximum principle of Degla [4] and a result on the principal eigenvalue of multi-point Boundary Value Problems (BVP's) of Degla [5] which allow the use of cone theoretic arguments and of the well-known general result on bifurcation from infinity; see Coyle [1], Mawhin [6] and Rabinowitz [7].

Furthermore, in our abstract setting, the nonlinear Krein-Rutman Theorem resets an important result on the simplicity of positive eigenvalues [8] by avoiding some inconclusive argument [8] (page 3086, lines 29-37) also misused in [9] (page 550, lines 15-27). However the gap in their arguments under their assumptions, remains an open question.

2. Preliminary Definitions and Notations

We say that a nonempty subset $\mathcal{K}$ of a Banach space X is a *cone* if it is closed and

1) $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$,
2) $\lambda\mathcal{K} \subset \mathcal{K}, \forall \lambda \geq 0$ and
3) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

In other words, the cones considered here are closed convex cones with vertex at 0.

A cone $\mathcal{K}$ of a Banach space X induces a partial ordering on X by the relation

$$x \preceq y \quad \text{if and only if } y - x \in \mathcal{K},$$

and it follows that

$$x \succeq 0 \quad \text{if and only if } x \in \mathcal{K}.$$

Therefore $(X, \mathcal{K})$ is called an *ordered Banach space* with $\mathcal{K}$ as the *positive cone* of X. Note that we write $x \prec y$ when $x \preceq y$ and $x \neq y$; *i.e.*,

$$x \prec y \quad \text{if and only if } y - x \in \mathcal{K} \setminus \{0\}.$$

A cone $\mathcal{K}$ of a Banach space X is said to be *generating* if $X = \mathcal{K} - \mathcal{K}$, and *total* if $X = \overline{\mathcal{K} - \mathcal{K}}$.

Given a Banach space X with dual X', if a cone $\mathcal{K}$ of X is generating, then the set defined by

$$\mathcal{K}' := \left\{ f \in X' / \langle f, u \rangle \geq 0, \forall u \in \mathcal{K} \right\}$$

is a cone of X' called the *dual cone* of $\mathcal{K}$.

The positive cone $\mathcal{K}$ of an ordered Banach space X is said to be *normal* if there exists a positive constant $c \geq 1$ such that

$$\forall x, y \in \mathcal{K}, \; x \preceq y \Rightarrow \|x\| \leq c \|y\|.$$

When $c = 1$, such an ordered Banach space is said to be *monotone*.

Let $(X, \mathcal{K})$ be an ordered Banach space. Then

- A linear operator $T : X \to X$ is said to be *positive* if

$$T(\mathcal{K}) \subset \mathcal{K},$$

and *strongly positive* if

$$T(\mathcal{K} \setminus \{0\}) \subset \overset{\circ}{\mathcal{K}}.$$

- An arbitrary operator $T : X \to X$ is said to be *increasing* if

$$\forall x, y \in X, \; x \preceq y \Rightarrow Tx \preceq Ty,$$

strictly increasing if

$$\forall x, y \in X, \; x \prec y \Rightarrow Tx \prec Ty,$$

and *strongly increasing* if

$$\forall x, y \in X, \; x \prec y \Rightarrow Ty - Tx \in \overset{\circ}{\mathcal{K}}.$$

We shall say that T is *increasing on* $\mathcal{K}$ if

$$\forall x, y \in \mathcal{K}, \; x \preceq y \Rightarrow Tx \preceq Ty.$$

Observe that if an operator T is increasing on $\mathcal{K}$ and satisfies $T(0) = 0$, then it leaves invariant $\mathcal{K}$.

Besides in our applications, we shall use the following terminology based on Degla [4] [5], Elias [10] and Coppel [11]. Given fixed positive integers n, m and $k_1, \cdots, k_m$ such that $2 \leq m \leq \sum_{i=1}^{m} k_i = n$, and real numbers $a = a_1 < \cdots < a_m = b$, we shall denote by P the Levin's polynomial defined by $P(t) = \prod_{i=1}^{m} (t - a_i)^{k_i}$ and we shall deal with disconjugate n^{th} order differential operators on $[a, b]$ of the form

$$Lx := x^{(n)} + p_1(t) x^{(n-1)} + \cdots + p_n(t) x$$

where the coefficients $p_1, \cdots, p_n$ are given continuous functions on $[a,b]$, that is, an nth-order differential linear operator L such that every nontrivial solution of the differential equation $Lx = 0$ has less than n zeros counting their multiplicities.

