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The Second Hochschild Cohomology Group for One-Parametric Self-Injective Algebras

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ABSTRACT

In this paper, we determine the second Hochschild cohomology group for a class of self-injective algebras of tame representation type namely, which are standard one-parametric but not weakly symmetric. These were classified up to derived equivalence by Bocian, Holm and Skowroński in [1]. We connect this to the deformation of these algebras.

Keywords: Hochschild Cohomology; Self-Injective Algebras; Socle Deformation

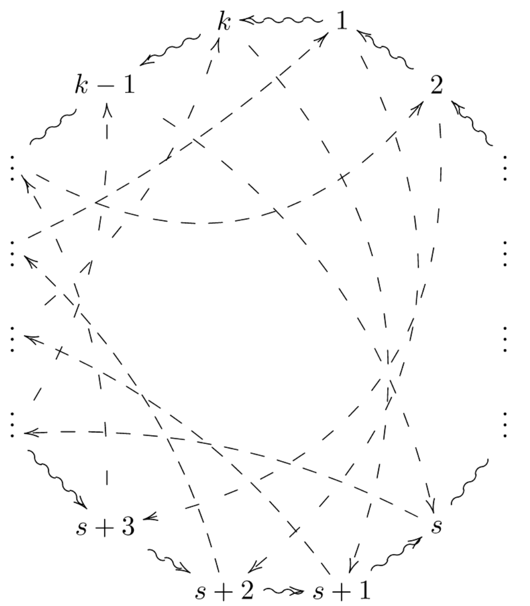
1. Introduction

This paper determines the second Hochschild cohomology group for all standard one-parametric but not weakly symmetric self-injective algebras of tame representation type. Bocian, Holm and Skowroński give, in [1], a classification of these algebras by quiver and relations up to derived equivalence. The algebras in [1] are divided into two types, namely the algebra $\Lambda=\Lambda(p,q,k,s,\lambda)$ where p,q,s,k are integers such that $p,\ q\geq 0,\ \ k\geq 2$, $1\leq s\leq k-1$, $\gcd(s,k)=1$, $\gcd(s+2,k)=1$ and $\lambda\in K\setminus\{0\}$ and the algebra $\Lambda=\Gamma^*(n)$ where $n\geq 1$. Thus the second Hochschild cohomology group will be known for all the classes of the algebras given in [1]. We remark that an algebra of the type $\Lambda(p,q,k,s,\lambda)$ is never isomorphic to an algebra of the type $\Gamma^*(n)$ as their stable Auslander-Reiten quivers are not isomorphic. We refer the reader to [1] which gives precise conditions for two algebras of the same type $\Lambda(p,q,k,s,\lambda)$ or $\Gamma^*(n)$ to be isomorphic.

We start, in Section 2, by introducing the algebras Λ, for both types, by quiver and relations. Section 3 of this paper describes the projective resolution of [2] which we use to find $HH^2(\Lambda)$. In the third section, we determine $HH^2(\Lambda)$ for the algebra $\Lambda=\Lambda(p,q,k,s,\lambda)$, considering separately the cases $1\leq s\leq k-2$ and $s=k-1$. The main result in this section is Theorem 4.9, which shows that $HH^2(\Lambda)$ has dimension 1 for $1\leq s\leq k-1$. This group measures the infinitesimal deformations of the algebra Λ; that is, if $HH^2(\Lambda)=0$ then Λ has no non-trivial deformations, which is not the case here. We include, in Section 4, Theorem 4.10 where we find a non-trivial deformation Λ_η of Λ associated to our non-zero element η in $HH^2(\Lambda)$. This illustrates the connection between the second Hochschild cohomology group and deformation theory. In the final section, we determine $HH^2(\Lambda)$ for $\Lambda=\Gamma^*(n)$. The main result in Section 5 is Theorem 5.4 which shows that $\dim HH^2(\Lambda)=2$. The results we found in this paper are in contrast to the majority of self-injective algebras of finite representation type (see [3]). Since Hochschild cohomology is invariant under derived equivalence, the second Hochschild cohomology group is now known for the standard one-parametric but not weakly symmetric self-injective algebras of tame representation type which are derived equivalent to the algebra of the type $\Lambda(p,q,k,s,\lambda)$ or $\Gamma^*(n)$.

2. The One-Parametric Self-Injective Algebras

In this chapter we describe the algebras of [1]. We start with the algebra $\Lambda=\Lambda(p,q,k,s,\lambda)$. Let K be an algebraically closed field and let p,q,s,k be integers such that $p,\ q\geq 0$, $k\geq 2$, $1\leq s\leq k-1$, $\gcd(s,k)=1$, $\gcd(s+2,k)=1$ and $\lambda\in K\setminus\{0\}$. From [1, Section 5], $\Lambda(p,q,k,s,\lambda)$ has quiver $\mathcal{Q}(p,q,k,s)$:



where, for any $i \in \{1,2,\cdots,k\}$, $i \sim\sim\sim\sim\sim\sim > i-1$ denotes the path

$$i \xrightarrow{\alpha_{(i,0)}} (i,1) \xrightarrow{\alpha_{(i,1)}} (i,2) \xrightarrow{\alpha_{(i,2)}} [r] \cdots$$
$$\xrightarrow{\alpha_{(i,q-1)}} (i,q) \xrightarrow{\alpha_{(i,q)}} i-1,$$

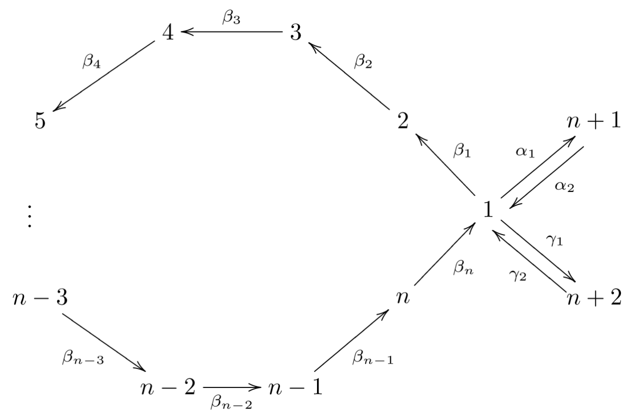
and $i-1----> i+s$ denotes the path

$$i-1 \xrightarrow{\beta_{i^0}} i^1 \xrightarrow{\beta_{i^1}} i^2 \xrightarrow{\beta_{i^2}} \cdots$$
$$\xrightarrow{\beta_{i^{p-1}}} i^p \xrightarrow{\beta_{i^p}} i+s.$$

Then $\Lambda = KQ(p,q,k,s)/I(p,q,k,s,\lambda)$ where $I(p,q,k,s,\lambda)$ is the ideal generated by the relations

- $\beta_{i^p}\beta_{(s+i+1)^0}$, for $i = 1,2,\cdots,k$,
- $\alpha_{(i,q)}\alpha_{(i-1,0)}$, for $i = 1,2,\cdots,k$,
- $\alpha_{(i,t')}\alpha_{(i,t'+1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)}\cdots\alpha_{(s+i,t')}$, for $t' = 0,1,\cdots,q$, $i = 1,2,\cdots,k$,
- $\beta_{i^j}\beta_{i^{j+1}}\cdots\beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)}\cdots\alpha_{(s+i,q)}\beta_{(s+i)^0}\beta_{(s+i)^1}\cdots$ $\beta_{(s+i)^j}$, for $j = 0,1,\cdots,p$, $i = 1,2,\cdots,k$,
- $\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p} - \beta_{(i+1)^0}\beta_{(i+1)^1}\cdots$ $\beta_{(i+1)^p}\alpha_{(s+i+1,0)}\alpha_{(s+i+1,1)}\cdots\alpha_{(s+i+1,q)}$, for $i = 2,\cdots,k$, and
- $\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p} - \lambda\beta_{2^0}\beta_{2^1}\cdots$ $\beta_{2^p}\alpha_{(s+2,0)}\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q)}$, where $\lambda \in K\setminus\{0\}$.

Next we describe the algebra $\Lambda = \Gamma^*(n)$ For $n \geq 1$, $\Gamma^*(n)$ is given in [1, Section 6] by the quiver $Q(n)$:



Then $\Lambda = KQ(n)/I(n)$ where $I(n)$ is the ideal generated by the relations:

1) $\alpha_1\alpha_2 = (\beta_1\beta_2\cdots\beta_n)^2 = \gamma_1\gamma_2$,

2) $\beta_n\alpha_1 = 0,\quad \beta_n\gamma_1 = 0,$

$\alpha_2\beta_1 = 0,\quad \gamma_2\beta_1 = 0,$

$\alpha_2\alpha_1 = 0,\quad \gamma_2\gamma_1 = 0,$

3) for all $j \in \{2,\cdots,n\}$,

$$\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\beta_1\cdots\beta_{j-1}\beta_j = 0.$$

Note that we write our paths from left to right.

In order to compute $HH^2(\Lambda)$, the next section gives the necessary background required to find the first terms of the projective resolution of Λ as a Λ,Λ-bimodule. Section 4 and Section 5 uses this part of a minimal projective bimodule resolution for our algebras to determine the second Hochschild cohomology group and provides the main results of this paper.

