Strong Laws of Large Numbers for Arrays of Rowwise Conditionally Negatively Dependent Random Variables

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ABSTRACT

Let $\{X_{nk}\}$ be an array of rowwise conditionally negative dependent random variables. Complete convergence of $n^{-1}\sum_{k=1}^{n} X_{nk}$ to 0 is obtained by using various conditions on the moments and conditional means.

Keywords: Negative Dependence; Complete Convergence

1. Introduction and Preliminaries

Concepts of negative dependence have been useful in developing laws of large numbers (cf: Taylor, Patterson and Bozorgnia [1]). Chung-type laws of large numbers for arrays of independent random variables were developed by Taylor, Patterson and Bozorgnia in [2].

Definition 1.1 Two random variables X and Y are pairwise negatively dependent (ND) if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \qquad (1)$$

for all $x, y \, \varepsilon \Re$.

Let (Ω, F, P) denote a probability space.

Definition 1.2. The sequence of random variables $\{X_n\}$ is said to be conditionally negatively dependent if there exists a sub σ-field ζ of F such that for each positive integer m

$$P\left[\bigcap_{i=1}^{m} P\left[X_i \, \varepsilon \, B_i \middle| \zeta\right]\right] \le \prod_{i=1}^{m} P\left[X_i \, \varepsilon \, B_i \middle| \zeta\right] \qquad (2)$$

where $P\left[X_i \, \varepsilon \, B_i \middle| \zeta\right]$ denotes the conditional probability of the random variable X being in the Boral set B_i given the sub-σ field ζ. Negatively dependent random variable are conditionally negatively dependent with respect to the trivial σ-field $\{\phi, F\}$.

Throughout this paper $\{X_{nk} : 1 \le k \le n, n \ge 1\}$ will denote rowwise conditionally independent random variables such that $EX_{nk} = 0$ for all n and k. The major result of this paper shows that

$$\frac{1}{n}\sum_{k=1}^{n} X_{nk} \to 0 \quad \text{completely} \qquad (.3)$$

where complete convergence is defined (Hsu and Robbins [3]) by

$$\sum_{n=1}^{\infty} P\left[\left\|\frac{1}{n}\sum_{k=1}^{n} X_{nk}\right\| > \eta\right] < \infty. \qquad (4)$$

Here $\|\cdot\|$ is a function on a separable Banach space to $\Re$. In the next section of this paper, strong laws of large numbers for arrays of rowwise conditionally negatively dependent random variables.

2. Strong Law for Random Variables

In this section, several lemmas are used in the proof of the major result. The first lemma will be presented without proof.

Lemma 2.1. Let X and Y be pairwise negatively dependent random variables. Then

$$\text{cov}(X, Y) \le 0$$

Lemma 2.2. Let X and Y be pairwise negatively dependent random variables. Then

$$\text{var}(X + Y) \le \text{var}(X) + \text{var}(Y)$$

Proof: For X and Y negatively dependent, we have by Lemma 2.1

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$$
$$\le \text{var}(X) + \text{var}(Y)$$

□

Theorem 2.1 Let $\{X_{nk}\}$ be an array of rowwise conditionally negatively dependent random variables. If

$$a)\ \sup_{1\leq k\leq n} E\left|X_{nk}\right|^2 = O(n)^\alpha,\ \alpha < 0 \qquad (5)$$

and for all $\eta > 0$

$$b)\ \sum_{n=1}^\infty P\left[\left|\frac{1}{n}\sum_{k=1}^n E_\zeta X_{nk}\right| > \eta\right] < \infty \qquad (6)$$

where E_ζ is the conditional expectation with respect to an appropriate σ-field that gives conditional negative dependence. Then

$$\frac{1}{n}\sum_{k=1}^n X_{nk} \to 0\ \text{completely}.$$

Proof. Let $\eta > 0$ be given. By Markov' inequality

$$\sum_{n=1}^\infty P\left[\left|\frac{1}{n}\sum_{k=1}^n X_{nk}\right| > \eta\right]$$

$$\leq \sum_{n=1}^\infty P\left[\left|\frac{1}{n}\sum_{k=1}^n X_{nk} - E_\zeta X_{nk}\right| > \frac{\eta}{2}\right]$$