Recall that an nth-order differential linear operator L;

$$Lx \equiv x^{(n)} + p_1(t) x^{(n-1)} + \cdots + p_n(t) x$$

with $p_1, \cdots, p_n \in \mathcal{C}([a,b])$, is disconjugate on $[a,b]$ if and only if L has a Polya factorization; that is, there exist n smooth positive functions $v_i \in \mathcal{C}^{n-i+1}([a,b])$, $1 \le i \le n$, such that

$$Lx = v_1 \cdots v_n D \frac{1}{v_n} D \cdots D \frac{1}{v_1} x \quad \text{for every } x \in \mathcal{C}^n([a,b]);$$

$$\text{where} \quad D = \frac{\mathrm{d}}{\mathrm{d}t}$$

cf. [11].

Furthermore G will denote the Green function associated to the Boundary Value Problems (in short BVP's)

$$\begin{cases} Lx = 0, \\ x^{(j)}(a_i) = 0, \ 1 \le i \le m, \ 0 \le j \le k_i - 1, \end{cases}$$

and besides, given $f \in \mathcal{C}([a,b])$, we shall adopt the notation $\|f\|_\infty = \sup_{a \le t \le b} |f(t)|$ and

$$S_f(t) = \begin{cases} \dfrac{f(t)}{|f(t)|} & \text{if } f(t) \ne 0 \\ 0 & \text{if } f(t) = 0 \end{cases}.$$

As in [5], we shall also consider the Banach space

$$X = \left\{ u \in \mathcal{C}([a,b]) : |u(t)| \le c |P(t)|, \ a \le t \le b; \text{ for a positive constant } c = c(u) \right\}$$

equipped with the norm

$$\|u\|_P = \|u\|_\infty + \sup_{t \ne a_1, \cdots, a_m} \left| \frac{u(t)}{P(t)} \right|$$

and ordered by the cone

$$\mathcal{K} = \left\{ u \in X : S_P(t) u(t) \ge 0, \ a \le t \le b \right\}.$$

Now we are ready to state a variant of *nonlinear Krein-Rutman theorems*.

Proposition 1.1. *Let X be a real Banach space, $\mathcal{K}$ a nontrivial cone in X and assume that $T : X \to X$ is a positively 1-homogeneous, compact and continuous operator.*

a) *If T is increasing on $\mathcal{K}$ and there exist a positive vector $u \succ 0$, a positive real number ω and a positive integer m, such that*

$$\omega T^m u \succeq u, \tag{i}$$

then T has a positive eigenvalue $\lambda_o \ge \omega^{-\frac{1}{m}}$ with a positive eigenvector.

In case that T is linear, its spectral radius $r(T)$ is such a positive eigenvalue and satisfies

$$r(T) = \max \left\{ \alpha^{-\frac{1}{m}}; \alpha > 0, \exists v \succ 0, \alpha T^m v \succeq v \right\}.$$

b) *If $\mathcal{K}$ has a nonempty interior $\overset{\circ}{\mathcal{K}}$ and $T \not\equiv 0$ with the property*

$$\forall x_1, x_2 \in \mathcal{K}, \ x_2 - x_1 \in \mathcal{K} \Rightarrow Tx_2 - Tx_1 \in \overset{\circ}{\mathcal{K}} \cup \{0\}, \tag{ii}$$

then T has a unique positive eigenvalue and a unique positive normalized eigenvector.

