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On a Unification of Generalized Mittag-Leffler Function and Family of Bessel Functions

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ABSTRACT

In the present work, a unification of certain functions of mathematical physics is proposed and its properties are studied. The proposed function unifies Lommel function, Struve function, the Bessel-Maitland function and its generalization, Dotsenko function, generalized Mittag-Leffler function etc. The properties include absolute and uniform convergence, differential recurrence relation, integral representations in the form of Euler-Beta transform, Mellin-Barnes transform, Laplace transform and Whittaker transform. The special cases namely the generalized hypergeometric function, generalized Laguerre polynomial, Fox H-function etc. are also obtained.

Keywords: Generalized Mittag-Leffler Function; Recurrence Relation; Wiman's Function

1. Introduction

In the present work, we propose an extension of a generalization of the Mittag-Leffler function due to A. K. Shukla and J. C. Prajapati [1], defined as

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)n!}, \tag{1.1}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$; $\operatorname{Re}(\alpha,\operatorname{Re}(\beta,\gamma))>0$ and $q\in(0,1)\cup\mathbb{N}$. This is an entire function of order $(\operatorname{Re}\alpha-q+1)^{-1}$ if $\operatorname{Re}\alpha>q-1$ and absolutely convergent in $\{|z|<R,R<1\}$ if $\operatorname{Re}\alpha=q-1$. In fact (1.1) contains the $E_\alpha(z)$-Mittag-Leffler function [2], $E_{\alpha,\beta}(z)$-the generalized Mittag-Leffler function [3] and the function $E_{\alpha,\beta}^{\gamma}(z)$ due to Prabhakar [4].

Gorenflo *et al.* [5], Saigo and Kilbas [6] studied several interesting properties of these functions.

Another generalization of Mittag-leffler function due to T. O. Salim [7], given by

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)(\delta)_n}, \tag{1.1'}$$

where $z,\alpha,\beta,\gamma,\delta\in\mathbb{C}$ and

$$\min\{\operatorname{Re}(\alpha,\beta,\delta)\}>0.$$

We state below the extended version in the form:

$$E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}, \tag{1.2}$$

where $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$, $\operatorname{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$, $\delta,\mu,p>0$. The function defined by (1.2) reduces to the one in (1.1) and (1.1') if $\rho=1$, $p=1$, $r=0$, $s=1$ and $\delta=1$, $\rho=\delta$, $p=1$, $r=0$, $s=1$ respectively.

It is noteworthy that the function in (1.2), besides containing the generalizations of the Mittag-Leffler function, also includes certain functions belonging to the family of Bessel function. To see this, take $s=0$, $r=0$, $\rho=1$, $p=1$, $\alpha=1$, $\beta=\nu+1$, and replaced z by $\frac{-z^2}{4}$ in (1.2), then we find the well known Bessel function [8]:

$$J_\nu(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{z}{2}\right)^{\nu+2n}.$$

When $s=0$, $r=0$, $\alpha=\mu$, $\beta=\nu+1$, and z is replaced by $(-z)$ then we get the Bessel Maitland Function [8] given by $J_\nu^\mu(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\nu+n\mu+1)n!}$. For $s=1$, $r=1$, $\rho=1$, $p=1$, $\alpha=\mu$, $\beta=\nu+\lambda+1$, $\mu=1$, $\lambda=\lambda+1$, $\gamma=1$, $\delta=1$, and z is replaced by $\frac{-z^2}{4}$, we obtain the Generalized Bessel Maitland function [8]:

$$J_{\nu,\lambda}^{\mu}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\nu+n\mu+\lambda+1)\Gamma(n+\lambda+1)}\left(\frac{z}{2}\right)^{\nu+2\lambda+2n}.$$

The Dotsenko Function [8]:

$$_2R_1(a,b;c,\omega;\mu;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+n\frac{\omega}{\mu}\right)}{\Gamma\left(c+n\frac{\omega}{\mu}\right)}\frac{z^n}{n!}$$

occurs by substituting $s=1$, $r=-1$, $\rho=1$, $p=1$, $\alpha=c$, $\beta=\frac{\omega}{\mu}$, $\mu=\frac{\omega}{\mu}$, $\lambda=b$, $\gamma=a$, $\delta=1$ in (1.2).

The Lommel Function defined by [9]:

$$S_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}{}_1F_2\left(\begin{matrix}1 & -\frac{z^2}{4}\\ 1/2(\mu-\nu+3), & 1/2(\mu+\nu+3),\end{matrix}\right)$$

is the special case $s=1$, $r=1$, $\rho=1$, $p=1$, $\alpha=1$, $\beta=\frac{1}{2}(\mu-\nu+3)$, $\mu=1$, $\lambda=\frac{1}{2}(\mu+\nu+3)$, $\gamma=1$, $\delta=1$, and z is replaced by $\frac{-z^2}{4}$ of (1.2). On making substitutions $s=1$, $r=1$, $\rho=1$, $p=1$, $\alpha=1$, $\beta=3/2$, $\mu=1$, $\lambda=3/2+\nu$, $\gamma=1$, $\delta=1$, and $z=\mp z^2/4$ in (1.2), provides us respectively, the Struve Function $H_\nu(z)$ [9] given by

$$H_\nu(z)=\frac{(z/2)^{\nu+1}}{\Gamma(3/2)\Gamma(3/2+\nu)}{}_1F_2\left(\begin{matrix}1 & -\frac{z^2}{4}\\ 3/2 & 3/2+\nu\end{matrix}\right).$$

and the Modified Struve Function [9]:

$$L_\nu(z)=\frac{(z/2)^{\nu+1}}{\Gamma(3/2)\Gamma(3/2+\nu)}{}_1F_2\left(\begin{matrix}1 & -\frac{z^2}{4}\\ 3/2 & 3/2+\nu\end{matrix}\right).$$

