

Trial Equation Method for Solving the Improved Boussinesq Equation

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ABSTRACT

Trial equation method is a powerful tool for obtaining exact solutions of nonlinear differential equations. In this paper, the improved Boussinesq is reduced to an ordinary differential equation under the travelling wave transformation. Trial equation method and the theory of complete discrimination system for polynomial are used to establish exact solutions of the improved Boussinesq equation.

KEYWORDS

The Nonlinear Partial Differential Equation; Complete Discrimination System for Polynomial; Trial Equation Method; Traveling Wave Transform; The Improved Boussinesq Equation

1. Introduction

In every field of engineering technology, science research, natural world and human society activities, nonlinear phenomena occupy an important position. The investigation of exact solutions of nonlinear evolution equations helps us understand these phenomena better. With the development of soliton theory and the application of computer symbolic system such as Matlab and Mathematica, many powerful methods for obtaining exact solutions of nonlinear evolution equations are presented, such as the inverse scattering method [1], Hirota's bilinear transformation [2,3], the tanh method [4], sine-cosine method [5], homogeneous balance method [6,7], exp-function method [8], and so on. Recently, Professor Liu proposed a powerful method named trial equation method [9-11] for finding exact solutions to nonlinear differential equations.

In coastal engineering, Boussinesq-type equations are frequently used in computer models for the simulation of water waves in shallow seas and harbours. The objective of this paper is to apply Liu's method and the theory of complete discrimination system for polynomial [12-16] to find the exact solutions of the nonlinear differential equation.

2. Description of Trial Equation Method

The objective of this section is to outline the use of trial equation method for solving a nonlinear partial differential equation (PDE). Suppose we have a nonlinear PDE for $u(x, t)$ in the form

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \dots) = 0 \quad (1)$$

where P is a polynomial, which includes nonlinear terms and the highest order derivatives and so on.

Step 1 Taking the wave transformation $u = u(\xi_1)$, $\xi_1 = kx + \omega t$, reduces Equation (1) to the ordinary differential equation (ODE).

$$M(u, ku', \omega u', k^2 u'', \omega^2 u'', k\omega u'', \dots) = 0 \quad (2)$$

Step 2 Take trial equation method

$$u'' = F(u) = a_0 + a_1u + \dots + a_mu^m. \quad (3)$$

Integrating the Equation (3) with respect to ξ_1 once, we get

$$(u')^2 = H(u) = \frac{2a_m}{m+1}u^{m+1} + \dots + a_1u^2 + 2a_0u + d \quad (4)$$

where m , a_i and integration constant d are to be determined. Substituting Equations (3), (4) and other derivative terms into Equation (2) yields a polynomial $G(u)$ of μ . According to the balance principle we can determine the value of m . Setting the coefficients of $G(u)$ to zero, we get a system of algebraic equations. Solving this system, we can determine values of a_0, a_1, \dots, a_m and integration constant.

Step 3 Rewrite Equation (4) by the integral form

$$\pm(\xi - \xi_0) = \int \frac{du}{\sqrt{H(u)}}. \quad (5)$$

According to the complete discrimination system of the polynomial, we classify the roots of $H(u)$ and solve the integral Equation (5). Thus we obtain the exact solutions to Equation (1).

3. Application of Trial Equation Method

The improved Boussinesq equation [17,18] reads as

$$u_{tt} - u_{xx} - uu_{xx} - (u_x)^2 - u_{xxx} = 0. \quad (6)$$

Taking the traveling wave transformation $u = u(\xi_1)$ and $\xi_1 = kx + \omega t$, we can obtain the corresponding reduced ODE.

$$(\omega^2 - k^2)u'' - k^2uu'' - k^2(u')^2 - k^2\omega^2u''' = 0. \quad (7)$$

we take the trial equation as follows

$$u'' = a_0 + a_1u + \dots + a_mu^m. \quad (8)$$

According to the trial equation method of rank homogeneous equation, balancing u''' with uu'' (or $(u')^2$) gets $m = 2$, so Equation (8) has the following specific form

$$u'' = a_0 + a_1u + a_2u^2. \quad (9)$$

Integrating Equation (9) with respect to ξ_1 once, we yield

$$(u')^2 = \frac{2}{3}a_2u^3 + a_1u^2 + 2a_0u + d. \quad (10)$$

where values of a_0, a_1, a_2 and the integration constant d are to be determined latter. By Equation (9) and Equation (10), we derive the following formula

$$u''' = \frac{10}{3}a_2^2u^3 + 5a_1a_2u^2 + (6a_0a_2 + a_1^2)u + a_0a_1 + 2a_2d. \quad (11)$$