In case that T is linear, this positive eigenvalue coincides with the spectral radius $r(T)$ of T, is algebraically simple and has the following variational characterization:

$$r(T) = \max\left\{\frac{1}{\alpha}; \alpha > 0, \exists v \succ 0, \alpha Tv \succeq v\right\}.$$

Remark 1.1. For a linear operator T, the condition (ii) of b) is equivalent to

$$\forall x \in X, \quad x \in \mathcal{K} \Rightarrow Tx \in \overset{\circ}{\mathcal{K}} \cup \{0\}. \tag{iii}$$

Furthermore the conclusion of b) can be heuristically motivated by the application of the Krein-Rutman theorem to the quotient space $X/_{\ker T}$.

Remark 1.2. The above theorem is readily applicable to any positively 1-homogeneous, compact and continuous operators that are strongly positive on the cone of an ordered Banach space.

Remark 1.3. The proof of Theorem 2 of [8] does not fully hold but is valid for strongly increasing operators. The reason is that its conclusion (2.9) is not correct and should be read $\underline{\lambda_0 \leq \lambda}$ which does not contradict the inequality (2.10) therein; that is $\lambda_0 \geq \lambda$.

The fact is that for instance in the Banach space $\mathbb{R}^2$ ordered by the cone

$$K = \mathbb{R}_+^2 = \{(x_1, x_2): x_1 \geq 0, x_2 \geq 0\},$$

we have

$$u := (2,2) \succ v := (1,2)$$

and so with $\alpha = \beta = 1 > 0$ it is clear that

$$\alpha u \succ \beta v \not\Rightarrow \alpha > \beta.$$

Likewise the inequality "$x_0 \succ -\delta_{x_0}(x_1)x_1$" of the paragraph 4 of the proof of theorem 4.8 of [9] does not contradict the definition of "$\delta_{x_0}(x_1)x_1$" as can be seen with $x_0 := (2,2)$ and $x_1 := -(1,2)$ for which $\delta_{x_0}(x_1) = 1$ by simply considering again the ordered Banach space $(\mathbb{R}^2, \mathbb{R}_+^2)$.

Therefore we are led to raise the following

Open Question: Does there exist a strictly increasing and positively 1-homogeneous compact operator of which positive eigenvalue is not simple?

Remark 1.4. For a positive compact linear operator T, the condition (i) of Part a) of Proposition 1.1 is equivalent to

$$r(T) > 0. \tag{iv}$$

The following example illustrates Proposition 1.1.

Example 1.5. Consider the system of boundary value problems:

$$\begin{cases} -x'' = \lambda Q(t)x, \\ x(0) = x(1) = 0, \end{cases}$$

with λ as a real parameter and

$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix}$$

where the $q_{i,j}$ are assumed to be nonnegative continuous functions on $[0,1]$ such that on the one hand $q_{1,1}$ and $q_{2,1}$ have a common support $\mathcal{S}_1$, and on the other hand $q_{1,2}$ and $q_{2,2}$ have a common support $\mathcal{S}_2$ such that $\mathcal{S}_1 \bigcup \mathcal{S}_2 \neq \varnothing$; *i.e.* $Q \not\equiv 0$, and where the unknown vector-valued function $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is clearly searched in $\mathcal{C}^2([0,1]; \mathbb{R}^2)$ with zero Dirichlet boundary condition.

Then this system has a unique normalized solution with positive component functions on the interval $(0,1)$ corresponding to a unique positive value of the parameter λ.

Justification. We shall make use of Proposition 1.1 for the sake of illustration that may motivate other interesting works. Indeed it is immediately seen that for nontrivial solutions, we have $\lambda \neq 0$, and the system of BVPs

$$\begin{cases} -x'' = \lambda Q x, \\ x(0) = x(1) = 0 \end{cases}$$

is equivalent to the integral equation

$$\int_0^1 g(t,s) Q(s) x(s) \, ds = \frac{1}{\lambda} x(t), \quad 0 \le t \le 1;$$

with

$$g(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Moreover by considering the special space of continuous vector-valued functions

$$X = \left\{ u \in \mathcal{C}\left([0,1], \mathbb{R}^2\right) : \frac{u(t)}{t(t-1)} \text{ is bounded on the interval } (0,1) \right\}$$

