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The p.q.-Baer Property of Skew Group Rings under Finite Group Action*

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ABSTRACT

In this paper, Let R is a ring, G be a finite group of ring automorphisms of R. $R*G$ denote the skew group ring of R under G. We investigate the right p.q.-Baer property of skew group rings under finite group action, Assume that R is a semiprime ring with a finite group G of X-outer ring automorphisms of R, then 1) $R*G$ is p.q.-Baer if and only if R is G-p.q.-Baer; 2) if R is p.q.-Baer, then $R*G$ is p.q.-Baer.

Keywords: p.q.-Baer Property; Skew Group Ring; Group Action

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let R is a ring, for a nonempty subset X of a ring R, $r_R(X)$ (resp., $l_R(X)$) denote a right (resp.,left) annihilator of X in R. A ring R is called right principally quasi-Baer (simply, right p.q.-Baer) if the right annihilator of every principal right ideal of R is generated, as a right ideal by an idempotent of R in [1]. A left principally quasi-Baer (simply, left p.q.-Baer) ring is defined similarly. Right p.q.-Baer rings have been initially studied in [1]. For more details on (right) p.q.-Baer rings, see [1-6]. A ring R is called quasi-Baer if the right annihilator of every right ideal is generated, as a right ideal by an idempotent of R in [7] (see also [8]. A ring R is called biregular, if for each $x \in R$, $RxR = eR$ for some central idempotent $e \in R$. We note that the class of right p.q.-Baer rings is a generalization of classes of quasi-Baer rings and biregular rings. $Q(R)$ denote a fixed maximal right ring of quotients of R. Recall from [9] an idempotent e of a ring R is called left (resp., right) semicentral if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. Equivalently, an idempotent e is left (resp., right) semicentral if and only if eR (resp., Re) is a two-sided ideal of R. $S_l(R)$ (resp., $S_r(R)$) denote the set of all left (resp., right) semicentral idempotents. An idempotent e of a ring R is called semicentral reduced if $S_l(eRe) = \{0, e\}$. According to [2] a ring R is called semicentral reduced if $S_l(R) = \{0,1\}$, *i.e.*, 1 is a semicentral reduced idempotent of R.

If R is a semiprime ring and I is a two-sided ideal of R, then $r_R(I) = l_R(I)$. For a right R-module M and a submodule N of M, we use $N_R \le^{ess} M_R$ and $N_R \le^{den} M_R$ to denote that N_R is essential in M_R and N_R is dense in M_R, respectively.

Let R is a ring, $\mathrm{Aut}(R)$ denote a group of ring automorphisms of R, G be a subgroup of $\mathrm{Aut}(R)$.

The skew group ring $R*G$ is defined to be

$$R * G = \oplus \sum_{g \in G} Rg$$

with addition given component wise and multiplication given as follows: if $a, b \in R$ and $g, h \in G$, then

$$(ab)(bh) = ab^{g^{-1}} gh \in Rgh.$$

We begin with the following example.

2. Preliminary

Example 2.1 There exist a ring R and a finite group G of ring automorphisms of R such that R is right p.q.-Baer but $R*G$ is not right p.q.-Baer.

Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ with a field F of characteristic 2, then R is right p.q.-Baer. Define $g \in \mathrm{Aut}(R)$ by

$$g\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

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Since characteristic of F is 2, Then $g^2=1$.

Now we show that $R*G$ is not right p.q.-Baer. Consider the right ideal $(1+g)(R*G)$ of $R*G$ generated by $1+g$. By computation, we have

$$r_{R*G}\left((1+g)(R*G)\right)$$
$$=\left\{\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} x & x+y \\ 0 & 0 \end{bmatrix}g \,\middle|\, x,y\in F\right\}$$

Suppose that

$$r_{R*G}\left((1+g)(R*G)\right)=e(R*G)$$

for some $e=e^2\in R*G$. Note that the idempotents of $R*G$ are 0, 1.

$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}g,\ \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}g$$

with $a,b\in F$. Since $e\in r_{R*G}\left((1+g)(R*G)\right)$, the only possible choice for e is 0. Thus if $R*G$ is right p.q.-Baer, then it follows that $r_{R*G}\left((1+g)(R*G)\right)=0$. This is a contradiction. Therefore $R*G$ is not right p.q.-Baer. Also we see that $R*G$ is not left p.q.-Baer.