In what follows, we shall use the following definitions and formulas. Euler (Beta) transform [10]:

$$B\{f(z):a,b\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)\mathrm{d}z. \quad (1.3)$$

Laplace transform [10]:

$$\mathcal{L}\{f(z)\}=\int_0^\infty \mathrm{e}^{-sz}f(z)\mathrm{d}z. \quad (1.4)$$

Mellin-Barnes transform [10]:

$$M[f(z);z]=\int_0^\infty z^{s-1}f(z)\mathrm{d}z=f^*(s),\ \mathrm{Re}(s)>0, \quad (1.5)$$

then

$$f(z)=M^{-1}[f^*(s);x]=\frac{1}{2\pi i}\int f^*(s)x^{-s}\mathrm{d}s. \quad (1.6)$$

Incomplete Gamma function [11]:

$$\gamma(\alpha,z)=\int_0^z \mathrm{e}^{-t}t^{\alpha-1}\mathrm{d}t,\ \mathrm{Re}(\alpha)>0. \quad (1.7)$$

The generalized hypergeometric function is denoted and defined by [11]

$$_pF_q\left[\begin{matrix}a_1,a_2,\cdots,a_p;\\ b_1,b_2,\cdots,b_q;\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}, \quad (1.8)$$

where $b_1,b_2,\cdots,b_q$ are neither zero nor negative integers, and

$$(\lambda)_n=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1).$$

The series is convergent for 1) $|z|<\infty$ if $p\le q$, 2) $|z|<1$ if $p=q+1$.

Wright generalized hypergeometric function [12]:

$$_p\psi_q\left(\begin{matrix}(\alpha_1,A_1),\cdots,(\alpha_p,A_p);\\ (\beta_1,B_1),\cdots,(\beta_q,B_q);\end{matrix}z\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha i+A_i n)}{\prod_{j=1}^{q}\Gamma(\beta j+B_j n)}\frac{z^n}{n!} \quad (1.9)$$

Laguerre polynomial [12]:

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}{}_1F_1\left[\begin{matrix}-n;x\\ 1+\alpha;\end{matrix}\right]. \quad (1.10)$$

2. Main Results

In this section, we prove the following results for the function defined in (1.2).

Theorem 2.1. The series represented by the function $E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)$ converges absolutely for $|z|<n^{1/p(\mathrm{Re}(\mu)r+\Re(\alpha)p-\mathrm{Re}(\delta)s+p)}$.

Proof: Consider,

$$E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}.$$

Take

$$u_n=\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}$$

then

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \times \frac{\Gamma\left(\alpha\left(p(n+1)+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu(n+1)}\right]^r (\rho)_{p(n+1)}}{\left[(\gamma)_{\delta(n+1)}\right]^s z^{p(n+1)+\rho-1}}\right|$$

$$= \left|\frac{\left[\Gamma(\gamma+\delta n)\right]^s \left[\Gamma(\lambda)\right]^r \Gamma(\rho) z^{pn+\rho-1}}{\left[\Gamma(\gamma)\right]^s \Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r \Gamma(\rho+pn)}\right.$$

$$\left. \times \frac{\left[\Gamma(\gamma)\right]^s \Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+\alpha p\right)\left[\Gamma(\lambda+\mu n+\mu)\right]^r \Gamma(\rho+pn+p)}{\left[\Gamma(\gamma+\delta n+\delta)\right]^s \left[\Gamma(\lambda)\right]^r \Gamma(\rho) z^{p(n+1)+\rho-1}}\right|$$

$$= \left|\frac{\left[\Gamma(\gamma+\delta n)\right]^s \Gamma(\alpha n+\beta+\alpha p)\left[\Gamma(\lambda+\mu n+\mu)\right]^r \Gamma(\rho+pn+p)}{\left[\Gamma(\gamma+\delta n+\delta)\right]^s \Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r \Gamma(\rho+pn) z^p}\right|$$

$$= \left|\frac{\left[(\lambda+\mu n)(\lambda+\mu n+1)(\lambda+\mu n+2)\cdots(\lambda+\mu n+\mu-1)\right]^r}{\left[(\gamma+\delta n)(\gamma+\delta n+1)(\gamma+\delta n+2)\cdots(\gamma+\delta n+\delta-1)\right]^s z^p}\right.$$