endowed with the norm $\|\cdot\|_P$ defined for any $u = (u_1, u_2) \in X$ by

$$\|u\|_P = \sum_{i=1}^{2} \left(\|u_i\|_\infty + \sup_{0 < t < 1} \left| \frac{u_i(t)}{t(t-1)} \right| \right),$$

which contains all possible solutions of our eigenvalue problem, and by letting

$$\mathcal{K} = \left\{ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in X : u_i \ge 0; i = 1, 2 \right\},$$

we see that $(X, \mathcal{K})$ is a normal ordered Banach space. Furthermore the non-zero linear operator $T : X \to X$; $u \mapsto Tu$ defined by

$$Tu(t) = \int_0^1 g(t,s) Q(s) u(s) \, ds, \quad 0 \le t \le 1,$$

is compact and satisfies

$$T(\mathcal{K}) \subset \mathring{\mathcal{K}} \cup \{0\};$$

with

$$\mathring{\mathcal{K}} = \left\{ u = (u_1, u_2) \in X : \inf_{0 < t < 1} \frac{u_i(t)}{t(1-t)} > 0; i = 1, 2 \right\},$$

by the strong classical maximum principle.

The conclusion follows. $\square$

3. Bifurcation from Infinity of Conjugate Multipoint BVPs

This part can be considered as a more elaborated application of the main result of the previous section.

In the sequel we shall make use of the notations mentioned in Section 2. According to this,

$$X = \left\{ u \in \mathcal{C}\left([a,b]\right) : |u(t)| \le c |P(t)|, a \le t \le b; \text{ for a positive constant } c = c(u) \right\}$$

equipped with the norm

$$\|u\|_P := \|u\|_\infty + \sup_{t \neq a_1, \cdots, a_m} \left| \frac{u(t)}{P(t)} \right|$$

and ordered by the cone

$$\mathcal{K} = \{u \in X : S_P(t)u(t) \geq 0, \ a \leq t \leq b\}$$

is an ordered Banach space.

Then the following theorem holds.

Theorem 2.1. *Let* $q \in \mathcal{C}([a,b])$ *satisfy*

$$q \not\equiv 0 \ \text{ and } \ S_P(t)q(t) \geq 0, \ \forall t \in [a,b].$$

Moreover let $h:[a,b] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

$$\lim_{|x| \to \infty} \frac{|h(t,x)|}{|x|} = 0 \ \text{ uniformly with respect to } \ a \leq t \leq b.$$

Then there exists a continuum $\mathcal{S}$ *of positive solutions of the BVPs*

$$Lx - \lambda q(t)x = h(t,x),$$
$$x^{(j)}(a_i) = 0, \ 1 \leq i \leq m, \ 0 \leq j \leq k_i - 1, \tag{E_λ}$$

and $\varepsilon_o > 0$ *such that*

1) *For each* $0 < \varepsilon < \varepsilon_o$, *there is a corresponding subcontinuum* $\mathcal{S}_\varepsilon$ *contained in*

$$U_\varepsilon = \left\{ (\lambda, u) : |\lambda - \lambda_1(q)| < \varepsilon, \|u\|_P > \frac{1}{\varepsilon} \right\}$$

which connects $(\lambda_1(q), \infty)$ *and* ∂U_ε.

2) *If* $(\mu_l, x_l) \in U_{\varepsilon_o}$ *with* $\|x_l\|_P \to \infty$ *as* $l \to \infty$, *then*

$$\mu_l \ \text{converges to} \ \lambda_1(q) \ \text{and} \ \frac{x_l}{\|x_l\|_P} \ \text{converges uniformly}$$

(in fact in X_P*) to the unique normalized nontrivial solution of*

$$\begin{cases} Lu = \lambda_1 q(t)u, \\ u^{(j)}(a_i) = 0, \ 1 \leq i \leq m, \ 0 \leq j \leq k_i - 1; \end{cases}$$

where $\lambda_1 := \lambda_1(q)$.

Remark 2.2. An analogue version of Theorem 2.1 can be stated with $q \in \mathbf{L}^1((a,b))$ satisfying the following property:

$$q \neq 0 \ \text{ on a set of positive measure and } \ S_P(t)q(t) \geq 0 \ \text{ for a.e. } \ t \in (a,b).$$

Remark 2.3. It is worth observing that Theorem 2.1 is a generalized version of a result of [1] since this Theorem 2.1 concerns multipoint conjugate boundary conditions and deals with a function q that may vanish on subintervals of $[a,b]$.