Definition 2.2 Let R be a semiprime ring. For $g\in \mathrm{Aut}(R)$, let

$$\phi_g=\left\{x\in Q_m(R)\,\middle|\, xr^g=rx \text{ for each } r\in R\right\},$$

where $Q_m(R)$ is the Martindale right ring of quotients of R (see [10] for more on $Q_m(R)$). We say that g is X-outer if $\phi_g=0$. A subgroup G of $\mathrm{Aut}(R)$ is called X-outer on R if every $1\neq g\in G$ is X-outer. Assume that R is a semiprime ring, then for $g\in\mathrm{Aut}(R)$, let

$$\Phi_g=\left\{x\in Q(R)\,\middle|\, xr^g=rx \text{ for each } r\in R\right\}.$$

For $g\in\mathrm{Aut}(R)$, we claim that $\Phi_g=\phi_g$. Obviously $\phi_g\subseteq\Phi_g$. Conversely, if $x\in\Phi_g$ then $xR=Rx$. There exists $I_R\leq^{den}R_R$ such that $xI\subseteq R$. Therefore $RI\lhd R$, $(RI)_R\leq^{den}R_R$, and $xRI=RxI\subseteq R$. Thus $x\in Q_m(R)$, hence $x\in\phi_g$. Therefore $\Phi_g=\phi_g$. So if G is X-outer on R, then G can be considered as a group of ring automorphismms of $Q(R)$ and G is X-outer on $Q(R)$. For more details for X-outer ring automorphisms of a ring, etc., see [10, p. 396] and [11].

We say that a ring R has no nonzero n-torsion (n is a positive integer) if $na=0$ with $a\in R$ implies $a=0$.

Lemma 2.3 Let R be a semiprime ring and G a group of ring automorphisms of R.

1) [11,12] If G is X-outer, then every nonzero two-sided ideal of $R*G$ intersects R nontrivially. Hence $R*G$ is semiprime.

2) [11] If G is finite and R has no nonzero $|G|$-torsion, Then $R*G$ is semiprime.

For a ring R, we use $\mathrm{Cen}(R)$ to denote the center of R.

Lemma 2.4 For a semiprime ring R, let G be a group of X-outer ring automorphisms of R.

Then $\mathrm{Cen}(R*G)=\mathrm{Cen}(R^G)$.

Proof.

Let $\alpha=a_1 1+a_2g_2+\cdots+a_ng_n\in\mathrm{Cen}(R)$ with $a_i\in R$, 1 the identity of G, and $g_i\in G$.

The

$$(a_1 1+a_2g_2+\cdots+a_ng_n)b=b(a_1 1+a_2g_2+\cdots+a_ng_n)$$

for all $b\in R$. So $a_1b=ba_1, a_2b^{g_2}=ba_2,\cdots,a_nb^{g_n}=ba_n$ for all $b\in R$. Since G is X-outer, it follows that $a_2=\cdots=a_n=0$. Hence $\alpha=a_1 1=a_1\in R$. Also since $\alpha b=b\alpha$ for all $b\in R$, we have that $a_1\in\mathrm{Cen}(R)$. Note that for all $g\in G$, $a_1g=ga_1=a_1^{g^{-1}}g$ implies $a_1=a_1^{g^{-1}}$. So $\alpha=a_1\in\mathrm{Cen}(R)^G$. Thus

$$\mathrm{Cen}(R*G)\subseteq\mathrm{Cen}(R)^G.$$

Conversely, $\mathrm{Cen}(R)^G\subseteq\mathrm{Cen}(R*G)$ is clear.

Therefore $\mathrm{Cen}(R*G)=\mathrm{Cen}(R^G)$.

Lemma 2.5 [13,14] Let R be a ring and G a finite group of ring automorphisms of R. Then $Q(R)*G$ is the maximal right ring of quatients of R.

