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Chaotic Properties on Time Varying Map and Its Set Valued Extension

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ABSTRACT

Every autonomous dynamical system (X, f) induces a set-valued dynamical system $(\kappa(X), \overline{f})$ on the space of compact subsets of *X*. In this paper we have investigated some chaotic relations between a nonautonomous dynamical system and its set valued extension.

1. Introduction

There are two main types of dynamical systems: differential equations and iterated maps(also called difference equation). Differential equation describes the continuous time evaluation of the system, whereas difference equation describes the discrete time evaluation of the system. Iterated maps are the tools for analyzing periodic and chaotic solution of differential equation. Again, there are two types of difference equation: autonomous and nonautonomous, called as autonomous and nonautonomous discrete dynamical system. During the past few decades, there has been increasing interest in the study of discrete dynamical system (or difference equation) of the form,

$$x_{n+1} = f_n(x_n), \ n \geq 0, \tag{1}$$

where $f_n : X_n \to X_{n+1}$ is a map and (X_n, d_n) is a metric space or all $n \in N$. In particular, if $f_n = f$, $X_n = X$ and $d_n = d$ for all $n \in N$, then (1) reduces to,

$$x_{n+1} = f(x_n), \ n = 0, 1, 2, \cdots. \tag{2}$$

where $f : X \to X$ is a map.

The system (1) is called a nonautonomous discrete dynamical system, which is governed by the sequence of maps $\{f_n\}$. While the dynamical system (2), governed by the single map *f*, called an autonomous discrete dynamical system.

Chaos of system (2) or a time-invariant map

$f : X \to X$ has been discussed thoroughly in [1-5]. However, evolutions of certain physical, biological, and economical complex systems are necessarily described by a nonautonomous systems whose dimensions vary with time in some cases. In [6], Chen and Tian study the chaos of system (1) (with $X_n = X$ and $d_n = d$ for all $n \in N$) by introducing several new concepts. In 2009, Chen and Shi in [7] introduced some basic concepts, including chaos in the sense of Devaney, Wiggins and in a strong sense of Li-Yorke and studied their behavior under topological conjugacy. In [8], the author introduced a new type of subsystem of a nonautonomous discrete dynamical system, which is a partial compositions of a given sequence of maps(from which nonautonomous dynamical system is generated), and the concept of chaos in the strong sense of Wiggins is introduced. Also, some Li-Yorke and Wiggins chaotic connections (in the strong sense) between a given dynamical system and its subsystems have been studied.

The main task to investigate the dynamical system (X, f) is, how the points of *X* move under the iterate of *f*. Nevertheless, in many fields or problems such as biological species, demography, numerical simulation and attractors, etc, it is not enough to know only how the points of X move, one should know how the subsets of X move. So it is also necessary to study the set valued dynamical system $(\kappa(X), \overline{f})$ associated to the system (X, f), where $f : X \to X$ is a continuous map on a compact metric space (X, d) and $\overline{f}$ is a natural

extension of f on $\kappa(X)$ (collection of all non-empty compact subsets of X). Many papers [9-13] has been devoted to the study of chaotic relation between autonomous system (X, f) and its set valued extension $(\kappa(X), \bar{f})$. Normally, we come across so many natural phenomena which explicitly depend on time where the starting point is just as important as the time elapsed. We would like to know what would be the collective dynamics of such system in relation to the individual dynamics. This paper is an endevour to investigate the relations between the individual dynamics and collective dynamics for time dependent discrete systems.

So, here we have considered the set-valued extension of a nonautonomous system (1), as

$$K_{n+1} = \bar{f}_n(K_n), n \geq 0, \qquad (3)$$

where $\bar{f}_n : \kappa(X_n) \to \kappa(X_{n+1})$. It is clear that this system is governed by the sequence of maps $\{\bar{f}_n\}$ on set-valued extension of X_n, i.e. $\kappa(X_n)$. So far, there is no investigation has been done on the chaotic relationship between systems (1) and (3).

In present paper, we investigate the relation between $F = \{f_n\}$ and $\bar{F} = \{\bar{f}_n\}$ in the related chaotic dynamical properties such as transitivity, sensitivity, dense set of periodic points, weak mixing, mixing and topological exactness, with $X_n = X$ and $d_n = d$ for all $n \in N$.