For a proof of this Theorem 2.1, we need the lemma below which can also be deduced from Proposition 1.1.

Lemma A. [5]

If $q \in \mathbf{L}^1((a,b))$ satisfies $q \neq 0$ on a set of positive measure and $S_P(t)q(t) \geq 0$ for a.e. $t \in (a,b)$, then the eigenvalue BVPs

$$\begin{cases} Lx = \lambda q(t)x, \ \text{ for almost every } t \\ x^{(j)}(a_i) = 0, \ 1 \leq i \leq m, \ 0 \leq j \leq k_i - 1 \end{cases} \tag{Q_λ}$$

has a positive eigenvalue λ_1 which is simple with an eigenfunction u_1 such that $\displaystyle\inf_{t \neq a_i} \frac{u_1(t)}{P(t)} > 0$.

Now we recall a standard result on bifurcation theory which together with Lemma A will prove our Theorem 2.1 which is about a bifurcation from infinity for conjugate multipoint BVPs.

Lemma B. [1] [6] [7] [12]

Let E be a real Banach space with norm $\|\cdot\|$. Assume that

$$T : [a,b] \times E \to E$$

is such that for each $\lambda \in [a,b]$, $T(\lambda, \cdot)$ is a compact linear operator, and for each $u \in E$, $T(\cdot, u)$ is differentiable on $[a,b]$. Let $\mathcal{P} \subset E$ be a cone with nonempty interior, $\overset{\circ}{\mathcal{P}}$.

Moreover suppose that

$$H : [a,b] \times E \to E$$

is a completely continuous map satisfying

$$\lim_{\|u\| \to \infty} \frac{H(\lambda, u)}{\|u\|} = 0 \quad \text{uniformly in } \lambda,$$

and consider the equation

$$u = T(\lambda, u) + H(\lambda, u).$$

If $\lambda_1 \in]a,b[$, $u_1 \in \overset{\circ}{\mathcal{P}}$ with $\|u_1\| = 1$,

$$\ker\left(I - T(\lambda_1, \cdot)\right) = \mathrm{span}(u_1), \quad \text{and} \quad \frac{\partial T}{\partial \lambda}(\lambda_1, u_1) \notin \mathrm{R}\left(I - T(\lambda_1, \cdot)\right);$$

then there exist $\varepsilon_o > 0$ and a continuum $\mathcal{S} \subset [a,b] \times \mathcal{P}$ such that for any $0 < \varepsilon < \varepsilon_o$, there exists a corresponding subcontinuum $\mathcal{S}_\varepsilon \subset \mathcal{S}$ contained in

$$U_\varepsilon = \left\{ (\lambda, u) : |\lambda - \lambda_1| < \varepsilon, \|u\| > \frac{1}{\varepsilon} \right\}$$

which connects (λ_1, ∞) and ∂U_ε. Moreover if $(\mu_l, x_l) \in U_{\varepsilon_o}$ with $\|x_l\| \to \infty$ as $l \to \infty$; then

$$\mu_l \to \lambda_1 \quad \text{and} \quad \frac{x_l}{\|x_l\|} \to u_1.$$

Proof of Theorem 2.1. First note that all possible solutions of the BVP's (E_λ) lie in X since they are of the form $t \mapsto \int_a^b G(t,s) f(s) \mathrm{d}s$, $a \le t \le b$; where f is continuous and G is the Green function of the BVPs

$$\begin{cases} Lx = 0, & \text{for almost every } t \\ x^{(j)}(a_i) = 0, & 1 \le i \le m, \ 0 \le j \le k_i - 1 \end{cases}$$

with the property that $(t,s) \mapsto \dfrac{G(t,s)}{P(t)}$ is bounded on $\bigcup_{i=1}^{m-1}(a_i, a_{i+1}) \times [a,b]$.