$$\times(\alpha pn+\alpha\rho-\alpha+\beta)(\alpha pn+\alpha\rho-\alpha+\beta+1)\cdots(\alpha pn+\alpha\rho-\alpha+\beta+\alpha-1)$$

$$\left.\times(\rho+pn)(\rho+pn+1)(\rho+pn+2)\cdots(\rho+pn+p-1)\right|$$

$$= \left|\frac{n^{\mu r}\left[\left(\frac{\lambda}{n}+\mu\right)\left(\frac{\lambda+1}{n}+\mu\right)\left(\frac{\lambda+2}{n}+\mu\right)\cdots\left(\frac{\lambda+\mu-1}{n}+\mu\right)\right]^r}{n^{\delta s}\left[\left(\frac{\gamma}{n}+\delta\right)\left(\frac{\gamma+1}{n}+\delta\right)\left(\frac{\gamma+2}{n}+\delta\right)\cdots\left(\frac{\gamma+\delta-1}{n}+\delta\right)\right]^s z^p}\right.$$

$$\times n^{\alpha p}\left(\alpha p+\frac{\alpha\rho-\alpha+\beta}{n}\right)\left(\alpha p+\frac{\alpha\rho-\alpha+1}{n}\right)\cdots\left(\alpha p+\frac{\alpha\rho-\alpha+\beta+\alpha p-1}{n}\right)$$

$$\left.\times n^p\left(\frac{\rho}{n}+p\right)\left(\frac{\rho+1}{n}+p\right)\left(\frac{\rho+2}{n}+p\right)\cdots\left(\frac{\rho+p-1}{n}+p\right)\right|.$$

Thus,

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|n^{\mu r+\alpha p-\delta s+p}/z^p\right|.$$

Hence,

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right|>1 \Rightarrow \lim_{n\to\infty}\left|n^{\mu r+\alpha p-\delta s+p}/z^p\right|>1 \Rightarrow |z|<n^{1/p\left(\mathrm{Re}(\mu)r+\Re(\alpha)-\mathrm{Re}(\delta)s+p\right)}.$$

Theorem 2.2. For $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$; $\mathrm{Re}(\alpha)>0$ and $\delta,\mu,p>0$ the differential recurrence relation form:

$$\beta E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)+\alpha z\frac{\mathrm{d}}{\mathrm{dz}}E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)=E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r).$$

Proof.
Consider,

$$\beta E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}(z;s,r)+\alpha z\frac{d}{dz}E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}(z;s,r)$$

$$=\beta\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s} z^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}+\alpha z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(z)^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}$$

$$=\beta\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(z)^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}+\alpha z\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(pn+\rho-1)(z)^{pn+\rho-2}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+1\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}(z)^{pn+\rho-1}}{\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}}=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)$$

As the series given in (1.2) converges uniformly in any compact subset of $\mathbb{C}$, the use of term by term differentiation under the sign of summation leads us to the following theorem.

Theorem 2.3. If $m\in\mathbb{N}$, $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$, $\text{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$ and $\delta,\mu,p>0$ then

$$\left(\frac{d}{dz}\right)^{m}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)=\frac{z^{pm-m}\left[(\gamma)_{\delta m}\right]^{s}\Gamma(\rho)}{\left[(\lambda)_{\mu m}\right]^{r}\left[(\rho)_{pm-m}\right]}E^{\gamma+\delta m,\delta}_{\alpha,\beta+\alpha pm,\lambda+\mu m,\mu,\rho+pm-m,p}(z;s,r), \tag{2.3.1}$$

$$\left(\frac{d}{dz}\right)^{m}\left[z^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(\omega z^{\alpha};s,r\right)\right]=z^{\beta-m-1}E^{\gamma,\delta}_{\alpha,\beta-m,\lambda,\mu,\rho,p}\left(\omega z^{\alpha};s,r\right),\text{ if Re}(\beta-m)>0. \tag{2.3.2}$$

Proof. Consider

$$\left(\frac{d}{dz}\right)^{m}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,rho,p}(z;s,r)=\left(\frac{d}{dz}\right)^{m}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}z^{pn+\rho-1}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^{r}\left[(\rho)_{pn}\right]}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^{r}\left[(\rho)_{pn}\right]}\left(\frac{d}{dz}\right)^{m}\left(z^{pn+\rho-1}\right)$$

$$=\sum_{n=m}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^{s}\Gamma(\rho)z^{pn+\rho-m-1}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^{r}\Gamma\left(pn+\rho-m\right)}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta(n+m)}\right]^{s}\Gamma(\rho)z^{pn+pm+\rho-m-1}}{\left[\Gamma\left(\alpha\left(pn+pm+\rho-1\right)+\beta\right)\right]\left[(\lambda)_{\mu(n+m)}\right]^{r}\Gamma\left(pn+pm+\rho-m\right)}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta m}\right]^{s}\left[(\gamma+\delta m)_{\delta n}\right]^{s}\Gamma(\rho)z^{pm-m}}{\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+\alpha pm\right)\right]\left[(\lambda)_{\mu m}\right]^{r}\left[(\lambda+\mu m)_{\mu n}\right]^{r}}\frac{z^{pn+\rho-1}}{\Gamma\left(pn+pm+\rho-m\right)}\frac{\Gamma\left(\rho+pm-m\right)}{\Gamma\left(\rho+pm-m\right)}$$