$$+ \sum_{n=1}^\infty P\left[\left|\frac{1}{n}\sum_{k=1}^n E_\zeta X_{nk}\right| > \frac{\eta}{2}\right] \qquad (7)$$

$$\leq C\sum_{n=1}^\infty EE_\zeta\left|\frac{1}{n}\sum_{k=1}^n X_{nk} - E_\zeta X_{nk}\right|^2$$

$$+ \sum_{n=1}^\infty P\left[\left|\frac{1}{n}\sum_{k=1}^n E_\zeta X_{nk}\right| > \frac{\eta}{2}\right]$$

By Lemma 2.2, the first term in Equation (7) is bounded by

$$C\sum_{n=1}^\infty EE_\zeta\left|\frac{1}{n}\sum_{k=1}^n X_{nk} - E_\zeta X_{nk}\right|^2$$

$$\leq C\sum_{n=1}^\infty \frac{1}{n^2} EE_\zeta\left(\sum_{k=1}^n \left(X_{nk} - E_\zeta X_{nk}\right)^2\right)$$

$$\leq C\sum_{n=1}^\infty \frac{1}{n^2}\cdot 2^2\sum_{k=1}^n \left|EX_{nk}\right|^2 \leq C\sum_{n=1}^\infty \frac{n\cdot n^\alpha}{n^2}$$

$$= C\sum_{n=1}^\infty n^{-1+\alpha} < \infty$$

The second term of Equation (7) is finite by Equation (6). Thus, the result follows. □

3. Strong Law for Random Elements in $\mathfrak{R}^m$

Theorem 1.2 can be extended to $\mathfrak{R}^m$. The next definition is a crucial type p inequality used to define a form of negative dependence (cf. Patterson, Taylor, and Bozorgnia [4]).

Definition 3.1. Random elements $X_k, k \geq 1$, in a type p Banach space $1 \leq p \leq 2$ are said to be type p negatively dependent if $E\|X_k\| < \infty, 1 \leq k \leq n$ and if there exist a finite positive constant C such that

$$E\left\|\sum_{k=1}^n \left(X_k - EX_k\right)\right\|^p < C\sum_{k=1}^n E\|X_k - EX_k\|^p \qquad (8)$$

for all $n \geq 1$.

Coordinatewise (with respect to the standard basis) negative dependence in $\mathfrak{R}^m$ can yield type 2 negative dependence. To see this for rowwise random elements X_{nk}, let $X_{nk} = (X_{nk1},\cdots,X_{nkm}), n, k \geq 1$ be random elements in $\mathfrak{R}^m$ such that $E(X_{nki}^2) < \infty$ for $1 \leq i \leq m, n, k \geq 1$. Then

$$E\left\|\sum_{k=1}^n \left(X_k - EX_k\right)\right\|^p = E\sum_{i=1}^m\left(\sum_{k=1}^n \left(X_{nki} - EX_{nki}\right)\right)^2$$

$$\leq \sum_{i=1}^m\sum_{k=1}^n E\left(X_{nki} - EX_{nki}\right)^2 = \sum_{i=1}^n E\sum_{k=1}^m \left(X_{nki} - EX_{nki}\right)^2 \quad (9)$$

$$= \sum_{k=1}^n E\|X_{nki} - EX_{nki}\|^2$$

Theorem 3.1 Let $\{X_{nk}\}$ be an array of rowwise conditionally coordinatewise negatively random elements in $\mathfrak{R}^m$. If

$$a)\ \sup_{1\leq k\leq n} E\|X_{nk}\|^2 = O(n)^\alpha,\ \alpha < 0 \qquad (10)$$

and for all $\eta > 0$

$$b)\ \sum_{n=1}^\infty P\left[\left\|\frac{1}{n}\sum_{k=1}^n E_\zeta X_{nk}\right\| > \eta\right] < \infty \qquad (11)$$

where E_ζ is the conditional expectation with respect to an appropriate σ-field that gives conditional negative dependence, then

$$\frac{1}{n}\sum_{k=1}^n X_{nk} \to 0\ \text{completely}$$

Proof. The proof is similar to that of **Theorem 2.1**. □

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