Now (E_λ) is equivalent, by the properties of the Green function G, to the following equation:

$$u - \lambda T_q u = Hu; \quad u \in X.$$

where $T_q u(t) = \int_a^b G(t,s) q(s) u(s) \mathrm{d}s, \ a \le t \le b$

and $Hu(t) = \int_a^b G(t,s) h(s, u(s)) \mathrm{d}s, \ a \le t \le b.$

Moreover as seen in the proof of Lemma A [5], the operator $T_q : X \to X$ is a non-zero positive compact linear operator satisfying

$$T_q(\mathcal{K}) \subset \overset{\circ}{\mathcal{K}} \cup \{0\},$$

while $H : X \to X$ is completely continuous and satisfies

$$\lim_{\|u\|_P \to \infty} \frac{Hu}{\|u\|_P} = 0$$

by the assumptions on h. Indeed:

1) We show that $H : X \to X$ is completely continuous.

Step 1. H maps bounded subsets into compact subsets. Let $(u_i)_{i \geq 1}$ be a sequence of elements of X of which norms are bounded, say by a constant real number $C > 0$. Let

$$M_o = (b-a) \sup \{ |h(\theta, \xi)| : a \leq \theta \leq b, |\xi| \leq C \}.$$

Then, on one hand,

$$\|u_i\|_\infty \leq C, \quad \forall i,$$

and on the other hand, we have for all $t_1, t_2 \in [a,b]$:

$$|Hu_i(t_2) - Hu_i(t_1)| \leq \int_a^b |G(t_2,s) - G(t_1,s)| |h(s, u_i(s))| ds$$

$$\leq \frac{M_o}{b-a} \int_a^b |G(t_2,s) - G(t_1,s)| ds$$

$$\leq M_o \sup_{a \leq s \leq b} |G(t_2,s) - G(t_1,s)|.$$

Hence for all i

$$|Hu_i(t_2) - Hu_i(t_1)| \leq M_o \omega_G(|t_2 - t_1|);$$

where ω_G is the modulus of continuity of G. Moreover as a continuous function, G is uniformly continuous on the compact set $[a,b] \times [a,b]$, and so $\lim_{\delta \to 0^+} \omega_G(\delta) = 0$. Therefore the Ascoli theorem implies the existence of a subsequence $(u_{i_l})_l$ of $(u_i)_i$ such that

$$Hu_{i_l} \to v \quad \text{uniformly on } [a,b]$$

for some $v \in C([a,b])$.

By applying again Ascoli theorem we see that there exists a subsequence of $(u_{i_l})_l$, still denoted by $(u_{i_l})_l$, such that

$$\int_a^b \frac{G(t,s)}{P(t)} h(s, u_{i_l}(s)) ds \to w(t)$$

uniformly on $\bigcup_{i=1}^{m-1} (a_i, a_{i+1})$ for a suitable $w \in C([a,b])$. Indeed, to realize this claim, let $\epsilon > 0$ and choose $\eta = \eta(\epsilon) > 0$ satisfying

$$\eta < \min \left\{ \frac{b-a}{2}, \frac{(b-a)\epsilon}{8c_1 M_o} \right\};$$

where c_1 is a finite upper-bound of the ratio $\frac{G(t,s)}{P(t)}$.

Consider now on the compact $[a,b] \times [a+\eta, b-\eta]$ the function g_η extending continuously the quotient function $(t,s) \mapsto \frac{G(t,s)}{P(t)}$. The function g_η is uniformly continuous on $[a,b] \times [a+\eta, b-\eta]$ and so there exists $\delta = \delta(\epsilon) > 0$ for which

$$\omega_{g_\eta}(\delta) < \frac{\epsilon}{2(M_o + 1)};$$

Therefore, denoting by w_l the continuous extension of $\frac{Hu_{i_l}}{P}$ to $[a,b]$, i.e.,

$$w_l : x \mapsto \int_a^b \frac{G(t,s)}{P(t)} h(s, u_{i_l}(s)) ds$$

we have for all $t_1, t_2 \in [a, b]$ satisfying $|t_2 - t_1| < \delta$:

$$\left| w_l(t_2) - w_l(t_1) \right| \leq \frac{4 c_1 M_o}{b - a} \eta + \frac{M_o}{b - a} \int_{a+\eta}^{b-\eta} \left| g_\eta(t_2, s) - g_\eta(t_1, s) \right| ds$$

$$\leq \frac{\epsilon}{2} + \sup_{a+\eta \leq s \leq b-\eta} \left| g_h(t_2, s) - g_h(t_1, s) \right| M_o$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon, \quad \forall l.$$