$$=\frac{z^{pm-m}\left[(\gamma)_{\delta m}\right]^{s}\Gamma(\rho)}{\left[(\lambda)_{\mu m}\right]^{r}\left[(\rho)_{pm-m}\right]}\times\sum_{n=0}^{\infty}\frac{\left[(\gamma+\delta m)_{\delta n}\right]^{s}z^{pn+\rho-1}}{\left[(\lambda+\mu m)_{\mu n}\right]^{r}\left[\Gamma\left(\alpha\left(pn+\rho-1\right)+\beta+\alpha pm\right)\right](\rho+pm-m)_{pn}}$$

$$=\frac{z^{pm-m}\left[(\gamma)_{\delta m}\right]^{s}\Gamma(\rho)}{\left[(\lambda)_{\mu m}\right]^{r}\left[(\rho)_{pm-m}\right]}E^{\gamma+\delta m,\delta}_{\alpha,\beta+\alpha pm,\lambda+\mu m,\mu,\rho+pm-m,p}(z;s,r).$$

Now consider,

$$\left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^m\left[z^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(\omega z^\alpha;s,r\right)\right]=\left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^m\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{\alpha(pn+\rho-1)}z^{\beta-1}\omega^{pn+\rho-1}}{\left[\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\omega^{pn+\rho-1}}{\left[\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^m\left(z^{\alpha(pn+\rho-1)+\beta-1}\right)$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\left(\omega z^\alpha\right)^{(pn+\rho-1)}z^{\beta-m-1}}{\left[\Gamma\left(\alpha(pn+\rho-1)+\beta-m\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}=z^{\beta-m-1}E^{\gamma,\delta}_{\alpha,\beta-m,\lambda,\mu,\rho,p}\left(\omega z^\alpha;s,r\right)$$

Next, taking $f(z)=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}\left(zu^\alpha;s,r\right)$, in the Euler (Beta) transform (1.3), one finds the following

Theorem 2.4. If $\alpha,\beta,\gamma,\lambda,\rho,\sigma,\eta,\nu\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho,\sigma,\eta,\nu)>0$ and $\delta,\mu,p>0$ then

$$\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(zu^\alpha;s,r\right)\mathrm{d}u=E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}\left(z;s,r\right),\tag{2.4.1}$$

$$\frac{1}{\Gamma(\eta)}\int_t^x(x-t)^{\eta-1}(s-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left[\nu(s-t)^\alpha;s,r\right]\mathrm{d}s=(x-t)^{\eta+\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left[\nu(x-t)^\alpha;s,r\right],\tag{2.4.2}$$

$$\int_0^z t^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left[\omega t^\alpha;s,r\right]\mathrm{d}t=z^bE^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu,\rho,p}\left[\omega t^\alpha;s,r\right],\tag{2.4.3}$$

$$\frac{1}{\Gamma(\sigma)}\int_0^1 z^{\sigma-1}(1-z)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(x(1-z)^\alpha;s,r\right)\mathrm{d}z=E^{\gamma,\delta}_{\alpha,\beta+\sigma,\lambda,\mu,\rho,p}\left(x;s,r\right).\tag{2.4.4}$$

Proof.
In (2.4.1),

$$L.H.S.=\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(zu^\alpha s,r\right)\mathrm{d}u=\frac{1}{\Gamma(\eta)}\int_0^1 u^{\beta-1}(1-u)^{\eta-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s u^{\alpha(pn+\rho-1)}z^{pn+\rho-1}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}u$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^n}{\left[\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}\int_0^1 u^{\alpha(pn+\rho-1)+\beta-1}(1-u)^{\eta-1}\mathrm{d}u$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^n}{\left[\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\right]\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}\frac{\Gamma(\eta)\left[\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\right]}{\left[\Gamma\left(\alpha(pn+\rho-1)+\beta\right)+\eta\right]}$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s z^{pn+\rho-1}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}=E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}\left(z;s,r\right)=R.H.S.$$

Now, denoting the *L.H.S.* of (2.4.2) by I, we have

$$I=\frac{1}{\Gamma(\eta)}\int_t^x(x-t)^{\eta-1}(s-t)^{\beta-1}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left[\nu(s-t)^\alpha;s,r\right]\mathrm{d}s$$

$$=\frac{1}{\Gamma(\eta)}\int_t^x(x-t)^{\eta-1}(s-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s(s-t)^{\alpha(pn+\rho-1)}\nu^{pn+\rho-1}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}s$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s\nu^{pn+\rho-1}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}\Gamma(\eta)}\int_t^x(x-t)^{\eta-1}(s-t)^{\alpha(pn+\rho-1)+\beta-1}\mathrm{d}s.$$

Here, introducing u as a new variable of integration, by means of the relation

$$u = \frac{s-t}{x-t},$$

The further simplification gives,

$$I = \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \nu^{pn+\rho-1} (x-t)^{\alpha(pn+\rho-1)+\beta+\eta-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn} \Gamma(\eta)} \int_0^1 (1-u)^{\eta-1} u^{\alpha(pn+\rho-1)+\beta-1} \mathrm{d}u$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \nu^{pn+\rho-1} (x-t)^{\alpha(pn+\rho-1)+\beta+\eta-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn} \Gamma(\eta)} \frac{\Gamma(\eta)\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]}{\left[\Gamma(\alpha(pn+\rho-1)+\beta+\eta)\right]}$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \nu^{pn+\rho-1} (x-t)^{\alpha(pn+\rho-1)+\beta+\eta-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta+\eta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}}$$

$$= (x-t)^{\eta+\beta-1} E_{\alpha,\beta+\eta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left[\nu(x-t)^{\alpha}; s, r\right].$$

as desired.