2. Basic Definition and Notation

Let $f : X \to X$ be a continuous self map on a compact metric space (X, d).

Definition 2.1 *A map f is said to be **transitive** if for every pair of open sets U and V, there exists an $n_0 \in N$ such that $f^{n_0}(U) \bigcap V \neq \varnothing$.*

Definition 2.2 *A point $x \in X$ is said to be **periodic** if there exists $p \in \mathbb{N}$ such that $f^p(x) = x$. The least such p is called the period(prime period) of the point x.*

Definition 2.3 *f is said to have **sensitive dependence on initial conditions** (**sensitive**), if there exist $\delta > 0$ (sensitivity constant) such that for every point $x \in X$ and for each $\epsilon > 0$ there is $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) \geq \delta$.*

A continuous map *f* is chaotic in the sense of Devaney (**Devaney chaotic**) if:

1) *f* is topological transitive;

2) *f* has dense set of periodic points;

3) *f* has sensitive dependence on initial conditions.

It is known that condition (1) together with (2) implies (3) on compact metric spaces, see [3]. Further, for interval maps it is known that transitivity alone implies chaos [2].

Definition 2.4 *Map f is **weakly mixing** if for any pair of non-empty open sets U, V in X, there exists a positive integer k, such that $f^k(U) \bigcap V \neq \varnothing$ and $f^k(V) \bigcap V \neq \varnothing$.*

Definition 2.5 *f is **topologically mixing**(**mixing**) if for any pair of non-empty open sets U, V in X, there exists an integer $m_0 > 0$ such that $f^n(U) \bigcap V \neq \varnothing$, for all $n > m_0$.*

Definition 2.6 *A map f is **topological exact or locally eventually onto**(**leo**) if for any non empty open set $U \subset X$, there exist an integer $n > 0$ such that $f^n(U) = X$.*

3. Set-Valued Extension

Define the hyperspace $\kappa(X)$ as the collection of all the non-empty compact subsets of X. If $A \in \kappa(X)$ we define the ϵ-neighbourhood of A as the set

$$N(A, \epsilon) = \{x \in X : d(x, A) < \epsilon\},$$

where $d(x, A) = \inf_{a \in A} d(x, a)$.

The Hausdorff metric on $\kappa(X)$ is defined as

$$\mathcal{H}(A, B) = \inf\{\epsilon > 0 : A \subseteq N(B, \epsilon) \text{ and } B \subseteq N(A, \epsilon)\}.$$

It is well known that $(\kappa(X), \mathcal{H})$ is a compact metric space, if (X, d) is a compact metric space.

For any finite collection $U_1, U_2, \cdots, U_n$ of non empty subsets of X, define

$$\langle U_1, U_2, \cdots, U_n \rangle$$
$$= \left\{ A \in \kappa(X) : A \subset \bigcup_{i=1}^{n} U_i, A \bigcap U_i \neq \varnothing \right\}. \qquad (4)$$

Collection of these kind of sets form a base for the topology on $\kappa(X)$, called Vietoris topology (also called hit and miss topology [9] given by Leopold Vietoris). It is worth noting that if X is a compact metric space then Hausdorff topology coincide with Vietoris topology.

Let A be a subset of *X*. Define the extension of *A* to $\kappa(X)$ as, $e(A) = \{E \in \kappa(X) / E \subset A\}$.

Remark 3.1 [12] *It is clear that $e(A) = \varnothing$ if and only if $A = \varnothing$.*

Result 3.2 [12] *Let A be a non-empty open subset of X. Then $e(A)$ is a non-empty open subset of $\kappa(X)$.*

Result 3.3 [12] *Let A and B be two non-empty subset of X and $f : X \to X$ is continuous. Then,*

1) $e(A \bigcap B) = e(A) \bigcap e(B)$.

2) $\bar{f}(e(A)) \subseteq e(f(A))$.

3) $\bar{f}^p = \overline{f^p}$, for every $p \in N$.