This shows that the sequence of functions $(w_l)_l$ is equicontinuous on $[a, b]$ and proves the claim since the functions w_l are also uniformly bounded as

$$\left\| w_l \right\|_\infty = \left\| \frac{H u_{i_l}}{P} \right\|_\infty \leq M_o c_1.$$

Now from the former convergence; *i.e.* $H u_{i_l} \to v$, we deduce that for all $t \neq a_1, \cdots, a_m$,

$$\frac{H u_{i_l}(t)}{P(t)} \to \frac{v(t)}{P(t)}.$$

Thus $w(t) = \dfrac{v(t)}{P(t)}$ for $t \in \bigcup_{i=1}^{m-1} (a_i, a_{i+1})$ and it follows that $\left(H u_{i_l} \right)_l$ converges to v in X.

Step 2. If $(u_i)_i$ converges to some u in $\left(X, \|\cdot\|_P \right) \rightarrowtail \left(X, \|\cdot\|_\infty \right)$, then the Lebesgue dominated convergence theorem implies that $\left(H u_i(t) \right)_i$ converges to $H u(t)$ for each $t \in [a, b]$.

It follows from the combination of Steps 1 & 2 that H is completely continuous; *i.e.*, H maps bounded sets into compact sets and is continuous.

2) We show that $H u = o\left(\|u\|_P \right)$ as $\|u\|_P \to \infty$.

To this end, let $\varepsilon > 0$ be arbitrary. Then by assumption there exists $A > 0$ such that

$$\left| h(t, x) \right| \leq \frac{\varepsilon}{4(b-a)\left(c_1 + \|G\|_\infty\right)} |x| \quad \text{for all } |x| \geq A \text{ and } a \leq t \leq b;$$

where $c_1 > 0$ is an upper-bound of $\dfrac{G(t, s)}{P(t)}$ on $\bigcup_{i=1}^{m-1} (a_i, a_{i+1}) \times [a, b]$. By setting

$M := \max \left\{ \left| h(t, x) \right| : a \leq t \leq b, |x| \leq A \right\}$, we have at once

$$\left| h(t, x) \right| \leq M + \frac{\varepsilon}{4(b-a)\left(c_1 + \|G\|_\infty\right)} |x| \quad \text{for all } a \leq t \leq b \text{ and } x \in \mathbb{R}.$$

Therefore for every $u \in X$, we have on one hand

$$\left| H u(t) \right| \leq \int_a^b \left| G(t, s) h(s, u(s)) \right| ds \leq M \|G\|_\infty (b-a) + \frac{\varepsilon}{4} \|u\|_\infty, \quad a \leq t \leq b,$$

and on the other hand

$$\left| \frac{H u(t)}{P(t)} \right| \leq M c_1 (b-a) + \frac{\varepsilon}{4} \|u\|_\infty, \quad \forall t \in \bigcup_{i=1}^{m-1} \,]a_i, a_{i+1}[.$$

Thus

$$\left\| H u \right\|_P \leq M(b-a)\left(c_1 + \|G\|_\infty\right) + \frac{\varepsilon}{2} \|u\|_\infty.$$

Now by putting

$$\alpha := 2(b-a)\left(c_1 + \|G\|_\infty\right) \frac{M}{\varepsilon}$$

we see clearly that

$$\forall u \in X, \ \|u\|_P > \alpha \Rightarrow \frac{\|Hu\|_P}{\|u\|_P} < \varepsilon.$$

That is

$$\lim_{\|u\|_P \to \infty} \frac{\|Hu\|_P}{\|u\|_P} = 0.$$

The result follows by applying Lemma B. $\square$

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