To prove (2.4.3) we begin with

$$\int_0^z t^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(\omega t^{\alpha}; s, r\right) \mathrm{d}t$$

$$= \int_0^z t^{\beta-1} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s t^{\alpha(pn+\rho-1)} \omega^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s \omega^{n}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \int_0^z t^{\alpha(pn+\rho-1)+\beta-1} \mathrm{d}t$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s z^{\alpha(pn+\rho-1)+\beta} (\omega)^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn} (\alpha(pn+\rho-1)+\beta)}$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s z^{\alpha(pn+\rho-1)+\beta} \omega^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta+1)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} = z^{\beta} E_{\alpha,\beta+1,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(\omega z^{\alpha}; s, r\right).$$

Hence the result.

Now, consider

$$\frac{1}{\Gamma(\sigma)} \int_0^1 z^{\sigma-1} (1-z)^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}\left(x(1-z)^{\alpha}; s, r\right) \mathrm{d}z$$

$$= \frac{1}{\Gamma(\sigma)} \int_0^1 z^{\sigma-1} (1-z)^{\beta-1} \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s (1-z)^{\alpha(pn+\rho-1)} x^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn}} \mathrm{d}z$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn} \Gamma(\sigma)} \int_0^1 (1-z)^{\alpha(pn+\rho-1)+\beta-1} z^{\sigma-1} \mathrm{d}z$$

$$= \sum_{n=0}^{\infty} \frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1}}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]\left[(\lambda)_{\mu n}\right]^r (\rho)_{pn} \Gamma(\sigma)} \frac{\Gamma(\sigma)\left[\Gamma(\alpha(pn+\rho-1)+\beta)\right]}{\left[\Gamma(\alpha(pn+\rho-1)+\beta)+\sigma\right]}$$

simplification of above series yields (2.4.4).

3. Mellin-Barnes Integral Representation of $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r)$

Theorem 3.1. Let $\alpha\in\mathbb{R}_+$; $\beta,\gamma,\lambda,\rho,\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$ and $\delta,\mu>0$, $p\in\mathbb{N}$. Then the function $E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)$ is represented by the Mellin-Barnes integral as

$$E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)=\frac{[\Gamma(\lambda)]^r\Gamma(\rho)pz^{\rho-1}}{2\pi i[\Gamma(\gamma)]^s}\int_L\frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\mathrm{d}S, \tag{3.1.1}$$

where $|\arg z|<\pi$; the contour of integration beginning at $-i\infty$ and ending at $+i\infty$, and indented to separate the poles of integrand at $S=-n$ for all $n\in\mathbb{N}_0$ (to the left) from those at $S=\frac{\gamma+n}{\delta}$ for all $n\in\mathbb{N}_0$ (to the right).

Proof.

We shall evaluate the integral on the *R.H.S.* of (3.1.1) as the sum of the residues at the poles $S=0,-1,-2,\cdots$ In fact, in view of the definition of residue, we have

$$\begin{aligned}I&=\frac{1}{2\pi i}\int_L\frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\mathrm{d}S\\&=\sum_{n=0}^{\infty}\underset{S=-n}{S\mathrm{Res}n}\left[\frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\right]\\&=\sum_{n=0}^{\infty}\lim_{S\to-n}\frac{\pi(S+n)}{\sin\pi pS}\frac{[\Gamma(\gamma-\delta S)]^s(-z)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r\Gamma(\rho-pS)}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{pn}}{p}\frac{[\Gamma(\gamma-\delta S)]^s}{\Gamma(\beta+\alpha(pn+\rho-1))[\Gamma(\lambda+\mu n)]^r\Gamma(\rho+pn)}(-z)^{pn}.\end{aligned}$$

This gives,

$$\begin{aligned}I&=\frac{[\Gamma(\gamma)]^s}{[\Gamma(\lambda)]^r\Gamma(\rho)pz^{\rho-1}}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r(\rho)_{pn}}\\&=\frac{[\Gamma(\gamma)]^s}{[\Gamma(\lambda)]^r\Gamma(\rho)pz^{\rho-1}}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r).\end{aligned}$$

4. Integral Transforms of $E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)$

In this section, we discussed some useful integral transforms like Euler transforms, Laplace transforms, Mellin transforms, Whittaker transforms,