It has been proved that the collection $\{e(U) : U$ open in $X\}$ generate a topology on $\kappa(X)$, called w^e-topology(also called Upper Vietoris topology). So if $\mathcal{U}$ is any non-empty open set in $\kappa(X)$ (with w^e-topology) then by the above result, there exists non-empty

open subsets U_i of X such that, $\mathcal{U} = \bigcup_{i=1}^{r} e(U_i)$.

4. Dynamical Properties on Time-Varying Map

Let (X, d) be a metric space and $f_n : X \rightarrow X$ be a sequence of maps, $n = 1, 2, \cdots$. For a point $x_0 \in X$, define a sequence as follows:

$$x_1 = f_1(x_0), x_2 = f_2(x_1), \cdots, x_{k+1} = f_{k+1}(x_k), \cdots,$$
$$k = 0, 1, 2, \cdots,$$

then the sequence $O(x_0) = \{x_k\}_{k=0}^{\infty}$ is said to be an orbit of the sequence $F = \{f_n\}_{n=1}^{\infty}$ of the maps (starting at x_0) or an orbit of F in the iterative way.

In addition, for any point $x_0 \in X$, define a sequence as follows:

$$x_1 = f_1(x_0), x_2 = f_2(x_0), \cdots, x_{n+1} = f_{n+1}(x_0), \cdots,$$
$$n = 0, 1, 2, \cdots,$$

then, the sequence $O(x_0) = \{x_k\}_{k=0}^{\infty}$ is said to be an orbit of the sequence $F = \{f_n\}_{n=1}^{\infty}$ of the maps (starting at x_0) or an orbit of F in the successive way.

Now on for convenience, for any sequence $F = \{f_k\}_{k=0}^{\infty}$ of maps defined on a metric space (X, d), denote maps $F_k : X \rightarrow X$ for any $k \in \mathbb{N}$, by

$$F_k(x) = f_k\left(f_{k-1}\left(\cdots\left(f_1(x)\right)\cdots\right)\right)$$
$$= f_k \circ \cdots \circ f_2 \circ f_1(x), \text{ for } x \in X,$$

and $F_0(x) = x$, for any $x \in X$. It is obvious that any orbit $O(x_0) = \{x_k\}_{k=0}^{\infty}$ of $F = \{f_n\}_{n=1}^{\infty}$ in the iterative way is an orbit $O(x_0) = \{x_k\}_{k=0}^{\infty}$ of $\hat{F} = \{F_n\}_{n=1}^{\infty}$ in the successive way.

Definition 4.1 $F = \{f_n\}_{n=1}^{\infty}$ is said to be **transitive in** iterative(*or successive*) *way if for every open set* U *of* X, *there exists a positive integer* n_0 *such that* $F_n(U) \bigcap V \neq \varnothing$ (*or* $f_n(U) \bigcap V \neq \varnothing$).

Definition 4.2 *Let* $x \in X$, x *is said to be **periodic in** iterative(*or successive*) for* $F = \{f_n\}_{n=1}^{\infty}$, *if there exists a integer* $m > 0$ *such that* $F_{m+i}(x) = F_i(x)$ (*or* $f_{m+i}(x) = f_i(x)$) *and for any interger* $k < m$, $F_{m+i}(x) \neq F_i(x)$ (*or* $f_{m+i}(x) \neq f_i(x)$).

Definition 4.3 *If there exists a constant* $\delta > 0$ *such that for any point* $x \in X$ *and any* $\epsilon > 0$, *the ball* $B(x, \epsilon)$ *contains a point* y *and there exists a positive integer* n_0 *such that* $d(F_{n_0}(x), F_{n_0}(y)) > \delta$ *or* $d(f_{n_0}(x), f_{n_0}(y)) > \delta$, *then the sequence* $F = \{f_n\}_{n=1}^{\infty}$ *of maps is said to be **sensitive dependence on initial***

condition (*on* X) *in the iterative or successive way.*

The sequence $F = \{f_n\}_{n=1}^{\infty}$ of maps is said to be chaotic (on X) in the iterative(or successive) way, in the sense of Devaney, if

1) F is transitive (on X) in the iterative (or successive) way.

2) The set of periodic points of F is dense in X in iterative (or successive) way.