3. Projective Resolutions

To find the second Hochschild cohomology group $HH^2(\Lambda)$, we could use the bar resolution given in [4]. This bar resolution is not a minimal projective resolution of Λ as Λ,Λ-bimodule. In practice, it is easier to compute the Hochschild cohomology group if we use a minimal projective resolution. So here we use the projective resolution of [2]. More generally, let $\Lambda = KQ/I$ be a finite dimensional algebra, where *K* is an algebraically closed field, Q is a quiver, and *I* is an admissible ideal of KQ. Fix a minimal set f^2 of generators for the ideal *I*. Let $x \in f^2$. Then $x = \sum_{j=1}^{r} c_j a_{1j}\cdots a_{kj}\cdots a_{s_j j}$, that is, *x* is a linear combination of paths $a_{1j}\cdots a_{kj}\cdots a_{s_j j}$ for $j = 1,\cdots,r$ and $c_j \in K$ and there are unique vertices *v* and *w* such that each path $a_{1j}\cdots a_{kj}\cdots a_{s_j j}$ starts at *v* and ends at *w* for all *j*. We write $\mathfrak{o}(x) = v$ and $\mathfrak{t}(x) = w$. Similarly $\mathfrak{o}(a)$ is the origin of the arrow *a* and $\mathfrak{t}(a)$ is the end of *a*.

In [2, Theorem 2.9], it is shown that there is a minimal projective resolution of Λ as a Λ,Λ-bimodule which begins:

$$\cdots \to Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0,$$

where the projective Λ,Λ-bimodules Q^0, Q^1, Q^2 are given by

$$Q^0 = \bigoplus_{v,\text{vertex}} \Lambda v \otimes v\Lambda,$$

$$Q^1 = \bigoplus_{a,\text{arrow}} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda, \text{ and}$$

$$Q^2 = \bigoplus_{x\in f^2} \Lambda\mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda,$$

and the maps g, A_1, A_2 and A_3 are Λ,Λ-bimodule homomorphisms, defined as follows. The map $g: Q^0 \to \Lambda$ is the multiplication map so is given by $v\otimes v \mapsto v$. The map $A_1: Q^1 \to Q^0$ is given by

$$\mathfrak{o}(a)\otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a)\otimes \mathfrak{o}(a)a - a\mathfrak{t}(a)\otimes \mathfrak{t}(a)$$

for each arrow a. With the notation for $x\in f^2$ given above, the map $A_2: Q^2 \to Q^1$ is given by

$$\mathfrak{o}(x)\otimes \mathfrak{t}(x) \mapsto \sum_{j=1}^r c_j \left(\sum_{k=1}^{s_j} a_{1j}\cdots a_{(k-1)j} \otimes a_{(k+1)j}\cdots a_{s_j j}\right),$$

where $a_{1j}\cdots a_{(k-1)j} \otimes a_{(k+1)j}\cdots a_{s_j j} \in \Lambda\mathfrak{o}(a_{kj})\otimes \mathfrak{t}(a_{kj})\Lambda$.

In order to describe the projective bimodule Q^3 and the map A_3 in the Λ,Λ-bimodule resolution of Λ in [2], we need to introduce some notation from [5]. Recall that an element $y\in KQ$ is uniform if there are vertices v,w such that $y = vy = yw$. We write $\mathfrak{o}(y)=v$ and $\mathfrak{t}(y)=w$. In [5], Green, Solberg and Zacharia show that there are sets f^n in KQ, for $n\ge 3$, consisting of uniform elements $y\in f^n$ such that

$$y = \sum_{x\in f^{n-1}} x r_x = \sum_{z\in f^{n-2}} z s_z$$

for unique elements $r_x, s_z \in KQ$ such that $s_z\in I$. These sets have special properties related to a minimal projective Λ-resolution of $\Lambda/\mathfrak{r}$, where $\mathfrak{r}$ is the Jacobson radical of Λ. Specifically the *n*-th projective in the minimal projective Λ-resolution of $\Lambda/\mathfrak{r}$ is $\bigoplus_{y\in f^n} \mathfrak{t}(y)\Lambda$.

In particular, to determine the set f^3, we follow explicitly the construction given in [5, §1]. Let f^1 denote the set of arrows of Q. Consider the intersection $\left(\bigoplus_i f_i^2 KQ\right)\cap\left(\bigoplus_j f_j^1 I\right)$. Set this intersection equal to some $\left(\bigoplus_l f_l^{3*} KQ\right)$. We then discard all elements of the form f^{3*} that are in $\bigoplus_i f_i^2 I$; the remaining ones form precisely the set f^3.

Thus, for $y\in f^3$ we have that $y\in\left(\bigoplus_i f_i^2 KQ\right)\cap\left(\bigoplus_j f_j^1 I\right)$. So we may write $y=\sum f_i^2 p_i = \sum q_i f_i^2 r_i$ with $p_i, q_i, r_i \in KQ$, such that p_i, q_i are in the ideal generated by the arrows of KQ, and p_i unique. Then [2] gives that $Q^3 = \bigoplus_{y\in f^3} \Lambda Q(y)\otimes Q(y)\Lambda$ and, for $y\in f^3$ in the notation above, the component of $A_3(\mathfrak{o}(y)\otimes\mathfrak{t}(y))$ in the summand $\Lambda\mathfrak{o}(f_i^2)\otimes\mathfrak{t}(f_i^2)\Lambda$ of Q^2 is $\mathfrak{o}(y)\otimes p_i - q_i\otimes r_i$.

Applying $\mathrm{Hom}(-,\Lambda)$ to this part of a minimal projective bimodule resolution of Λ gives us the complex

$$0 \to \mathrm{Hom}(Q^0,\Lambda) \xrightarrow{d_1} \mathrm{Hom}(Q^1,\Lambda) \xrightarrow{d_2}$$

$$\mathrm{Hom}(Q^2,\Lambda) \xrightarrow{d_3} \mathrm{Hom}(Q^3,\Lambda)$$

where d_i is the map induced from A_i for $i=1,2,3$. Then $HH^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2$.

Throughout, all tensor products are tensor products over K, and we write $\otimes$ for $\otimes_K$. When considering an element of the projective Λ,Λ-bimodule $Q^1 = \bigoplus_{a,\text{arrow}} \Lambda\mathfrak{o}(a)\otimes\mathfrak{t}(a)\Lambda$ it is important to keep track of the individual summands of Q^1. So to avoid confusion we usually denote an element in the summand $\Lambda\mathfrak{o}(a)\otimes\mathfrak{t}(a)\Lambda$ by $\lambda\otimes_a\lambda'$ using the subscript "a" to remind us in which summand this element lies. Similarly, an element $\lambda\otimes_{f_i^2}\lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^2)\otimes\mathfrak{t}(f_i^2)\Lambda$ of Q^2 and an element $\lambda\otimes_{f_i^3}\lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^3)\otimes\mathfrak{t}(f_i^3)\Lambda$ of Q^3. We keep this notation for the rest of the paper.

4. $HH^2(\Lambda)$ for $\Lambda = \Lambda(p,q,k,s,\lambda)$

We have given $\Lambda=\Lambda(p,q,k,s,\lambda)$ by quiver and relations in Section 2. However, these relations are not minimal. So next we will find a minimal set of relations f^2 for this algebra.

Let

$$f_{1,1}^2 = \alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p} - \lambda\beta_{2^0}\beta_{2^1}\cdots\beta_{2^p}\alpha_{(s+2,0)}\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q)},$$

$$f_{1,i}^2 = \alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p} - \beta_{(i+1)^0}\beta_{(i+1)^1}\cdots\beta_{(i+1)^p}\alpha_{(s+i+1,0)}\alpha_{(s+i+1,1)}\cdots\alpha_{(s+i+1,q)} \quad \text{for } i\in\{2,\cdots,k\},$$

$$f_{2,i}^2 = \beta_{i^p}\beta_{(s+i+1)^0} \quad \text{for } i\in\{1,\ldots,k\},$$

$$f_{3,i}^2 = \alpha_{(i,q)}\alpha_{(i-1,0)} \quad \text{for } i\in\{1,\ldots,k\},$$

$$f_{4,i,j}^2 = \beta_{i^j}\beta_{i^{j+1}}\cdots\beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)}\cdots\alpha_{(s+i,q)}\beta_{(s+i)^0}\beta_{(s+i)^1}\cdots\beta_{(s+i)^j} \quad \text{where } j\in\{1,\ldots,p-1\} \text{ and } i\in\{1,\ldots,k\},$$

$$f_{5,i,t'}^2 = \alpha_{(i,t')}\alpha_{(i,t'+1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)}\cdots\alpha_{(s+i,t')} \quad \text{where } t'\in\{1,\ldots,q-1\} \text{ and } i\in\{1,\ldots,k\}.$$

The remaining relations given in Section 2 are all linear combinations of the above relations. For example, the relation $\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(s+i,0)}\alpha_{(s+i,1)}\cdots\alpha_{(s+i,q)}\beta_{(s+i)^0}$ can be written as

$$\alpha_{(i-1,0)}\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q)}\beta_{(i-1)^0}\beta_{(i-1)^1}\cdots$$
$$\beta_{(i-1)^{p-1}}f^2_{2,i-1}-f^2_{1,i-1}\beta_{(s+i)^0}.$$

So this relation is in *I* and is not in f^2.

Proposition 4.1 *For* $\Lambda=\Lambda(p,q,k,s,\lambda)$ *and with the above notation, the minimal set of relations is*

$$f^2=\left\{f^2_{1,i},f^2_{2,i},f^2_{3,i},f^2_{4,i,j},f^2_{5,i,t'}\right\}.$$

In contrast to the majority of self-injective algebras of finite representation type, we will show that the algebra $\Lambda(p,q,k,s,\lambda)$ has non-zero second Hochschild cohomology group (see [3, Theorem 6.5]). Recall that $HH^2(\Lambda)=\mathrm{Ker}d_3/\mathrm{Im}d_2$, where

$$d_3:\mathrm{Hom}(Q^2,\Lambda)\to\mathrm{Hom}(Q^3,\Lambda)$$

is induced by $A_3:Q^3\to Q^2$.

First we will find $\mathrm{Im}d_2$. Since

$$d_2:\mathrm{Hom}(Q^1,\Lambda)\to\mathrm{Hom}(Q^2,\Lambda),$$

let $f\in\mathrm{Hom}(Q^1,\Lambda)$ so that $d_2f=fA_2$. We consider the cases $1\le s\le k-2$ and $s=k-1$ separately.