For the convenience, we introduce the Notation:

$$(\beta,\alpha)^r=\frac{[\Gamma(\beta+\alpha n)]^r}{[\Gamma(\beta)]^r}$$

Theorem 4.1. (*Euler*(*Beta*) *transforms*)

$$\begin{aligned}&\int_0^1 z^{a-1}(1-z)^{b-1}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(xz^{\sigma};s,r)\mathrm{d}z\\&=\frac{x^{\rho-1}[\Gamma(\lambda)]^r\Gamma(b)\Gamma(\rho)}{[\Gamma(\gamma)]^s}\cdot{}_{s+2}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\sigma\rho-\sigma+a,\sigma p), & (1,1); & x\\(\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p),\ (a+b,\sigma); & \end{matrix}\right],\end{aligned}$$

where $\alpha,\beta,\gamma,\lambda,\sigma,a,b\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,a,b)>0$ and $\delta,\mu p>0$.

Proof.

$$\int_0^1 z^{a-1}(1-z)^{b-1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(xz^{\sigma};s,r\right)\mathrm{d}z = \int_0^1 z^{a-1}(1-z)^{b-1}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1} z^{\sigma(pn+\rho-1)}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}z$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\int_0^1 z^{\sigma(pn+\rho-1)+a-1}(1-z)^{(b-1)}\mathrm{d}z$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma\left[(\gamma+\delta n)\right]^s\left[\Gamma(\lambda)\right]^r\Gamma(b)\Gamma\left(\sigma(pn+\rho-1)+a\right)x^{pn+\rho-1}}{\left[\Gamma(\gamma)\right]^s\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r\Gamma\left(\sigma(pn+\rho-1)+a+b\right)(\rho)_{pn}}$$

$$=\frac{x^{\rho-1}\left[\Gamma(\lambda)\right]^r\Gamma(b)\Gamma(\rho)}{\left[\Gamma(\lambda)\right]^s}\times{}_{s+2}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\sigma\rho-\sigma+a,\sigma p), & (1,1); & x\\ (\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p),(a+b,\sigma); & \end{matrix}\right].$$

Theorem 4.2. (*Laplace transforms*)

$$\int_0^{\infty} z^{a-1}\mathrm{e}^{-sz}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(xz^{\sigma};s,r\right)\mathrm{d}z$$

$$=\frac{s^{-\sigma(\rho-1)}x^{(\rho-1)-a}\left[\Gamma(\lambda)\right]^r\Gamma(\rho)}{\left[\Gamma(\gamma)\right]^s}\,{}_{s+2}\Psi_{r+2}\left[\begin{matrix}(\gamma,\delta)^s, & (a,\sigma), & (1,1); & \frac{x}{s^{\sigma p}}\\ (\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p); & \end{matrix}\right],$$

where $\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b)>0$ and $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b)>0$.

Proof. We begin with

$$I=\int_0^{\infty} z^{a-1}\mathrm{e}^{-sz}E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}\left(xz^{\sigma};s,r\right)\mathrm{d}z=\int_0^{\infty} z^{a-1}\mathrm{e}^{-sz}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{pn+\rho-1}z^{\sigma(pn+\rho-1)}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\mathrm{d}z$$

$$=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n}]^s x^{(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r(\rho)_{pn}}\int_0^{\infty}\mathrm{e}^{-sz}z^{\sigma(pn+\rho-1)+a-1}\mathrm{d}z.$$

On making substitution $t=sz$, we get

$$I=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{(pn+\rho-1)}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\int_0^{\infty}\mathrm{e}^{-t}(t/s)^{\sigma(pn+\rho-1)+a-1}(1/s)\mathrm{d}t$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{\delta n}\right]^s x^{(pn+\rho-1)}s^{-\sigma(pn+\rho-1)-a}}{\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[(\lambda)_{\mu n}\right]^r(\rho)_{pn}}\Gamma\left(\sigma(pn+\rho-1)+a\right)$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma\left[(\gamma+\delta n)\right]^s\left[\Gamma(\lambda)\right]^r x^n s^{-\sigma n-a}\Gamma\left(\sigma(pn+\rho-1)+a\right)\Gamma(\rho)\Gamma(n+1)}{\left[\Gamma(\gamma)\right]^s\Gamma\left(\alpha(pn+\rho-1)+\beta\right)\left[\Gamma(\lambda+\mu n)\right]^r\Gamma(\rho+pn)n!}$$

$$=\frac{s^{-\sigma(\rho-1)}x^{\rho-1}\left[\Gamma(\lambda)\right]^r}{\left[\Gamma(\gamma)\right]^s}\,{}_{s+2}\Psi_{r+2}\left[\begin{matrix}(\gamma,\delta)^s, & (a,\sigma), & (1,1); & \frac{x}{s^{\sigma p}}\\ (\alpha\rho-\alpha+\beta,\alpha p), & (\lambda,\mu)^r, & (\rho,p), & \end{matrix}\right].$$

In proving the following theorem we use the integral formula involving the Whittaker function:

$$\int_0^{\infty} t^{\nu-1}\mathrm{e}^{-t/2}W_{\lambda,\mu}(t)\mathrm{d}t=\frac{\Gamma(1/2+\mu+\nu)\Gamma(1/2-\mu+\nu)}{\Gamma(1/2-\lambda+\nu)},\ \mathrm{Re}(\nu\pm\mu)>-1/2.$$