Substituting Equations (9), (10) and (11) into Equation (7), we have

$$r_3u^3 + r_2u^2 + r_1u + r_0 = 0. \quad (12)$$

where

$$r_0 = \omega^2a_0 - k^2d - k^2a_0 - k^2\omega^2(a_0a_1 + 2a_2d). \quad (13)$$

$$r_1 = -3k^2a_0 - k^2a_1 + \omega^2a_1 - k^2\omega^2(a_1^2 + 6a_0a_2). \quad (14)$$

$$r_2 = -2k^2a_1 - k^2a_2 + \omega^2a_2 - 5k^2\omega^2a_1a_2. \quad (15)$$

$$r_3 = -\frac{5}{3}k^2a_2 - \frac{10}{3}k^2\omega^2a_2^2. \quad (16)$$

Let the coefficient $r_i = 0 (i = 0, 1, 2, 3)$ be zero, we will yield nonlinear algebraic equations. Solving the equations, we will determine the values of a_0, a_1, a_2, d . We get $a_1 = \frac{1}{k^2} - \frac{1}{\omega^2}, a_2 = -\frac{1}{2\omega^2}, a_0$ and d are two arbitrary constants. When the above conditions are satisfied, we use the complete discrimination system for the third order polynomial and have the following solving process.

Let

$$v = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} u, \xi = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} \xi_1, d_2 = a_1 \left(\frac{2}{3}a_2\right)^{\frac{2}{3}}, d_1 = 2a_0 \left(\frac{2}{3}a_2\right)^{\frac{1}{3}}, d_0 = d. \quad (17)$$

Then Equation (10) becomes

$$(v')^2 = v^3 + d_2 v^2 + d_1 v + d_0. \quad (18)$$

where v is a function of ξ . The integral form of Equation (18) is

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{v^3 + d_2 v^2 + d_1 v + d_0}}. \quad (19)$$

Denote

$$F(v) = v^3 + d_2 v^2 + d_1 v + d_0. \quad (20)$$

$$\Delta = -27 \left(\frac{2d_2^3}{27} + d_0 - \frac{d_1 d_2}{3} \right)^2 - 4(d_1 - d_2^2)^3, D_1 = d_1 - \frac{d_2^2}{3}. \quad (21)$$

According to the complete discrimination system, we give the corresponding single traveling wave solutions to Equation (6).

Case1. $\Delta = 0, D_1 < 0. F(v) = 0$ has a double real root and a simple real root. Then we have

$$F(v) = (v - \lambda_1)^2 (v - \lambda_2), \lambda_1 \neq \lambda_2. \quad (22)$$

When $v > \lambda_2$, the corresponding solutions are

$$u_1 = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} \left\{ (\lambda_1 - \lambda_2) \tanh^2 \left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2} \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} (kx + \omega t - \xi_0) \right] + \lambda_2 \right\}, (\lambda_1 > \lambda_2); \quad (23)$$

$$u_2 = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} \left\{ (\lambda_1 - \lambda_2) \coth^2 \left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2} \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} (kx + \omega t - \xi_0) \right] + \lambda_2 \right\}, (\lambda_1 > \lambda_2); \quad (24)$$

$$u_3 = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} \left\{ (-\lambda_1 + \lambda_2) \sec^2 \left[\frac{\sqrt{-\lambda_1 + \lambda_2}}{2} \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} (kx + \omega t - \xi_0) \right] + \lambda_1 \right\}, (\lambda_1 < \lambda_2). \quad (25)$$

Case2. $\Delta = 0, D_1 = 0. F(v) = 0$ has a triple root. Then we have

$$F(v) = (v - \lambda)^3. \quad (26)$$

The corresponding solution is

$$u_4 = 4 \left(\frac{2}{3}a_2\right)^{\frac{2}{3}} (kx + \omega t - \xi_0)^{-2} + \lambda. \quad (27)$$