Let $1\le s\le k-2$ and

$$f\left(e_i\otimes_{\beta_{(i+1)^0}}e_{(i+1)^1}\right)=c_{1,i}\beta_{(i+1)^0},$$

$$f\left(e_{(i+1)^j}\otimes_{\beta_{(i+1)^j}}e_{(i+1)^{j+1}}\right)=c_{2,i+1,j}\beta_{(i+1)^j}$$
for $j\in\{1,\ldots,p-1\}$,

$$f\left(e_{(i+1)^p}\otimes_{\beta_{(i+1)^p}}e_{s+i+1}\right)=c_{2,i+1,p}\beta_{(i+1)^p},$$

$$f\left(e_i\otimes_{\alpha_{(i,0)}}e_{(i,1)}\right)=c_{3,i}\alpha_{(i,0)},$$

$$f\left(e_{(i,t')}\otimes_{\alpha_{(i,t')}}e_{(i,t'+1)}\right)=c_{4,i,t'}\alpha_{(i,t')}$$
for $t'\in\{1,\ldots,q-1\}$ and

$$f\left(e_{(i,q)}\otimes_{\alpha_{(i,q)}}e_{i-1}\right)=c_{4,i,q}\alpha_{(i,q)},$$

where all coefficients $c_{1,i},c_{2,i+1,j}$ for $j\in\{1,\cdots,p-1\}$, $c_{2,i+1,p},c_{3,i},c_{4,i,t'}$ for $t'\in\{1,\ldots,q-1\}$, $c_{4,i,q}\in K$. Now we find fA_2.

First we have,

$$fA_2\left(e_1\otimes_{f^2_{1,1}}e_{s+1}\right)$$
$$=f\left(e_1\otimes_{\alpha_{(1,0)}}e_{(1,1)}\right)\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p}+\alpha_{(1,0)}f\left(e_{(1,1)}\otimes_{\alpha_{(1,1)}}e_{(1,2)}\right)\alpha_{(1,2)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p}$$
$$+\cdots+\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q-1)}f\left(e_{(1,q)}\otimes_{\alpha_{(1,q)}}e_k\right)\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p}+\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}f\left(e_k\otimes_{\beta_{1^0}}e_{1^1}\right)\beta_{1^1}\cdots\beta_{1^p}$$
$$+\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}f\left(e_{1^1}\otimes_{\beta_{1^1}}e_{1^2}\right)\beta_{1^2}\cdots\beta_{1^p}+\cdots+\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^{p-1}}f\left(e_{1^p}\otimes_{\beta_{1^p}}e_{s+1}\right)$$
$$-\lambda\Big[f\left(e_1\otimes_{\beta_{2^0}}e_{2^1}\right)\beta_{2^1}\cdots\beta_{2^p}\alpha_{(s+2,0)}\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q)}+\beta_{2^0}f\left(e_{2^1}\otimes_{\beta_{2^1}}e_{2^2}\right)\beta_{2^2}\cdots\beta_{2^p}\alpha_{(s+2,0)}\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q)}$$
$$+\cdots+\beta_{2^0}\beta_{2^1}\cdots\beta_{2^{p-1}}f\left(e_{2^p}\otimes_{\beta_{2^p}}e_{s+2}\right)\alpha_{(s+2,0)}\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q)}+\beta_{2^0}\beta_{2^1}\cdots\beta_{2^p}f\left(e_{s+2}\otimes_{\alpha_{(s+2,0)}}e_{(s+2,1)}\right)\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q)}$$
$$+\beta_{2^0}\beta_{2^1}\cdots\beta_{2^p}\alpha_{(s+2,0)}f\left(e_{(s+2,1)}\otimes_{\alpha_{(s+2,1)}}e_{(s+2,2)}\right)\alpha_{(s+2,2)}\cdots\alpha_{(s+2,q)}+\cdots$$
$$+\beta_{2^0}\beta_{2^1}\cdots\beta_{2^p}\alpha_{(s+2,0)}\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q-1)}f\left(e_{(s+2,q)}\otimes_{\alpha_{(s+2,q)}}e_{s+1}\right)\Big]$$
$$=\left(c_{3,1}+c_{4,1,1}+\cdots+c_{4,1,q}+c_{1,k}+c_{2,1,1}+\cdots+c_{2,1,p}\right)\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}$$
$$\beta_{1^1}\cdots\beta_{1^p}$$
$$-\lambda\left(c_{1,1}+c_{2,2,1}+\cdots+c_{2,2,p}+c_{3,s+2}+c_{4,s+2,1}+\cdots+c_{4,s+2,q}\right)\beta_{2^0}\beta_{2^1}\cdots\beta_{2^p}$$
$$\alpha_{(s+2,0)}\alpha_{(s+2,1)}\cdots\alpha_{(s+2,q)}$$
$$=\left(c_{3,1}+c_{4,1,1}+\cdots+c_{4,1,q}+c_{1,k}+c_{2,1,1}+\cdots+c_{2,1,p}-c_{1,1}-c_{2,2,1}-\cdots\right.$$
$$\left.-c_{2,2,p}-c_{3,s+2}-c_{4,s+2,1}-\cdots-c_{4,s+2,q}\right)\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p}.$$

Similarly for $i\in\{2,\cdots,k\}$,

$$fA_2\left(e_i \otimes_{f_{1,i}^2} e_{s+i}\right) = \left(c_{3,i} + c_{4,i,1} + \cdots + c_{4,i,q} + c_{1,i-1} + c_{2,i,1} + \cdots + c_{2,i,p} - c_{1,i} - c_{2,i+1,1} - \cdots - c_{2,i+1,p} - c_{3,s+i+1} - c_{4,s+i+1,1} - \cdots - c_{4,s+i+1,q}\right)\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}.$$

For the remaining terms, $fA_2(\mathfrak{o}(x)\otimes_x \mathfrak{t}(x)) = 0$ where $x \in \left\{f_{2,i}^2, f_{3,i}^2, f_{4,i,j}^2, f_{5,i,t'}^2\right\}$ for all $i \in \{1,\cdots,k\}$, $j \in \{1,\cdots,p-1\}$ and $t' \in \{1,\cdots,q-1\}$.

Let

$$c_{i'} = c_{3,i} + c_{4,i,1} + \cdots + c_{4,i,q} + c_{1,i-1} + c_{2,i,1} + \cdots + c_{2,i,p} - c_{1,i} - c_{2,i+1,1} - \cdots - c_{2,i+1,p} - c_{3,s+i+1} - c_{4,s+i+1,1} - \cdots - c_{4,s+i+1,q}$$

for $i = 1,\cdots,k$ and

$$\rho_i = \alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}$$

for $i = 1,\cdots,k.$

Thus for $i \in \{1,\cdots,k\}$ and $1 \le s \le k-2$, fA_2 is given by

$$fA_2\left(e_i \otimes_{f_{1,i}^2} e_{s+i}\right) = c_{i'}\rho_i,$$

$$fA_2\left(e_{i^p} \otimes_{f_{2,i}^2} e_{(s+i+1)^1}\right) = 0,$$

$$fA_2\left(e_{(i,q)} \otimes_{f_{3,i}^2} e_{(i-1,1)}\right) = 0,$$

$$fA_2\left(e_{i^j} \otimes_{f_{4,i,j}^2} e_{(s+i)^{j+1}}\right) = 0 \text{ where } j \in \{1,\cdots,p-1\} \text{ and}$$

$$fA_2\left(e_{(i,t')} \otimes_{f_{5,i,t'}^2} e_{(s+i,t'+1)}\right) = 0 \text{ where } t' \in \{1,\cdots,q-1\},$$

where $c_{1'},\cdots,c_{k'} \in K$ with $\Sigma_{i=1}^k c_{i'} = 0$. So $\dim \mathrm{Im} d_2 = k-1.$

For $s = k-1$, we let

$$f\left(e_i \otimes_{\beta_{(i+1)^0}} e_{(i+1)^1}\right) = c_{1,i}\beta_{(i+1)^0},$$

$$f\left(e_{(i+1)^j} \otimes_{\beta_{(i+1)^j}} e_{(i+1)^{j+1}}\right) = c_{2,i+1,j}\beta_{(i+1)^j}$$

for $j \in \{1,\cdots,p-1\}$,

$$f\left(e_{(i+1)^p} \otimes_{\beta_{(i+1)^p}} e_i\right) = c_{2,i+1,p}\beta_{(i+1)^p},$$

$$f\left(e_i \otimes_{\alpha_{(i,0)}} e_{(i,1)}\right) = c_{3,i}\alpha_{(i,0)},$$

$$f\left(e_{(i,t')} \otimes_{\alpha_{(i,t')}} e_{(i,t'+1)}\right) = c_{4,i,t'}\alpha_{(i,t')}$$

for $t' \in \{1,\cdots,q-1\}$ and

$$f\left(e_{(i,q)} \otimes_{\alpha_{(i,q)}} e_{i-1}\right) = c_{4,i,q}\alpha_{(i,q)} + d_{1,i}\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p},$$

where for all $i \in \{1,\cdots,k\}$ the coefficients $c_{1,i}, c_{2,i+1,j}$ for $j \in \{1,\ldots,p-1\}, c_{2,i+1,p}$ $c_{3,i}, c_{4,i,t'}$ for $t' \in \{1,\cdots,q-1\}, c_{4,i,q}$, $d_{1,i}$ are in K.