Assume that a group G of ring automorphisms of a ring R is finite. Then for $a\in R$, let $tr(a)=\sum_{g\in G}a^g$, which is called the trace of a. Also for a right ideal I of R, the right ideal $tr(I)=\left\{\sum_{g\in G}a^g\,\middle|\,a\in I\right\}$ of R^G is called the trace of I. Say $G=\{g_1,\cdots,g_n\}$. we put $t=g_1+\cdots+g_n\in R*G$. For $r\in R$ and $\alpha=a_1g_1+a_2g_2+\cdots+a_ng_n\in R*G$ with $a_i\in R$, define $r\cdot\alpha=r^{g_1}a_1^{g_1}+\cdots+r^{g_n}a_n^{g_n}$. Then R is a right $R*G$-module. Moreover, we see that ${}_{R^G}R_{R*G}$ is an $(R^G,R*G)$-bimodule.

Lemma 2.6 Assume that R is a semiprime ring and $e\in B(Q(R))$. Let I be a two-sided ideal of R such that $I_R\leq^{ess}eR_R$ and $r_R(I)=fR$ with $f\in B(R)$. Then $e=1-f$.

Proof. Since R is semiprime,

$$I_R\leq^{ess}l_R\left(r_R(I)\right)_R=(1-f)R.$$

Thus

$$I_R\leq^{ess}(1-f)Q(R)_R.$$

As $I_R\leq^{ess}eR_R$, $I_R\leq^{ess}eQ(R)_R$. We note that e and $1-f$ are in $B(Q(R))$. So we have that $e=1-f$.

Proposition 2.7 [1] Let R be a semiprime ring. Then the following are equivalent.

1) R is right p.q.-Baer.

2) Every principal two-sided ideal of R is right essential in a ring direct summand of R.

3) Every finitely generated two-sided ideal of R is right essential in a ring direct summand of R.

4) Every principal two-sided ideal of R that is closed as a right ideal is a direct summand of R.

5) For every principal two-sided ideal I of R, $r_R(I)$ is right essential in a direct summand of R.

6) R is left p.q.-Baer.

For a ring R with a group G of ring automorphisms of R, we say that a right ideal I of R is G-invariant if $I^g \subseteq I$ for every $g \in G$, where $I^g = \{a^g \mid a \in I\}$. Assume that R is a semiprime ring with a group G of ring automorphisms of R. We say that R is G-p.q.-Baer if the right annihilator of every finitely generated G-invariant two-sided ideal is generated by an idempotent, as a right ideal. By Proposition 2.7, if a ring R is semiprime p.q.-Baer with a group G of ring automorphisms of R, then R is G-p.q.-Baer.

A ring R is called right Rickart if the right annihilator of each element is generated by an idempotent of R. A left Rickart ring is defined similarly. A ring R is called Rickart if R is both right and left Rickart. A ring R is said to be reduced if R has no nonzero nilpotent element. We note that reduced Rickart rings are p.q.-Baer rings.

We put

$$B_p(Q(R)) = \{e \in B(Q(R)) \mid \text{there exists } x \in R \text{ with } RxR_R \leq^{ess} eR_R\}$$

Let $\hat{Q}_{pqB}(R)$ be the subring of $Q(R)$ generated by R and $B_p(Q(R))$.

Lemma 2.8 [15] Assume that R is a semiprime ring. Then:

1) The ring $\hat{Q}_{pqB}(R)$ is the smallest right ring of quotients of R which is p.q.-Baer.

2) R is p.q.-Baer if and only if $B_p(Q(R)) \subseteq R$.

With these preparations, in spite of Example 2.1, we have the following result for p.q.-Baer property of $R*G$ on a semiprime ring R for the case when G is finite and X-outer.

3. Main Results

Theorem 3.1 Let R be a semiprime ring with a finite group G of X-outer ring automorphisms of R. Then $R*G$ is p.q.-Baer if and only if R is G-p.q.-Baer.