Case3. $\Delta > 0, D_1 < 0. F(v) = 0$ has three different real roots. Then we have

$$F(v) = (v - \lambda_1)(v - \lambda_2)(v - \lambda_3), \lambda_1 < \lambda_2 < \lambda_3. \quad (28)$$

When $\lambda_1 < v < \lambda_2$, we take the transformation as follows

$$v = \lambda_1 + (\lambda_2 - \lambda_1) \sin^2 \varphi. \quad (29)$$

According to the Equation (19), we have

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{F(v)}} = \frac{2}{\sqrt{\lambda_3 - \lambda_1}} \int \frac{d\varphi}{\sqrt{1 - m^2 \sin^2 \varphi}}. \quad (30)$$

where $m^2 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}$. On the basis of Equation (30) and the definition of the Jacobi elliptic sine function, we have

$$v = \lambda_1 + (\lambda_2 - \lambda_1) \operatorname{sn}^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} (\xi_1 - \xi_0), m \right). \quad (31)$$

The corresponding solutions is

$$u_5 = \left(\frac{2}{3} a_2 \right)^{\frac{1}{3}} \left[\lambda_1 + (\lambda_2 - \lambda_1) \operatorname{sn}^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} \left(\frac{2}{3} a_2 \right)^{\frac{1}{3}} (kx + \omega t - \xi_0), m \right) \right]. \quad (32)$$

when $v > \lambda_3$, we take the transformation as follows

$$v = \frac{-\lambda_2 \sin^2 \varphi + \lambda_3}{\cos^2 \varphi}. \quad (33)$$

The corresponding solutions is

$$u_6 = \left(\frac{2}{3} a_2 \right)^{\frac{1}{3}} \frac{\left[\lambda_3 - \lambda_2 \operatorname{sn}^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} \left(\frac{2}{3} a_2 \right)^{\frac{1}{3}} (kx + \omega t - \xi_0), m \right) \right]}{\operatorname{cn}^2 \left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} \left(\frac{2}{3} a_2 \right)^{\frac{1}{3}} (kx + \omega t - \xi_0), m \right)}. \quad (34)$$

where $m^2 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}$.

Case 4. $\Delta < 0, F(v) = 0$ has only a real root. Then we have

$$F(v) = (v - \lambda)(v^2 + pv + q), p^2 - 4q < 0. \quad (35)$$

when $v > \lambda_1$, we take the transformation as follows

$$v = \lambda + \sqrt{\lambda^2 + p\lambda + q} \tan^2 \frac{\varphi}{2}. \quad (36)$$

According to the Equation (19), we have

$$\text{where } m^2 = \frac{1}{2} \left(1 - \frac{\lambda + \frac{p}{2}}{\sqrt{\lambda^2 + p\lambda + q}} \right). \quad (37)$$

On the basis of Equation (37) and the definition of the Jacobi elliptic cosine function, we have

$$v = \lambda + \frac{2\sqrt{\lambda^2 + p\lambda + q}}{1 + \operatorname{cn} \left((\lambda^2 + p\lambda + q)^{\frac{1}{4}} (\xi_1 - \xi_0), m \right)} - \sqrt{\lambda^2 + p\lambda + q}. \quad (38)$$

The corresponding solutions is

$$u_7 = \left(\frac{2}{3}a_2\right)^{\frac{1}{3}} \left[\lambda + \frac{2\sqrt{\lambda^2 + p\lambda + q}}{1 + \operatorname{cn}\left(\left(\lambda^2 + p\lambda + q\right)^{\frac{1}{4}}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}(kx + \omega t - \xi_0), m\right)} - \sqrt{\lambda^2 + p\lambda + q} \right]. \quad (39)$$

In Equations (23), (24), (25), (27), (32), (34), and (39), the integration constant ξ_0 has been rewritten, but we still use it. The solutions $u_i (i = 1, \dots, 7)$ are all possible exact traveling wave solutions to Equation (6). We can see it is easy to write the corresponding solutions to the improved Boussinesq equation.

4. Conclusion

Trial equation method is a systematic method to solve nonlinear differential equations. The advantage of this method is that we can deal with nonlinear equations with linear methods. This method has the characteristics of simple steps and clear effectivity. Based on the idea of the trial equation method and the aid of the computerized symbolic computation, some exact traveling wave solutions to the improved Boussinesq equation have been obtained. With the same method, some of other equations can be dealt with.

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