Then we can find fA_2 for $i \in \{1,\cdots,k\}$ in the same way as the previous case to see that it is given by

$$fA_2\left(e_i \otimes_{f_{1,i}^2} e_{i-1}\right) = c_{i'}\rho_i \text{ where } c_{i'}, \rho_i \text{ as above},$$

$$fA_2\left(e_{i^p} \otimes_{f_{2,i}^2} e_{i^1}\right) = 0,$$

$$fA_2\left(e_{(i,q)} \otimes_{f_{3,i}^2} e_{(i-1,1)}\right) = d_{1,i}\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)},$$

$$fA_2\left(e_{i^j} \otimes_{f_{4,i,j}^2} e_{(i-1)^{j+1}}\right) = 0 \text{ where } j \in \{1,\cdots,p-1\} \text{ and}$$

$$fA_2\left(e_{(i,t')} \otimes_{f_{5,i,t'}^2} e_{(i-1,t'+1)}\right) = 0 \text{ where } t' \in \{1,\cdots,q-1\},$$

where $c_{1'},\ldots,c_{k'},d_{1,1},\ldots,d_{1,k} \in K$ with $\Sigma_{i=1}^k c_{i'} = 0$. Note that there is no dependency between the $d_{1,i}$. So $\dim \mathrm{Im} d_2 = 2k-1.$

Proposition 4.2 If $1 \le s \le k-2$, we have $\dim \mathrm{Im} d_2 = k-1$. If $s = k-1$, we have $\dim \mathrm{Im} d_2 = 2k-1.$

Next we find $\mathrm{Hom}\left(Q^2,\Lambda\right)$ and again consider the two cases separately. Let $1 \le s \le k-2$ and $h \in \mathrm{Hom}\left(Q^2,\Lambda\right)$. Then h is defined by

$$\mathfrak{o}\left(f_{1,i}^2\right)\otimes \mathfrak{t}\left(f_{1,i}^2\right) \mapsto d_i\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p} \text{ for } i \in \{1,2,\cdots,k\},$$
$$\text{else} \mapsto 0,$$

where $d_i \in K$.

Therefore $\dim\mathrm{Hom}\left(Q^2,\Lambda\right) = k.$ Hence, $\dim \mathrm{Ker} d_3 \le k.$

For $s = k-1$ and $i \in \{1,2,\cdots,k\}$, h is given by

$$\mathfrak{o}\left(f_{1,i}^2\right)\otimes \mathfrak{t}\left(f_{1,i}^2\right) \mapsto d_i\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p} + d_{i'}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)},$$
$$\mathfrak{o}\left(f_{3,i}^2\right)\otimes \mathfrak{t}\left(f_{3,i}^2\right) \mapsto d_{i''}\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)},$$
$$\text{else} \mapsto 0,$$

where $d_i, d_{i'}, d_{i''}$ are in K for $i \in \{1,\cdots,k\}$. Thus $\dim\mathrm{Hom}(Q^2,\Lambda) = 3k$.

Proposition 4.3 If $1 \le s \le k-2$, we have $\dim\mathrm{Hom}(Q^2,\Lambda) = k$. *If* $s = k-1$, $\dim\mathrm{Hom}(Q^2,\Lambda) = 3k$.

Corollary 4.4 If $1 \le s \le k-2$, we have $\dim\mathrm{Ker}d_3 \le k$. *If* $s = k-1$, $\dim\mathrm{Ker}d_3 \le 3k$.

In order to find $\mathrm{Ker}d_3$ and hence determine $HH^2(\Lambda)$ we start by giving a non-zero element in $HH^2(\Lambda)$ for all s.

Proposition 4.5 Define $h_1 \in \mathrm{Hom}(Q^2,\Lambda)$ by

$$\mathfrak{o}(f_{1,1}^2)\otimes\mathfrak{t}(f_{1,1}^2) = e_1 \otimes e_{s+1} \mapsto \alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p} = \rho_1,$$
$$\text{else} \mapsto 0.$$

Then h_1 is in $\mathrm{Ker}d_3$.

Proof. We note that $\rho_1 \ne 0$ so h_1 is a non-zero map. To show that $h_1 \in \mathrm{Ker}d_3$ we show that $h_1A_3 = 0$. First, observe that $\rho_1\beta_{(s+2)^0} = 0$ and $\rho_1\alpha_{(s+1,0)} = 0$. Hence $\rho_1\mathfrak{r} = 0$. Similarly we have $\mathfrak{r}\rho_1 = 0$.

Recall that $Q^3 = \coprod_{y\in f^3}\Lambda\mathfrak{o}(y)\otimes\mathfrak{t}(y)\Lambda$ where $y = \sum_u f_u^2 p_u = \sum_u q_u f_u^2 r_u$ and p_u, q_u are in the ideal generated by the arrows. For $y \in f^3$ the component of $A_3(\mathfrak{o}(y)\otimes\mathfrak{t}(y))$ in $\Lambda\mathfrak{o}(f_u^2)\otimes\mathfrak{t}(f_u^2)\Lambda$ is

$$\sum\left(\mathfrak{o}(y)\otimes_{f_u^2} p_u - q_u \otimes_{f_u^2} r_u\right).$$

Then

$$h_1A_3(\mathfrak{o}(y)\otimes\mathfrak{t}(y)) = \sum_u\left(h_1\left(\mathfrak{o}(y)\otimes_{f_u^2} p_u\right) - q_u h_1\left(\mathfrak{o}(f_u^2)\otimes_{f_u^2}\mathfrak{t}(f_u^2)\right)r_u\right).$$

Thus

$$h_1\left(\mathfrak{o}(y)\otimes_{f_u^2} p_u\right) = \begin{cases}\rho_1 p_u & \text{if } f_u^2 = f_{1,1}^2\\ 0 & \text{otherwise.}\end{cases}$$

As p_u is in the arrow ideal of KQ, $\rho_1 p_u \in \rho_1\mathfrak{r} = 0$. So we have $h_1(\mathfrak{o}(y)\otimes p_u) = 0$. Similarly $h_1\left(q_u \otimes_{f_u^2} r_u\right) = 0$ as $q_u\rho_1 r_u \in \mathfrak{r}\rho_1 r_u = 0$. Therefore $h_1A_3(\mathfrak{o}(y)\otimes\mathfrak{t}(y)) = 0$ for all $y \in f^3$ so $h_1A_3 = 0$. Thus $h_1 \in \mathrm{Ker}d_3$ as required.

Theorem 4.6 For $\Lambda = \Lambda(p,q,k,s,\lambda)$ where p,q are positive integers, $k \ge 2$, $1 \le s \le k-1$ with $\gcd(s+2,k) = 1 = \gcd(s,k)$ and $\lambda \in K\setminus\{0\}$, we have $HH^2(\Lambda) \ne 0$.

Proof. Consider the element $h_1 + \mathrm{Im}d_2$ of $HH^2(\Lambda)$ where h_1 is given as in Proposition 4.5 by

$$\mathfrak{o}(f_{1,1}^2)\otimes\mathfrak{t}(f_{1,1}^2) = e_1 \otimes e_{s+1} \mapsto \rho_1,$$
$$\text{else} \mapsto 0.$$

Suppose for contradiction that $h_1 \in \mathrm{Im}d_2$. Then $h_1(e_1 \otimes e_{s+1}) = fA_2(e_1 \otimes e_{s+1})$. So $\rho_1 = c_1'\rho_1$ and so $c_1' = 1$. Also $h_1(e_i \otimes e_{s+i}) = fA_2(e_i \otimes e_{s+i})$ where $i \in \{2,\cdots,k\}$. Then $0 = c_i'\rho_i$, where $i \in \{2,\cdots,k\}$. But this contradicts having $\Sigma_{i=1}^k c_{i'} = 0$. Therefore $h_1 \notin \mathrm{Im}d_2$, that is, $h_1 + \mathrm{Im}d_2 \ne 0 + \mathrm{Im}d_2$. So $h_1 + \mathrm{Im}d_2$ is a non-zero element in $HH^2(\Lambda)$. □

Note that we can also define maps $h_i : Q^2 \to \Lambda$ by

$$\mathfrak{o}(f_{1,i}^2)\otimes\mathfrak{t}(f_{1,i}^2) \mapsto \alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p} = \rho_i,$$
$$\text{else} \mapsto 0.$$

for $i = 2,\cdots,k$. However, $h_1, h_2, \cdots, h_k$ all represent the same element $h_1 + Imd_2$ of $HH^2(\Lambda)$.

As we have found a non-zero element in $HH^2(\Lambda)$ we know that $\dim HH^2(\Lambda) \ge 1$. In the case $1 \le s \le k-2$ we have the following result, the proof of which is immediate from Proposition 4.2, Corollary 4.4 and Theorem 4.6.

Proposition 4.7 For $\Lambda = \Lambda(p,q,k,s,\lambda)$ where $1 \le s \le k-2$, we have $\dim\mathrm{Ker}d_3 = k$ and $\dim HH^2(\Lambda) = 1$.