Theorem 4.3. (*Whittaker transforms*)

$$\int_0^\infty \mathrm{e}^{-qt/2} t^{\psi-1} W_{\eta,\nu}(qt) E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega t^\sigma; s,r)\mathrm{d}t$$

$$= \frac{[\Gamma(\lambda)]^r \omega^{\rho-1}\Gamma(\rho)}{[\Gamma(\gamma)]^s q^{\sigma(\rho-1)-\psi}} \times {}_{s+3}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\psi+\nu+1/2,\sigma), & (\psi-\nu+1/2,\sigma), & (1;1); & \dfrac{\omega^p}{q^{\sigma p}} \\ (\beta,\alpha), & (\lambda,\mu)^r, & (\psi-\eta+1,\sigma), & (\rho,p); & \end{matrix}\right],$$

where $\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\sigma,\rho,a,b)>0$ and $\delta,\mu,p>0$.

Proof. Let

$$I = \int_0^\infty \mathrm{e}^{-qt/2} t^{\psi-1} W_{\eta,\nu}(qt) E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega t^\sigma; s,r)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt/2} t^{\psi-1} W_{\eta,\nu}(qt) \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s \omega^{pn+\rho-1} t^{\sigma(pn+\rho-1)}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r (\rho)_{pn}}\mathrm{d}t$$

$$= \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s \omega^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r (\rho)_{pn}} \int_0^\infty \mathrm{e}^{-qt/2} t^{\sigma(pn+\rho-1)+\psi-1} W_{\eta,\nu}(qt)\mathrm{d}t$$

then using the substitution $\xi = qt$, we get

$$I = \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^s \omega^{pn+\rho-1} q^{-\sigma(pn+\rho-1)-\psi}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda)_{\mu n}]^r (\rho)_{pn}} \int_0^\infty \mathrm{e}^{-\xi/2} \xi^{\sigma(pn+\rho-1)+\psi-1} W_{\eta,\nu}(\xi)\mathrm{d}\xi$$

$$= \sum_{n=0}^\infty \frac{\Gamma[(\gamma+\delta n)]^s [\Gamma(\lambda)]^r \omega^{pn+\rho-1} q^{-\sigma(pn+\rho-1)-\psi}}{[\Gamma(\gamma)]^s \Gamma(\alpha(pn+\rho-1)+\beta)[\Gamma(\lambda+\mu n)]^r (\rho)_{pn}} \times \frac{\Gamma(\sigma(pn+\rho-1)+\psi+\nu+1/2)\Gamma(\sigma n+\psi-\nu+1/2)}{\Gamma(\sigma(pn+\rho-1)+\psi-\eta+1)}$$

$$= \frac{[\Gamma(\lambda)]^r \omega^{\rho-1}\Gamma(\rho)}{[\Gamma(\gamma)]^s q^{\sigma(\rho-1)-\psi}} \times {}_{s+3}\Psi_{r+3}\left[\begin{matrix}(\gamma,\delta)^s, & (\psi+\nu+1/2,\sigma), & (\psi-\nu+1/2,\sigma), & (1;1); & \dfrac{\omega^p}{q^{\sigma p}} \\ (\beta,\alpha), & (\lambda,\mu)^r, & (\psi-\eta+1,\sigma), & (\rho,p); & \end{matrix}\right],$$

Theorem 4.4. (*Mellin transforms*)

$$\int_0^\infty t^{pS-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(-\omega t; s,r)\mathrm{d}t = \frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s}{\omega^{pS}\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r \Gamma(\rho-pS)} \tag{4.4.1}$$

where $\alpha,\beta,\gamma,\lambda,\rho,S\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho,S)>0$ $\delta,\mu,p>0$.

Proof. Putting $z=-\omega t$ in (3.1.1), we get

$$\begin{aligned} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r) &= \frac{[\Gamma(\lambda)]^r \Gamma(\rho) p z^{\rho-1}}{2\pi i[\Gamma(\gamma)]^s} \int_L \frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s (-\omega t)^{-pS}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r \Gamma(\rho-pS)}\mathrm{d}S \\ &= \frac{[\Gamma(\lambda)]^r \Gamma(\rho) p z^{\rho-1}}{2\pi i[\Gamma(\lambda)]^s} \int_L f^*(S) t^{-pS}\mathrm{d}S, \end{aligned} \tag{4.4.2}$$

in which

$$f^*(S) = \frac{\Gamma(pS)\Gamma(1-pS)[\Gamma(\gamma-\delta S)]^s}{\omega^{pS}\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)[\Gamma(\lambda-\mu S)]^r \Gamma(\rho-pS)}$$

using (1.5) and (1.6) in (4.4.2), immediately leads us to (4.4.1).