3) F has sensitive dependence on initial condition in the iterative(or successive) way.

Definition 4.4 *If for any pair of non-empty open sets* U, V *in* X, *there exists a positive integer* k, *such that* $F_k(U) \bigcap V \neq \varnothing$ *and* $F_k(V) \bigcap V \neq \varnothing$ ($f_k(U) \bigcap V \neq \varnothing$ *and* $f_k(V) \bigcap V \neq \varnothing$), *then the sequence of maps* $F = \{f_n\}_{n=1}^{\infty}$ *is said to be **weakly mixing** in iterative* (*successive*) *way.*

Definition 4.5 *If for any non-empty open sets* U *and* V *in* X, *there exists a positive integer* n_0 *such that,*

$$F_n(U) \bigcap V \neq \varnothing \text{ or } f_n(U) \bigcap V \neq \varnothing, \text{ for all } n \geq n_0.$$

Then the sequence $F = \{f_n\}_{n=1}^{\infty}$ of maps is said to be **mixing** (on X) in the iterative or successive way.

Definition 4.6 *If for any non-empty open set* U *in* X, *there exists a positive integer* n_0 *such that,*

$$F_{n_0}(U) = X \text{ or } f_{n_0}(U) = X,$$

then the sequence $F = \{f_n\}_{n=1}^{\infty}$ of maps is said to be **topologically exact** (on X) in the iterative or successive way.

It easy to see that that chaotic properties defined for a autonomous system (2) which is governed by the single map f on a metric space X, is a particular case for the chaotic properties defined for nonautonomous system (1) in successive way.

5. Main Results

Consider a compact metric space (X, d) and its set-valued extension $(\kappa(X), \mathcal{H})$. Let $F = \{f_n\}_{n=1}^{\infty}$ and $\bar{F} = \{\bar{f}_n\}_{n=1}^{\infty}$ be the sequence of continuous maps representing the nonautonomous systems $x_{n+1} = f_n(x_n)$ and $K_{n+1} = \bar{f}_n(K_n)$ respectively, where $f_n : X \rightarrow X$ and $\bar{f}_n : \kappa(X) \rightarrow \kappa(X)$ for all $n \geq 0$. Here we will take w^e-topology on $\kappa(X)$ for proving all our results and examples.

Theorem 5.1 *Sequence of maps* $F = \{f_n\}_{n=1}^{\infty}$ *is transitivity in iterative* (*or successive*) *way on X iff* $\bar{F} = \{\bar{f}_n\}_{n=1}^{\infty}$ *is transitive in iterative* (*or successive*) *way*

on $\kappa(X)$.

Proof. We will do the proof for iterative way, for successive way it would be similar.

Take a pair of non-empty open sets $\mathcal{U} = \bigcup_{i=1}^{r} e(U_i)$, $\mathcal{V} = \bigcup_{j=1}^{s} e(V_j)$ in $\kappa(X)$, where U_i, V_j are open in X for $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Fix U_{i_0} and V_{j_0}. Since F is transitive in iterative way, we can find a $z \in U_{i_0}$ and $k \in N$ such that $F_k(z) \in V_{j_0}$, implies $\overline{F}_k(\{z\}) \in e(V_{j_0})$, where $\{z\} \in e(U_{j_0})$. Consequently, $\overline{F}_k(\mathcal{U}) \cap \mathcal{V} \neq \varnothing$.

Conversely, take a pair of non-empty open set U and V in X. Since X is compact, so for U open in X we can find a non-empty open set $U_0 \subset X$, such that $\overline{U}_0 \subset U$. Clearly, $e(U_0)$ and $e(V)$ will be open and non-empty in $\kappa(X)$, there exist an positive integer $k \in \mathbb{N}$ such that $\overline{F}_k(e(U_0)) \cap e(V) \neq \varnothing$, therefore,

$$e\left(\overline{F}_k(\overline{U}_0) \cap V\right) = e\left(\overline{F}_k(\overline{U}_0)\right) \cap e(V)$$
$$\supseteq \overline{F}_k(e(U_0)) \cap e(V) \neq \varnothing$$