For the case $s = k-1$, we need more details to find $\mathrm{Ker}d_3$. Following [5] we may choose the set f^3 to consist of the following elements:

$$\{f_{1,i}^3, f_{2,i}^3, f_{3,i,t'}^3, f_{4,i,j}^3\},$$

where

$$f_{1,i}^3 = f_{1,i}^2\alpha_{(i-1,0)}\alpha_{(i-1,1)} + \beta_{(i+1)^0}\beta_{(i+1)^1}\cdots\beta_{(i+1)^p}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q-1)}f_{3,i}^2\alpha_{(i-1,1)} = \alpha_{(i,0)}f_{5,i,1}^2 \in e_iKQe_{(i-1,2)} \text{ where } i \in \{2,\cdots,k\},$$

$$f_{1,1}^3 = f_{1,1}^2\alpha_{(k,0)}\alpha_{(k,1)} + \lambda\beta_{2^0}\beta_{2^1}\cdots\beta_{2^p}\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q-1)}f_{3,1}^2\alpha_{(k,1)} = \alpha_{(1,0)}f_{5,1,1}^2 \in e_1KQe_{(k,2)},$$

$$f_{2,i}^3 = f_{1,i}^2\beta_{i^0}\beta_{i^1} - \alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^{(p-1)}}f_{2,i}^2\beta_{i^1} = -\beta_{(i+1)^0}f_{4,i+1,1}^2 \in e_iKQe_{i^2} \text{ where } i \in \{2,\cdots,k\},$$

$$f_{2,1}^3 = f_{1,1}^2\beta_{1^0}\beta_{1^1} - \alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^{(p-1)}}f_{2,1}^2\beta_{1^1} = -\lambda\beta_{2^0}f_{4,2,1}^2 \in e_1KQe_{1^2},$$

$$f_{3,i,t'}^3 = f_{5,i,t'}^2\alpha_{(i-1,t'+1)} = \alpha_{(i,t')}f_{5,i,t'+1}^2 \in e_{(i,t')}KQe_{(i-1,t'+2)} \text{ where } i \in \{1,\cdots,k\} \text{ and } t' \in \{1,\cdots,q-2\},$$

$$f_{3,i,q-1}^3 = f_{5,i,q-1}^2\alpha_{(i-1,q)} - \alpha_{(i,q-1)}f_{3,i}^2\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q)}\beta_{(i-1)^0}\beta_{(i-1)^1}\cdots\beta_{(i-1)^p} = -\alpha_{(2,q-1)}\alpha_{(2,q)}f_{1,1}^2 \in e_{(i,q-1)}KQe_{i-2}$$
where $i \in \{1,3,\cdots,k,\}$

$$f_{3,2,q-1}^3 = \lambda f_{5,2,q-1}^2\alpha_{(1,q)} - \alpha_{(2,q-1)}f_{3,2}^2\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p} = -\alpha_{(2,q-1)}\alpha_{(2,q)}f_{1,1}^2 \in e_{(2,q-1)}KQe_k,$$

$$f_{3,i,q}^3 = f_{3,i}^2\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q)}\beta_{(i-1)^0}\beta_{(i-1)^1}\cdots\beta_{(i-1)^p}\alpha_{(i-2,0)} - \alpha_{(i,q)}f_{1,i-1}^2\alpha_{(i-2,0)} = \alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)}\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q-1)}f_{3,i-1}^2$$
$\in e_{(i,q)}KQe_{(i-2,1)}$ where $i \in \{1,3,\cdots,k\}$,

$$f_{3,2,q}^3 = f_{3,2}^2\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p}\alpha_{(k,0)} - \alpha_{(2,q)}f_{1,1}^2\alpha_{(k,0)} = \lambda\alpha_{(2,q)}\beta_{2^0}\beta_{2^1}\cdots\beta_{2^p}\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q-1)}f_{3,1}^2 \in e_{(2,q)}KQe_{(k,1)},$$

$$f_{4,i,j}^3 = f_{4,i,j}^2\beta_{(i-1)^{(j+1)}} = \beta_{i^j}f_{4,i,j+1}^2 \in e_{i^j}KQe_{(i-1)^{(j+2)}} \text{ where } i \in \{1,\cdots,k\} \text{ and } j \in \{1,\cdots,p-2\},$$

$$f_{4,i,p-1}^3 = f_{4,i,p-1}^2\beta_{(i-1)^p} - \beta_{i^{(p-1)}}f_{2,i}^2\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)}\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q)} = \beta_{i^{(p-1)}}\beta_{i^p}f_{1,i-1}^2 \in e_{i^{(p-1)}}KQe_{i-2} \text{ where } i \in \{1,3,\cdots,k\},$$

$$f_{4,2,p-1}^3 = f_{4,2,p-1}^2\beta_{1^p} - \lambda\beta_{2^{(p-1)}}f_{2,2}^2\beta_{2^1}\cdots\beta_{2^p}\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)} = \beta_{2^{(p-1)}}\beta_{2^p}f_{1,1}^2 \in e_{2^{(p-1)}}KQe_k,$$

$$f_{4,i,p}^3 = f_{2,i}^2\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)}\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q)}\beta_{(i-1)^0} + \beta_{i^p}f_{1,i-1}^2\beta_{(i-1)^0} = \beta_{i^p}\alpha_{(i-1,0)}\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q)}\beta_{(i-1)^0}\beta_{(i-1)^1}\cdots\beta_{(i-1)^{(p-1)}}f_{2,i-1}^2$$
$\in e_{i^p}KQe_{(i-1)^1}$ where $i \in \{1,3,\cdots,k\}$,

$$f_{4,2,p}^3 = f_{2,i}^2\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)}\alpha_{(i-1,1)}\cdots\alpha_{(i-1,q)}\beta_{(i-1)^0} + \beta_{i^p}f_{1,i-1}^2\beta_{(i-1)^0} = \beta_{2^p}\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^{(p-1)}}f_{2,1}^2 \in e_{2^p}KQe_{1^1}.$$

Thus the projective bimodule Q^3 is $\bigoplus_{y\in f^3}\Lambda\mathfrak{o}(y)\otimes\mathfrak{t}(y)\Lambda$

$$\begin{aligned}&= \bigoplus_{i=1}^k\Big[\Big(\Lambda e_i \otimes_{f_{1,i}^3} e_{(i-1,2)}\Lambda\Big)\oplus\Big(\Lambda e_i \otimes_{f_{2,i}^3} e_{i^2}\Lambda\Big)\bigoplus_{t'=1}^{q-2}\Big(\Lambda e_{(i,t')}\otimes_{f_{3,i,t'}^3} e_{(i-1,t'+2)}\Lambda\Big)\\ &\oplus\Big(\Lambda e_{(i,q-1)}\otimes_{f_{3,i,q-1}^3} e_{i-2}\Lambda\Big)\oplus\Big(\Lambda e_{(i,q)}\otimes_{f_{3,i,q}^3} e_{(i-2,1)}\Lambda\Big)\bigoplus_{j=1}^{p-2}\Big(\Lambda e_{i^j}\otimes_{f_{4,i,j}^3} e_{(i-1)^{(j+2)}}\Lambda\Big)\\ &\oplus\Big(\Lambda e_{i^{(p-1)}}\otimes_{f_{4,i,p-1}^3} e_{i-2}\Lambda\Big)\oplus\Big(\Lambda e_{i^p}\otimes_{f_{4,i,p}^3} e_{(i-1)^1}\Lambda\Big)\Big].\end{aligned}$$

Now we determine $\mathrm{Ker}\,d_3$ in the case $s = k-1$. Let $h \in \mathrm{Ker}\,d_3$, so $h \in \mathrm{Hom}(Q^2,\Lambda)$ and $d_3h = 0$. Recall that for $i \in \{1,\cdots,k\}$, h is given by

$$\begin{aligned}&\mathfrak{o}(f_{1,i}^2)\otimes\mathfrak{t}(f_{1,i}^2) \mapsto d_i\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p} + d_{i'}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)},\\ &\mathfrak{o}(f_{3,i}^2)\otimes\mathfrak{t}(f_{3,i}^2) \mapsto d_{i''}\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)}\\ &\qquad\text{else} \mapsto 0,\end{aligned}$$

where $d_i, d_{i'}, d_{i''}$ are in K.

Then for $i \in \{1,\cdots,k\}$, we have $hA_3\Big(e_i \otimes_{f_{1,i}^3} e_{(i-1,2)}\Big)$

$$\begin{aligned}&= h\Big(e_i\otimes_{f_{1,i}^2} e_{i-1}\Big)\alpha_{(i-1,0)}\alpha_{(i-1,1)} + \beta_{(i+1)^0}\beta_{(i+1)^1}\cdots\beta_{(i+1)^p}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q-1)}h\Big(e_{(i,q)}\otimes_{f_{3,i}^2}\alpha_{(i-1,1)}\Big) - \alpha_{(i,0)}h\Big(e_{(i,1)}\otimes_{f_{5,i,1}^2} e_{(i-1,2)}\Big)\\ &= d_i\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)}\alpha_{(i-1,1)} + d_{i'}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\alpha_{(i-1,0)}\alpha_{(i-1,1)}\\ &\quad + d_{i''}\beta_{(i+1)^0}\beta_{(i+1)^1}\cdots\beta_{(i+1)^p}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)}\\ &= 0\end{aligned}$$

In a similar way we can show that $hA_3\Big(e_1\otimes_{f_{1,1}^3} e_{(k,2)}\Big) = 0$.

For $i \in \{2,\cdots,k\}$, we have $hA_3\Big(e_i\otimes_{f_{2,i}^3} e_{i^2}\Big)$

$$
\begin{aligned}
&= h\left(e_i \otimes_{f_{1,i}^2} e_{i-1}\right)\beta_{i^0}\beta_{i^1} - \alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^{(p-1)}} h\left(e_{i^p} \otimes_{f_{2,i}^2} e_{i^1}\right)\beta_{i^1} + \beta_{(i+1)^0} h\left(e_{(i+1)^1} \otimes_{f_{4,i+1,1}^2} e_{i^2}\right) \\
&= d_i \alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\beta_{i^0}\beta_{i^1} + d_{i'}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1} \\
&= d_{i'}\alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}.
\end{aligned}
$$

As $h \in \operatorname{Ker} d_3$ we have $d_{i'} = 0$ for $i \in \{2,\cdots,k\}$. Similarly it can be shown that

$$hA_3\left(e_1 \otimes_{f_{2,1}^3} e_{1^2}\right) = d_{1'}\alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}$$

so that $d_{1'} = 0$.