Proof. Assume that $R*G$ is p.q.-Baer. Say

$$I = Ra_1R + \cdots + Ra_nR$$

is a finitely generated G-invariant two-sided ideal of R with $a_i \in R$. Then $I*G$ is a two-sided ideal of $R*G$. Moreover,

$$I*G = (R*G)a_1(R*G) + \cdots + (R*G)a_n(R*G),$$

Note that $R*G$ is semiprime by Lemma2.3, So Proposition 2.7 yields that there exists $e \in S_l(R*G)$ such that

$$(I*G)_{R*G} \leq^{ess} e(R*G)_{R*G}.$$

Since $R*G$ is semiprime, $e \in B(R*G)$ by [9]. Hence by Lemma2.4, $e \in \mathrm{Cen}(R)^G$. First, we see that $I_R \leq^{ess} eR_R$. For this, let $0 \neq er \in eR$ with $r \in R$. As $(I*G)_{R*G} \leq^{ess} e(R*G)_{R*G}$, there exists $\beta \in R*G$ such that $0 \neq er\beta \in I*G$.

Say $\beta = b_1g_1 + \cdots + b_ng_n$ with $b_i \in R$ and $g_i \in G$ for $i = 1, \cdots, n.$ Then

$$er\beta = (erb_1)g_1 + \cdots + (erb_n)g_n \in I*G.$$

Hence $0 \neq erb_j \in I$ for some j, so $I_R \leq^{ess} eR_R$. As $e = e^2 \in \mathrm{Cen}(R)^G$, $I \subseteq eRe$, and so $I_{eRe} \leq^{ess} eRe_{eRe}$.

Now we show that $r_R(I) = (1-e)R$. If $e = 0$, then $r_R(I) = R$. So we may assume that $e \neq 0$. Note that eRe is semiprime and $I_{eRe} \leq^{ess} eRe_{eRe}$, and so $r_{eRe}(I) = 0$. Hence

$$eR \cap r_R(I) = eRe \cap r_R(I) = 0.$$

As $I \subseteq eR$, $(1-e)R \subseteq r_R(I)$. From the modular law,

$$r_R(I) = (1-e)R \oplus (eR \cap r_R(I)).$$

But since $eR \cap r_R(I) = 0$, $r_R(I) = (1-e)R$. Therefore R is G-p.q.-Baer.

Conversely, let R be G-p.q.-Baer. Take

$$e \in B_p(Q(R)*G).$$

Then

$$e \in [\mathrm{Cen}(Q(R))]^G$$

by Lemma 2.4 since G is also X-outer on $Q(R)$ as was noted. Also there exists $\alpha \in R*G$ such that

$$(R*G)\alpha(R*G)_{R*G} \leq^{ess} e(R*G)_{R*G}$$

because $Q(R)*G$ is the maximal right ring of quotients of $R*G$ (Lemma 2.5) and $e \in B_p(Q(R)*G)$. Say $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ with $a_i \in R$ and $g_i \in G$ for $i = 1, 2, \cdots, n$. Then $\alpha \in e(R*G)(eR)*G$ and so $a_i \in eR$ for each $i = 1, 2, \cdots, n$. Consider $K = \sum_{i=1, g \in G}^{n} Ra_i^g R$. Then K is a finitely generated G-invariant two-sided ideal of R. Further, $K \subseteq eR$ because $e \in [\mathrm{Cen}(Q(R))]^G$. By the preceding argument, we see that $K_R \leq^{ess} eR_R$. From the assumption, there exists $f \in S_l(R) = B(R)$ such that $r_R(K) = fR$. Thus $e = 1 - f \in R$ by Lemma 2.6. Therefore $e \in R \subseteq R*G$, so $B_p(Q(R)*G) \subseteq R*G$. From

Lemma2.8, $R*G$ is p.q.-Baer.

Corollary 3.2 Let R be a semiprime ring with a finite group G of X-outer ring automorphisms of R. If R is p.q.-Baer, then $R*G$ is p.q.-Baer.

Proof. The proof follows immediately by Theorem 3.1.

4. Conclusion

In [16] researched quasi-Baer property of skew group rings under finite group actions on a semiprime ring and their applications to C^*-algebras (see also [17,18]). In this paper, we investigate the right p.q.-Baer property of skew group rings under finite group action. Assume that R is a semiprime ring with a finite group G of X-outer ring automorphisms of R, then 1) $R*G$ is p.q.-Baer if and only if R is G-p.q. -Baer; 2) if R is p.q.-Baer, then $R*G$ is p.q.-Baer.

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