Hence $F_k(U) \cap V \neq \varnothing$. $\square$

Example 5.2 *Consider the sequence of maps* $F = \{f_n\}_{n=1}^{\infty}$ *on the unit circle* S^1*, defined as*

$$f_n(\theta) = \theta + 2\pi \frac{\alpha}{n}, \text{ for } n \geq 0,$$

where $\alpha < 1$ *is an irrational.*

Then,

$$F_n(\theta) = f_n \circ f_{n-1} \circ \cdots \circ f_1(\theta) = \theta + 2\pi\alpha\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$ It

is not difficult to prove that F is transitive in iterative way but not in successive way. Hence the sequence $\overline{F} = \{\overline{f}_n\}_{n=1}^{\infty}$ on $\kappa(S^1)$ is transitive in iterative way but not in successive way (by Theorem 5.1).

Theorem 5.3 *The sequence of maps* $F = \{f_n\}_{n=1}^{\infty}$ *is topologically mixing in iterative (or successive) way on X iff* $\overline{F} = \{\overline{f}_n\}_{n=1}^{\infty}$ *is topologically mixing in iterative (or successive) way on* $\kappa(X)$.

Proof. The proof is similar to proof done for transitivity, with slight modifications.

Theorem 5.4 *Let* $\overline{F} = \{\overline{f}_n\}_{n=1}^{\infty}$ *and* $F = \{f_n\}_{n=1}^{\infty}$ *be the sequences of continuous maps on $\kappa(X)$ and X respectively. If $\overline{F}$ is sensitive in iterative (or successive) way, then F is sensitive in iterative (or successive) way.*

Proof. Let $\overline{F} = \{\overline{f}_n\}_{n=1}^{\infty}$ be sensitive in iterative way, with sensitive constant λ. Let $x \in X$ and $\epsilon > 0$, then as $\{x\} \in \kappa(X)$ there exists $A \in B_{\mathcal{H}}(\{x\}, \epsilon)$ and

$n_0 \in N$ such that

$$\mathcal{H}\left(\overline{F}_{n_0}(\{x\}), \overline{F}_{n_0}(A)\right) = \sup_{y \in A} d\left(F_{n_0}(x), F_{n_0}(y)\right) \geq \lambda.$$

Since A is compact and F_{n_0} is continuous, we can find a $y_0 \in A$ such that $d\left(F_{n_0}(x), F_{n_0}(y_0)\right) \geq \lambda$. Clearly $A \in B_{\mathcal{H}}(\{x\}, \epsilon)$ implies $A \in B_d(x, \epsilon)$, which implies $d(x, y_0) < \epsilon$. Hence F is sensitive in iterative way on X.

Similarty, we can prove it for successive way. $\square$

Theorem 5.5 *If* $F = \{f_n\}_{n=1}^{\infty}$ *has dense set of periodic points in iterative (or successive) way on X, then* $\overline{F} = \{\overline{f}_n\}_{n=1}^{\infty}$ *has dense set of periodic points in iterative (or successive) way on* $\kappa(X)$.

Proof. Let F has dense set of periodic points in successive way. Take any open set $\mathcal{U}$ in $\kappa(X)$, then $\mathcal{U} = \bigcup_{i=1}^{r} e(U_i)$, where U_i open in X. There exists $x_i \in U_i$ and a positive integer n_i correspondingly, such that $f_{n_i+k}(x_i) = f_k(x_i)$ for $i = 1, 2, \cdots, r$, $k \in N$. Take $m = lcm\{n_i \mid 1 \leq i \leq r\}$ and $G = \{x_i, x_2, \cdots, x_r\}$, then clearly $G \in \mathcal{U}$ and $\overline{f}_{m+k}(G) = \overline{f}_k(G)$, for all $k \in N$. Therefore, $\overline{F}$ has dense set of periodic points on $(\kappa(X), \mathcal{H})$ in successive way.

Proof in iterative way can be done likewise. $\square$

Here we give an example where the nonautonomous dynamical system don't have any periodic points in iterative (and successive) way but its set-valued extension has dense set of periodic points in successive way.