We also have $hA_3\left(\mathfrak{o}\left(f_{3,i,t'}^3\right) \otimes_{f_{3,i,t'}^3} \mathfrak{t}\left(f_{3,i,t'}^3\right)\right) = 0$ for $i \in \{1,\cdots,k\}$ and $t' \in \{1,\cdots,q\}$. Finally, putting

$$hA_3\left(\mathfrak{o}\left(f_{4,i,j}^3\right) \otimes_{f_{4,i,j}^3} \mathfrak{t}\left(f_{4,i,j}^3\right)\right) = 0$$

does not give any new information for $i \in \{1,\cdots,k\}$, $j \in \{1,\cdots,p\}$.

Thus h is given by

$$
\begin{aligned}
&\mathfrak{o}\left(f_{1,i}^2\right) \otimes \mathfrak{t}\left(f_{1,i}^2\right) \mapsto d_i \alpha_{(i,0)}\alpha_{(i,1)}\cdots\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p} \text{ for } i \in \{1,2,\cdots,k\}, \\
&\mathfrak{o}\left(f_{3,i}^2\right) \otimes \mathfrak{t}\left(f_{3,i}^2\right) \mapsto d_{i''}\alpha_{(i,q)}\beta_{i^0}\beta_{i^1}\cdots\beta_{i^p}\alpha_{(i-1,0)} \text{ for } i \in \{1,2,\cdots,k\}, \\
&\qquad \text{else} \mapsto 0,
\end{aligned}
$$

where $d_i, d_{i''}$ for $i \in \{1,\cdots,k\}$ are in K. It is clear that there is no dependency between $d_i, d_{i''}$, and therefore $\dim \operatorname{Ker} d_3 = 2k$.

Proposition 4.8 For $\Lambda = \Lambda(p,q,k,s,\lambda)$ and $s = k-1$, we have $\dim \operatorname{Ker} d_3 = 2k$.

Using Propositions 4.2, 4.7, 4.8 and Theorem 4.6 we get the main result of this section.

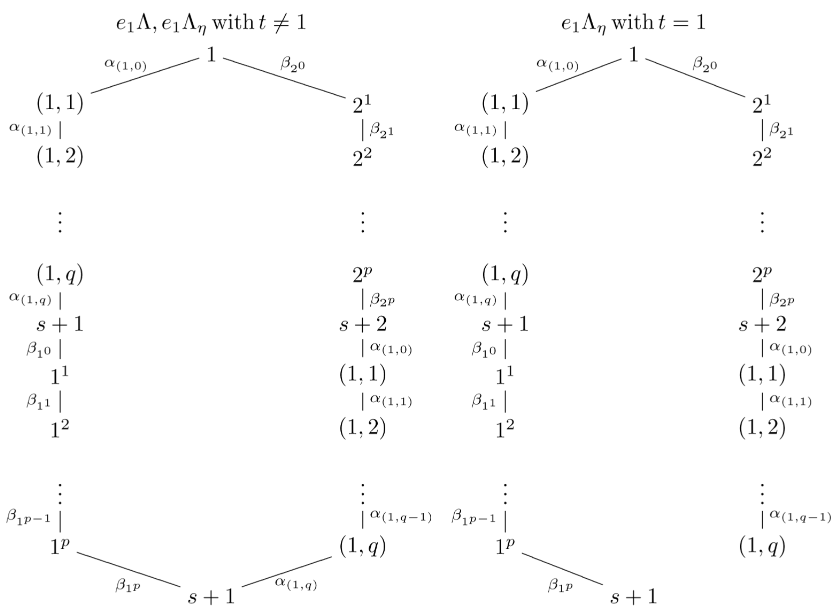
Theorem 4.9 For $\Lambda = \Lambda(p,q,k,s,\lambda)$ where p, q, s, k are integers such that $p,\ q \ge 0$, $k \ge 2$, $1 \le s \le k-1$, $\gcd(s,k) = 1$, $\gcd(s+2,k) = 1$ and $\lambda \in K \setminus \{0\}$, we have $\dim HH^2(\Lambda) = 1$.

We conclude this section by giving a deformation of Λ which arises from the non-zero element $h_1 + \operatorname{Im} d_2$ in $HH^2(\Lambda)$.

Let $\eta = h_1 + \operatorname{Im} d_2$. Recall that $\rho_1 = \alpha_{(1,0)}\alpha_{(1,1)}\cdots\alpha_{(1,q)}\beta_{1^0}\beta_{1^1}\cdots\beta_{1^p}$. We introduce a new parameter t and define the algebra Λ_η to be the algebra KQ/I_η where I_η is the ideal generated by the following elements:

1) $f_{1,1}^2 - t\rho_1,\ f_{1,j}^2$ where $j \in \{2,\cdots,k\}$,
2) for all $i \in \{1,\cdots,k\}$, $f_{2,i}^2, f_{3,i}^2, f_{4,i,j}^2, f_{5,i,t'}^2$, where $j \in \{1,\cdots,p-1\}$, $t' \in \{1,\cdots,q-1\}$,
3) $\rho_1 a$ for all arrows a with $\mathfrak{t}(\rho_1) = \mathfrak{o}(a)$,
4) $a\rho_1$ for all arrows a with $\mathfrak{t}(a) = \mathfrak{o}(\rho_1)$.

We now need to show that $\dim \Lambda_\eta = \dim \Lambda$ to verify that Λ_η is indeed a deformation of Λ. First of all, it is clear that $\dim e_j \Lambda_\eta = \dim e_j \Lambda$ for all t and for all vertices e_i with $e_j \ne e_1$. Now we consider $e_1\Lambda$ and $e_1\Lambda_\eta$ with $t \ne 1$, and $e_1\Lambda_\eta$ with $t = 1$. These projective modules are described as follows:



In each case we see that

$$\dim e_1\Lambda = \dim e_1\Lambda_\eta = 2p+2q+4$$

for all *t*. Hence $\dim\Lambda_\eta = \dim\Lambda$. Moreover, when $t=1$ the algebras Λ and Λ_η are not isomorphic since, in this case, Λ_η is not self-injective. Thus we have found a non-trivial deformation of Λ.

Theorem 4.10 With Λ, η, and Λ_η as defined above, then Λ_η is a non-trivial deformation of Λ. Moreover, the algebras Λ and Λ_η are socle equivalent.

5. $HH^2(\Lambda)$ for $\Lambda = \Gamma^*(n)$

We have given the algebra $\Lambda = \Gamma^*(n)$ by quiver and relations in Section 2. Note that these relations are not minimal. So we will find a minimal set of relations f^2 for this algebra.

Let

$$f_{1,1}^2 = \alpha_1\alpha_2 - \gamma_1\gamma_2, \quad f_{1,2}^2 = \alpha_1\alpha_2 - (\beta_1\beta_2\cdots\beta_n)^2$$

$$f_{2,1}^2 = \beta_n\alpha_1, \quad f_{2,2}^2 = \beta_n\gamma_1,$$

$$f_{2,3}^2 = \alpha_2\beta_1, \quad f_{2,4}^2 = \gamma_2\beta_1,$$

$$f_{2,5}^2 = \alpha_2\alpha_1, \quad f_{2,6}^2 = \gamma_2\gamma_1,$$

$$f_{3,j}^2 = \beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\beta_1\cdots\beta_{j-1}\beta_j,$$
$$\text{for } j\in\{2,\cdots,n-1\}.$$

The remaining relation $\beta_n(\beta_1\beta_2\cdots\beta^n)^2$ can be written as $f_{2,1}^2\alpha_2 - \beta_n f_{1,2}^2$. So this relation is in *I* and is not in f^2.

Proposition 5.1 For $\Lambda = \Gamma^*(n)$ and with the above notation, the minimal set of relations is

$$f^2 = \{f_{1,1}^2, f_{1,2}^2, f_{2,1}^2, f_{2,2}^2, f_{2,3}^2, f_{2,4}^2, f_{2,5}^2, f_{2,6}^2, f_{3,j}^2 \text{ for } j=2,\cdots,n-1\}.$$

Recall that the projective $Q^3 = \bigoplus_{y\in f^3}\Lambda\mathfrak{o}(y)\otimes\mathfrak{t}(y)\Lambda$. Thus we have

$$\begin{aligned}Q^3 &= (\Lambda e_1\otimes e_2\Lambda)\oplus(\Lambda e_1\otimes e_3\Lambda)\oplus(\Lambda e_1\otimes e_{n+1}\Lambda)\\ &\oplus(\Lambda e_1\otimes e_{n+2}\Lambda)\oplus(\Lambda e_{n+1}\otimes e_1\Lambda)\\ &\oplus(\Lambda e_{n+2}\otimes e_1\Lambda)\oplus(\Lambda e_{n-1}\otimes e_1\Lambda)\\ &\oplus\bigoplus_{m=2}^{n-2}(\Lambda e_m\otimes e_{m+2}\Lambda).\end{aligned}$$

(We note that the projective Q^3 is also described in [4] although Happel gives no description of the maps in the Λ,Λ-projective resolution of Λ.) Following [2], and with the notation introduced in Section 3, we may choose the set f^3 to consist of the following elements:

$$\{f_{1,1}^3, f_{1,2}^3, f_{1,3}^3, f_{1,4}^3, f_{n+1}^3, f_{n+2}^3, f_{n-1}^3, f_m^3\},$$

with $m\in\{2,\cdots,n-2\}$ where

$$f_{1,1}^3 = f_{1,1}^2\beta_1 = \alpha_1 f_{2,3}^2 - \gamma_1 f_{2,4}^2 \in e_1KQe_2,$$

$$f_{1,2}^3 = f_{1,2}^2\beta_1\beta_2 = \alpha_1 f_{2,3}^2\beta_2 - \beta_1 f_{3,2}^2 \in e_1KQe_3,$$

$$f_{1,3}^3 = f_{1,2}^2\alpha_1 = \alpha_1 f_{2,5}^2 - \beta_1\beta_2\cdots\beta_n\beta_1\cdots\beta_{n-1}f_{2,1}^2 \in e_1KQe_{n+1},$$

$$f_{1,4}^3 = f_{1,2}^2\gamma_1 - f_{1,1}^2\gamma_1 = \gamma_1 f_{2,6}^2 - \beta_1\beta_2\cdots\beta_n\beta_1\cdots\beta_{n-1}f_{2,2}^2 \in e_1KQe_{n+2},$$

$$f_{n+1}^3 = f_{2,5}^2\alpha_2 - f_{2,3}^2\beta_2\cdots\beta_n\beta_1\cdots\beta_n = \alpha_2 f_{1,2}^2 \in e_{n+1}KQe_1,$$

$$f_{n+2}^3 = f_{2,4}^2\beta_2\cdots\beta_n\beta_1\cdots\beta_n - f_{2,6}^2\gamma_2 = \gamma_2 f_{1,1}^2 - \gamma_2 f_{1,2}^2 \in e_{n+2}KQe_1,$$

$$f_{n-1}^3 = f_{3,n-1}^2\beta_n = \beta_{n-1}f_{2,1}^2\alpha_2 - \beta_{n-1}\beta_n f_{1,2}^2 \in e_{n-1}KQe_1,$$

$$f_m^3 = f_{3,m}^2\beta_{m+1} = \beta_m f_{3,m+1}^2 \in e_mKQe_{m+2}$$
$$\text{for } m\in\{2,\cdots,n-2\}.$$

We know that $HH^2(\Lambda) = \mathrm{Ker}d_3/\mathrm{Im}d_2$. First we will find $\mathrm{Im}d_2$. Let $f\in\mathrm{Hom}(Q^1,\Lambda)$ and so write

$$f(e_1\otimes_{\alpha_1}e_{n+1}) = c_1\alpha_1, \quad f(e_{n+1}\otimes_{\alpha_2}e_1) = c_2\alpha_2,$$

$$f(e_1\otimes_{\gamma_1}e_{n+2}) = c_3\gamma_1, \quad f(e_{n+2}\otimes_{\gamma_2}e_1) = c_4\gamma_2,$$

$$f(e_k\otimes_{\beta_k}e_{k+1}) = d_k\beta_k + d_k'\beta_k\cdots\beta_n\beta_1\cdots\beta_{k-1}\beta_k,$$
$$\text{for } k\in\{1,\cdots,n\},$$

where $c_1,c_2,c_3,c_4,d_k,d_k'\in K$ for $k\in\{1,\cdots,n\}$.

Now we find $fA_2 = d_2f$. We have

$$\begin{aligned}fA_2(e_1\otimes_{f_{1,1}^2}e_1) &= f(e_1\otimes_{\alpha_1}e_{n+1})\alpha_2 + \alpha_1 f(e_{n+1}\otimes_{\alpha_2}e_1)\\ &\quad - f(e_1\otimes_{\gamma_1}e_{n+2})\gamma_2 - \gamma_1 f(e_{n+2}\otimes_{\gamma_2}e_1)\\ &= c_1\alpha_1\alpha_2 + c_2\alpha_1\alpha_2 - c_3\gamma_1\gamma_2 - c_4\gamma_1\gamma_2\\ &= (c_1+c_2-c_3-c_4)\alpha_1\alpha_2.\end{aligned}$$

Also

$$\begin{aligned}fA_2(e_1\otimes_{f_{1,2}^2}e_1) &= f(e_1\otimes_{\alpha_1}e_{n+1})\alpha_2 + \alpha_1 f(e_{n+1}\otimes_{\alpha_2}e_1)\\ &\quad - f(e_1\otimes_{\beta_1}e_2)\beta_2\cdots\beta_n\end{aligned}$$

$$\beta_1\cdots\beta_n - \cdots - \beta_1\cdots\beta_{n-1}f(e_n\otimes_{\beta_n}e_1)\beta_1\cdots\beta_n - \beta_1\cdots$$
$$\beta_n f(e_1\otimes_{\beta_1}e_2)\beta_2\cdots\beta_n - \cdots - \beta_1\cdots\beta_n\beta_1\cdots\beta_{n-1}f(e_n\otimes_{\beta_n}e_1)$$
$$= c_1\alpha_1\alpha_2 + c_2\alpha_1\alpha_2 - d_1\beta_1\cdots\beta_n\beta_1\cdots\beta_n - \ldots - d_n\beta_1\cdots$$
$$\beta_n\beta_1\cdots\beta_n - d_1\beta_1\cdots\beta_n\beta_1\cdots\beta_n - \ldots - d_n\beta_1\cdots\beta_n\beta_1\cdots\beta_n$$
$$= (c_1+c_2)\alpha_1\alpha_2 - (2d_1+\ldots+2d_n)(\beta_1\cdots\beta_n)^2$$
$$= (c_1+c_2-2d_1-\ldots-2d_n)\alpha_1\alpha_2.$$

We can show by direct calculation that

$fA_2\left(\mathfrak{o}\left(f_j^2\right)\otimes \mathfrak{t}\left(f_j^2\right)\right)=0$ for all $f_j^2 \neq f_{1,1}^2, f_{1,2}^2$.

Thus fA_2 is given by

$$fA_2\left(e_1 \otimes_{f_{1,1}^2} e_1\right)=\left(c_1+c_2-c_3-c_4\right)\alpha_1\alpha_2=c'\alpha_1\alpha_2,$$

$$fA_2\left(e_1 \otimes_{f_{1,2}^2} e_1\right)=\left(c_1+c_2-2d_1-\ldots-2d_n\right)\alpha_1\alpha_2 = c''\alpha_1\alpha_2.$$

So $\dim \operatorname{Im} d_2 = 2$.

Proposition 5.2 For $\Lambda=\Gamma^*(n)$, we have $\dim \operatorname{Im} d_2 = 2.$

Now we determine $\operatorname{Ker} d_3$. Let $h\in \operatorname{Ker} d_3$, so $h\in \operatorname{Hom}\left(Q^2,\Lambda\right)$ and $d_3 h = 0$. Then $h:Q^2\to\Lambda$ is given by

$$h\left(e_1 \otimes_{f_{1,1}^2} e_1\right)=c_1e_1+c_2\alpha_1\alpha_2+c_3\beta_1\beta_2\cdots\beta_n,$$

$$h\left(e_1 \otimes_{f_{1,2}^2} e_1\right)=c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n,$$

$$h\left(\mathfrak{o}\left(f_{2,l}^2\right)\otimes_{f_{2,l}^2}\mathfrak{t}\left(f_{2,l}^2\right)\right)=0,\text{ for } l\in\{1,\cdots,4\},$$

$$h\left(e_{n+1} \otimes_{f_{2,5}^2} e_{n+1}\right)=c_7e_{n+1},$$

$$h\left(e_{n+2} \otimes_{f_{2,6}^2} e_{n+2}\right)=c_8e_{n+2} \text{ and}$$

$$h\left(\mathfrak{o}\left(f_{3,j}^2\right)\otimes_{f_{3,j}^2}\mathfrak{t}\left(f_{3,j}^2\right)\right)=d_j\beta_j+d_j'\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_j,$$

for $j\in\{2,\cdots,n-1\}$

for some $c_1,\cdots,c_8,d_j,d_j'\in K$ for $j\in\{2,\cdots,n-1\}$.

Then

$$\begin{aligned}
&hA_3\left(e_1\otimes_{f_{1,1}^3} e_2\right)\\
&=h\left(e_1\otimes_{f_{1,1}^2} e_1\right)\beta_1-\alpha_1 h\left(e_{n+1}\otimes_{f_{2,3}^2} e_2\right)\\
&\quad+\gamma_1 h\left(e_{n+2}\otimes_{f_{2,4}^2} e_2\right)\\
&=\left(c_1e_1+c_2\alpha_1\alpha_2+c_3\beta_1\beta_2\cdots\beta_n\right)\beta_1-0+0\\
&=c_1\beta_1+c_3\beta_1\beta_2\cdots\beta_n\beta_1.
\end{aligned}$$

As $h\in\operatorname{Ker} d_3$ we have $c_1=0$ and $c_3=0$.

$$\begin{aligned}
&hA_3\left(e_1\otimes_{f_{1,2}^3} e_3\right)\\
&=h\left(e_1\otimes_{f_{1,2}^2} e_1\right)\beta_1\beta_2-\alpha_1 h\left(e_{n+1}\otimes_{f_{2,3}^2} e_2\right)\beta_2\\
&\quad+\beta_1 h\left(e_2\otimes_{f_{3,2}^2} e_3\right)\\
&=\left(c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n\right)\beta_1\beta_2-0\\
&\quad+\beta_1\left(d_2\beta_2+d_2'\beta_2\cdots\beta_n\beta_1\beta_2\right)\\
&=\left(c_4+d_2\right)\beta_1\beta_2+\left(c_6+d_2'\right)\beta_1\beta_2\cdots\beta_n\beta_1\beta_2.
\end{aligned}$$

As $h\in\operatorname{Ker} d_3$ we have $c_4+d_2=0$ and $c_6+d_2'=0$. So $d_2=-c_4$ and $d_2'=-c_6$.