Example 5.6 *Consider the sequence space,* $\Sigma = \{(s_0, s_1, s_2, \cdots) \mid s_i = 0 \text{ or } 1\}$ *on two symbols. Let* $s = (s_0, s_1, s_2, \cdots)$, $t = (t_0, t_1, t_2, \cdots)$ *be any two elements of* Σ. *Define distance between them as* $d(s,t) = \Sigma_{n \geq 0} \cdot \frac{s_n - t_n}{2^n}$. *It has been proved that* (Σ, d) *is a metric space.*

Define a binary composition of addition on elements of Σ as

$$s + t = (q_0, q_1, q_3, \cdots),$$

where $q_i = s_i + t_i$ if $s_i + t_i < 2$, else $q_i = s_i + t_i - 2$ and carry 1 to next position. With this composition Σ is a compact topological group.

Consider a sequence of map $P = \{\sigma_n\}_{n=1}^{\infty}$ on Σ, defined as

$$\sigma_n(s) = t + \chi_n,$$

where $\chi_n = (\alpha_1, \alpha_2, \alpha_3, \cdots)$, $\alpha_i = 1$ if $i = n$, else 0.
$P_n(s) = \sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_1(s) = s + \chi_1 + \chi_2 + \cdots + \chi_n$, for all $n \in N$.

It can be seen that P has no periodic points in iterative and successive way. Consider an open set $e(U) \in \kappa(\Sigma)$, where U is open in Σ. Since the cylinder set,

$$[z_1, z_2, \cdots, z_t] = \left\{ (x_n)_{n=1}^{\infty} \in X : x_1 = z_1, \cdots, x_t = z_t \right\},$$

forms the basis for the topology on Σ, there exist $V = [z_1, z_2, \cdots, z_t] \subset U$, which is compact in Σ, hence $V \in e(U) \bigcap \kappa(\Sigma)$. We can find a $k \in N$ such that $\sigma_k(V) = \bar{\sigma}_k(V) = V$. Hence, $\kappa(\Sigma)$ has dense set of periodic points in successive way.

Theorem 5.7 *The sequence of maps* $F = \{f_n\}_{n=1}^{\infty}$ *is weakly mixing in iterative* (*or successive*) *way iff* $\bar{F} = \{\bar{f}_n\}_{n=1}^{\infty}$ *is weakly mixing in iterative* (*or successive*) *way.*

Proof. Let F is weakly mixing in successive way on X. Consider a pair of non-empty open sets $\mathcal{U} = \bigcup_{i=1}^{r} e(U_i)$, $\mathcal{V} = \bigcup_{j=1}^{s} e(V_j)$ in $\kappa(X)$, where U_i, V_j are open in X, for $i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, s$. Fix U_{i_0} and V_{j_0}, therefore there exist an positive integer n_0 such that $f_{n_0}(U_{i_0}) \bigcap V_{j_0} \neq \varnothing$ and $f_{n_0}(V_{i_0}) \bigcap V_{j_0} \neq \varnothing$, there exists $x \in U_{i_0}$ and $y \in V_{j_0}$ such that $f_{n_0}(x) \in V_{j_0}$ and $f_{n_0}(y) \in V_{j_0}$. So, $\{x\} \in \mathcal{U}$ and $\{y\} \in \mathcal{V}$, consequently implies $\bar{f}_{n_0}(\{x\}) \in \mathcal{V}$ and $\bar{f}_{n_0}(\{y\}) \in \mathcal{V}$.

Conversely, suppose that $\bar{F} = \{\bar{f}_n\}_{n=1}^{\infty}$ is weakly mixing in successive way. Take any pair of non-empty open sets U, V in X, then $e(U)$ and $e(V)$ will be open in $\kappa(X)$. We can find $n \in \mathbb{N}$ such that $\bar{f}_n(e(U)) \bigcap e(V) \neq \varnothing$ and $\bar{f}_n(e(V)) \bigcap e(V) \neq \varnothing$. Now

$$e(f_n(U) \bigcap V) = e(f_n(U)) \bigcap e(V)$$
$$\supseteq \bar{f}_n(e(U)) \bigcap e(V) \neq \varnothing$$

and

$$e(f_n(V) \bigcap V) = e(f_n(V)) \bigcap e(V)$$
$$\supseteq \bar{f}_n(e(V)) \bigcap e(V) \neq \varnothing$$

Hence $f_n(U) \bigcap V \neq \varnothing$ and $f_n(V) \bigcap V \neq \varnothing$.

The proof in iterative way can be done likewise. □

Theorem 5.8 *The sequence of maps* $F = \{f_n\}_{n=1}^{\infty}$ *is topologically exact in iterative* (*or successive*) *way on* X *iff* $\bar{F} = \{\bar{f}_n\}_{n=1}^{\infty}$ *is topologically exact in iterative* (*or successive*) *way on* $\kappa(X)$.

Proof. The proof is easy, hence omitted.

Example 5.9 *Consider* Z_2, *the cycle group with two elements and discrete topology. Binary operation of addition* ("+") *and subtraction* ("−") *is defined under modulo* 2. *Let* $X = \left\{ (x_n)_{n=1}^{\infty} : x_n \in Z_2 \right\}$. *It is well Known*

that X *is compact, perfect and has countable base containing clopen sets which can be chosen to consist of cylinder sets of the form*

$$[z_1, z_2, \cdots, z_t] = \left\{ (x_n)_{n=1}^{\infty} \in X : x_1 = z_1, \cdots, x_t = z_t \right\}$$

Define a sequence of maps $\{f_n\}_{n=1}^{\infty}$ on X, as $f_n\left(\{x_n\}_{n=1}^{\infty}\right) = \{y_n\}_{n=1}^{\infty}$, where

$$y_m = \begin{cases} x_{m+1} & \text{if } x_1 \neq x_{n+1} \\ 1 + x_{m+1} & \text{if } x_1 = x_{n+1} \end{cases} \quad \text{for all } m \in N.$$

It is clear that for every non-empty cylinder set $[z_1, z_2, \cdots, z_k]$,

$$F_k\left([z_1, z_2, \cdots, z_k]\right) = f_k \circ \cdots \circ f_2 \circ f_1\left([z_1, z_2, \cdots, z_k]\right) = X.$$

Therefore, F is topological exact in iterative way, clearly it can be seen that F in not topological exact in successive way on X.

Hence, $F = (f_n)_{n=1}^{\infty}$ is mixing, weakly mixing, transitive in iterative way on X and so is $\bar{F}$ on $\kappa(X)$. Also, in every cylinder set we can find a sequence of repetitive block of symbols, which are periodic in successive and iterative way under F. It is not difficult to see that F is sensitive with sensitivity constant $\frac{1}{2}$, in iterative ways.

It is interesting to see that for any open set $e(U)$, there exists cylinder sets $S = [z_1, z_2, \cdots, z_r]$ and $T = [z_1, z_2, \cdots, z_r, z_{r+1}, \cdots, z_{r+s}]$ in $U \bigcap \kappa(X)$, where $r, s \in N$. We can always find a positive integer k such that $\mathcal{H}\left(\bar{F}_k(S), \bar{F}_k(T)\right) \geq \frac{1}{2}$, hence $\bar{F}$ is sensitive on $\kappa(X)$ in iterative way.

6. Conclusion

In this article we have studied some chaotic properties on time-varying map (*i.e.* a sequence of time-invariant maps). We have investigated the relation between $F = \{f_n\}$ and $\bar{F} = \{\bar{f}_n\}$ defined on X and $\kappa(X)$ respectively, in the related chaotic dynamical properties such as transitivity, sensitivity, dense set of periodic points, weak mixing, mixing and topological exactness. In this endeavour, we proved that, F is transitive (weak mixing, mixing and leo, respectively) if and only if $\bar{F}$ is so in iterative (successive) way. Also an example is given to prove that denseness of periodic points for $\bar{F}$ doesn't imply the same for F, in successive way. The question which is still open is, does sensitivity of F implies sensitivity for $\bar{F}$, which we think may not be possible in general, as for autonomous map sensitivity on

original dynamical system doesn't imply sensitivity on hyperspace dynamical system. These kinds of investigations would be useful in understanding the relationship between the dynamics of individual movement and the dynamics of collective movements for the time-varying map (*i.e.* a sequence of time-invariant maps).

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