Next,

$$\begin{aligned}
&hA_3\left(e_1\otimes_{f_{1,3}^3} e_{n+1}\right)\\
&=h\left(e_1\otimes_{f_{1,2}^2} e_1\right)\alpha_1-\alpha_1 h\left(e_{n+1}\otimes_{f_{2,5}^2} e_{n+1}\right)\\
&\quad+\beta_1\beta_2\cdots\beta_n\beta_1\cdots\beta_{n-1}h\left(e_n\otimes_{f_{2,1}^2} e_{n+1}\right)\\
&=\left(c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n\right)\alpha_1-c_7\alpha_1+0\\
&=\left(c_4-c_7\right)\alpha_1.
\end{aligned}$$

So we have $c_4-c_7=0$ and hence $c_7=c_4$.

$$\begin{aligned}
&hA_3\left(e_1\otimes_{f_{1,4}^3} e_{n+2}\right)\\
&=h\left(e_1\otimes_{f_{1,2}^2} e_1\right)\gamma_1-h\left(e_1\otimes_{f_{1,1}^2} e_1\right)\gamma_1-\gamma_1 h\left(e_{n+2}\otimes_{f_{2,6}^2} e_{n+2}\right)\\
&\quad+\beta_1\beta_2\cdots\beta_n\beta_1\cdots\beta_{n-1}h\left(e_n\otimes_{f_{2,2}^2} e_{n+2}\right)\\
&=\left(c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n\right)\gamma_1\\
&\quad-\left(c_1e_1+c_2\alpha_1\alpha_2+c_3\beta_1\beta_2\cdots\beta_n\right)\gamma_1-c_8\gamma_1+0\\
&=\left(c_4-c_1-c_8\right)\gamma_1.
\end{aligned}$$

Therefore $c_8=c_4$ as $c_1=0$.

$$\begin{aligned}
&hA_3\left(e_{n+1}\otimes_{f_{n+1}^3} e_1\right)\\
&=h\left(e_{n+1}\otimes_{f_{2,5}^2} e_{n+1}\right)\alpha_2-h\left(e_{n+1}\otimes_{f_{2,3}^2} e_2\right)\beta_2\cdots\\
&\quad\beta_n\beta_1\cdots\beta_n-\alpha_2 h\left(e_1\otimes_{f_{1,2}^2} e_1\right)\\
&=c_7\alpha_2-0-\alpha_2\left(c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n\right)\\
&=\left(c_7-c_4\right)\alpha_2.
\end{aligned}$$

Thus again we have $c_7=c_4$.

$$\begin{aligned}
&hA_3\left(e_{n+2}\otimes_{f_{n+2}^3} e_1\right)\\
&=h\left(e_{n+2}\otimes_{f_{2,4}^2} e_2\right)\beta_2\cdots\beta_n\beta_1\cdots\beta_n\\
&\quad-h\left(e_{n+2}\otimes_{f_{2,6}^2} e_{n+2}\right)\gamma_2-\gamma_2 h\left(e_1\otimes_{f_{1,1}^2} e_1\right)\\
&\quad+\gamma_2 h\left(e_1\otimes_{f_{1,2}^2} e_1\right)\\
&=0-c_8\gamma_2-\gamma_2\left(c_1e_1+c_2\alpha_1\alpha_2+c_3\beta_1\beta_2\cdots\beta_n\right)\\
&\quad+\gamma_2\left(c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n\right)\\
&=\left(-c_8-c_1+c_4\right)\gamma_2.
\end{aligned}$$

As $c_1=0$ above, we have $c_8=c_4$ as we already know.

Also

$$\begin{aligned}
&hA_3\left(e_{n-1}\otimes_{f_{n-1}^3} e_1\right)\\
&=h\left(e_{n-1}\otimes_{f_{3,n-1}^2} e_n\right)\beta_n-\beta_{n-1}h\left(e_n\otimes_{f_{2,1}^2} e_{n+1}\right)\alpha_2\\
&\quad+\beta_{n-1}\beta_n h\left(e_1\otimes_{f_{1,2}^2} e_1\right)\\
&=\left(d_{n-1}\beta_{n-1}+d'_{n-1}\beta_{n-1}\beta_n\beta_1\cdots\beta_{n-1}\right)\beta_n\\
&\quad+\beta_{n-1}\beta_n\left(c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n\right)\\
&=d_{n-1}\beta_{n-1}\beta_n+d'_{n-1}\beta_{n-1}\beta_n\beta_1\cdots\beta_{n-1}\beta_n\\
&\quad+c_4\beta_{n-1}\beta_n+c_6\beta_{n-1}\beta_n\beta_1\beta_2\cdots\beta_n\\
&=\left(d_{n-1}+c_4\right)\beta_{n-1}\beta_n\\
&\quad+\left(d'_{n-1}+c_6\right)\beta_{n-1}\beta_n\beta_1\cdots\beta_{n-1}\beta_n.
\end{aligned}$$

So we have $d_{n-1}=-c_4$ and $d'_{n-1}=-c_6$.
Finally, for $2\le m\le n-2$, we have

$$\begin{aligned}
&hA_3\left(e_m\otimes_{f_m^3} e_{m+2}\right)\\
&=h\left(e_m\otimes_{f_{3,m}^2} e_{m+1}\right)\beta_{m+1}-\beta_m h\left(e_{m+1}\otimes_{f_{3,m+1}^2} e_{m+2}\right)\\
&=\left(d_m\beta_m+d'_m\beta_m\beta_{m+1}\cdots\beta_n\beta_1\cdots\beta_m\right)\beta_{m+1}\\
&\quad-\beta_m\left(d_{m+1}\beta_{m+1}+d'_{m+1}\beta_{m+1}\beta_{m+2}\cdots\beta_n\beta_1\cdots\beta_{m+1}\right)\\
&=\left(d_m-d_{m+1}\right)\beta_m\beta_{m+1}\\
&\quad+\left(d'_m-d'_{m+1}\right)\beta_m\beta_{m+1}\cdots\beta_n\beta_1\cdots\beta_m\beta_{m+1}.
\end{aligned}$$

Therefore we have $d_m=d_{m+1}$ and $d'_m=d'_{m+1}$. Hence $d_m=-c_4$ and $d'_m=-c_6$ for $m\in\{2,\cdots,n-1\}$ as we have above $d_2=d_{n-1}=-c_4$ and $d'_2=d'_{n-1}=-c_6$.
Thus h is given by

$$h\left(e_1\otimes_{f_{1,1}^2} e_1\right)=c_2\alpha_1\alpha_2,$$

$$h\left(e_1\otimes_{f_{1,2}^2} e_1\right)=c_4e_1+c_5\alpha_1\alpha_2+c_6\beta_1\beta_2\cdots\beta_n,$$

$$h\left(\mathfrak{o}\left(f_{2,l}^2\right)\otimes_{f_{2,l}^2}\mathfrak{t}\left(f_{2,l}^2\right)\right)=0,\text{ for } l\in\{1,\cdots,4\},$$

$$h\left(e_{n+1}\otimes_{f_{2,5}^2} e_{n+1}\right)=c_4e_{n+1},$$

$$h\left(e_{n+2}\otimes_{f_{2,6}^2} e_{n+2}\right)=c_4e_{n+2} \text{ and}$$

$$\begin{aligned}
&h\left(\mathfrak{o}\left(f_{3,j}^2\right)\otimes_{f_{3,j}^2}\mathfrak{t}\left(f_{3,j}^2\right)\right)\\
&=-c_4\beta_j-c_6\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_j,\\
&\text{for } j\in\{2,\cdots,n-1\}
\end{aligned}$$

for some $c_2,c_4,c_5,c_6\in K$.

Proposition 5.3 *For* $\Lambda=\Gamma^*(n)$, *we have* $\dim\operatorname{Ker}d_3=4$.

Therefore

$$\dim HH^2(\Lambda)=\dim\operatorname{Ker}d_3-\dim\operatorname{Im}d_2=4-2=2$$

and a basis is given by the maps η_1 and η_2 where η_1 is given by

$$\begin{aligned}
&e_1\otimes_{f_{1,2}^2} e_1\mapsto e_1,\\
&e_{n+1}\otimes_{f_{2,5}^2} e_{n+1}\mapsto e_{n+1},\\
&e_{n+2}\otimes_{f_{2,6}^2} e_{n+2}\mapsto e_{n+2},\\
&\mathfrak{o}\left(f_{3,j}^2\right)\otimes_{f_{3,j}^2}\mathfrak{t}\left(f_{3,j}^2\right)\mapsto-\beta_j,\text{ for } j\in\{2,\cdots,n-1\},\\
&\text{else}\mapsto 0,
\end{aligned}$$

η_2 is given by

$$\begin{aligned}
&e_1\otimes_{f_{1,2}^2} e_1\mapsto\beta_1\beta_2\cdots\beta_n,\\
&\mathfrak{o}\left(f_{3,j}^2\right)\otimes_{f_{3,j}^2}\mathfrak{t}\left(f_{3,j}^2\right)\mapsto-\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_j,\\
&\qquad\text{for } j\in\{2,\cdots,n-1\},\\
&\text{else}\mapsto 0.
\end{aligned}$$

From Proposition 5.2 and Proposition 5.3 we get the main result of this section.

Theorem 5.4 For $\Lambda=\Gamma^*(n)$ with $n\ge 1$ we have $\dim HH^2(\Lambda)=2$.

To connect this with deformations we use a similar discussion as Section 4. We introduce the parameter t and define the algebra Λ_{η_2} to be the algebra KQ/I_{η_2} where I_{η_2} is the ideal generated by the following elements:

1) $f_{1,1}^2$,
2) $f_{1,2}^2-t\beta_1\beta_2\cdots\beta_n$,
3) $f_{2,1}^2, f_{2,2}^2, f_{2,3}^2, f_{2,4}^2, f_{2,5}^2, f_{2,6}^2$,
4) $f_{2,1}^2, f_{2,2}^2, f_{2,3}^2, f_{2,4}^2, f_{2,5}^2, f_{2,6}^2$, for $j\in\{2,\cdots,n-1\}$.

We can show that $\dim\Lambda_{\eta_2}\ne\dim\Lambda$. Hence this algebra has no non-trivial deformation.

From Theorem 4.9 and Theorem 5.4 we have now found $HH^2(\Lambda)$ for all standard one-parametric but not weakly symmetric self-injective algebras of tame representation type.

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