5. Generalized Hypergeometric Function Representation of $E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)$

Taking $\alpha = a$, $\delta = l$, $\mu = m$ in (1.2), we get

$$E_{k,\beta,\lambda,m,\rho,p}^{\gamma,l}(z;s,r)=\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{ln}\right]^{s} z^{pn+\rho-1}}{\Gamma\left(a(pn+\rho-1)+\beta\right)\left[(\lambda)_{mn}\right]^{r}(\rho)_{pn}}=\frac{1}{\Gamma(\beta+a\rho-a)}\sum_{n=0}^{\infty}\frac{\left[(\gamma)_{ln}\right]^{s} z^{pn+\rho-1}(1)_{n}}{(\beta+a\rho-a)_{apn}\left[(\lambda)_{\mu n}\right]^{r}(\rho)_{pn}\,n!}$$

$$=\frac{z^{\rho-1}}{\Gamma(\beta+a\rho-a)}\sum_{n=0}^{\infty}\frac{\left[\prod_{i=1}^{l}\left(\frac{\gamma+i-1}{l}\right)_{n}\right]^{s}\left[\prod_{k=1}^{m}\left(\frac{\lambda+k-1}{m}\right)_{n}\right]^{-r}(1)_{n}}{\prod_{j=1}^{a}\left(\frac{\beta+a\rho-a+j-1}{ap}\right)_{n}\prod_{h=1}^{p}\left(\frac{\rho+h-1}{p}\right)_{n}}\times\frac{\frac{l^{lsn}z^{pn}}{m^{mrn}p^{pn}(ap)^{apn}}}{n!}$$

$$=\frac{z^{\rho-1}}{\Gamma(\beta+a\rho-a)}\times{}_{l^{s}}F_{ap+m^{r}+p}\left[\begin{matrix}\Delta(l,\gamma)^{s}, & 1; & \frac{l^{ls}z^{p}}{m^{mr}p^{p}(ap)^{ap}}\\ \Delta(ap,\beta+a\rho-a), & \Delta(m,\lambda)^{r},\Delta(p,\rho); & \end{matrix}\right]$$

where $\Delta(n;\alpha)$ is a n-tuple $\frac{\alpha}{n},\frac{\alpha+1}{n},\cdots,\frac{\alpha+n-1}{n}$.

6. Relationship with Some Known Special Functions (Generalized Laguerre Polynomial, Fox H-Function, Wright Hypergeometric Function)

6.1. Relationship with Generalized Laguerre Polynomials

Putting $\alpha=k,\ \beta=\nu+1,\ \gamma=-m,\ r=0,\ s=1,\ \rho=1,\ p=1$ and replacing δ by $q\in\mathbb{N}$ and z by z^k in (1.2), we get

$$\begin{aligned}E_{k,\nu+1,\lambda,\mu,1,1}^{-m,q}(z;1,0)&=\sum_{n=0}^{\left[\frac{m}{q}\right]}\frac{\left[(-m)_{qn}\right]}{\Gamma(kn+\nu+1)}\frac{z^{kn}}{n!}=\sum_{n=0}^{\left[\frac{m}{q}\right]}\frac{(-1)^{qn}m!}{(m-qn)!}\frac{1}{\Gamma(kn+\nu+1)}\frac{z^{kn}}{n!}\\&=\frac{\Gamma(m+1)}{\Gamma(km+\nu+1)}\sum_{n=0}^{\left[\frac{m}{q}\right]}\frac{(-1)^{qn}}{(m-qn)!}\frac{\Gamma(km+\nu+1)}{\Gamma(kn+\nu+1)}\frac{z^{kn}}{n!}=\frac{\Gamma(m+1)}{\Gamma(km+\nu+1)}Z_{\left[\frac{m}{q}\right]}^{(\nu)}(z,k)\end{aligned}\tag{6.1.1}$$

where $Z_{\left[\frac{m}{q}\right]}^{(\nu)}(z,k)$ is polynomial of degree $\left[\frac{m}{q}\right]$ in z^k.

In particular, $Z_{m}^{(\nu)}(z,1)=L_{m}^{(\nu)}(z)$, so that

$$E_{k,\nu+1,\lambda,\mu,1,1}^{-m,1}(z;1,0)=\frac{\Gamma(m+1)}{\Gamma(km+\nu+1)}L_{m}^{(\nu)}(z)\tag{6.1.2}$$

6.2. Relationship with Fox H-Function

From (3.1.1), we have

$$\begin{aligned}E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(z;s,r)&=\frac{\left[\Gamma(\lambda)\right]^{r}\Gamma(\rho)}{2\pi i\left[\Gamma(\gamma)\right]^{s}}\times\int_{L}\frac{\Gamma(S)\left[\Gamma(\gamma-\delta S)\right]^{s}}{\Gamma(\beta+\alpha\rho-\alpha-\alpha pS)\left[\Gamma(\lambda-\mu S)\right]^{r}\Gamma(\rho-pS)}(-z)^{-S}\,\mathrm{d}S\\&=\frac{\left[\Gamma(\lambda)\right]^{r}\Gamma(\rho)}{\left[\Gamma(\gamma)\right]^{s}}\times H_{s,r+3}^{1,s}\left(-z\left|\begin{matrix}\left[(1-\gamma,\delta)\right]^{s}; & & & \\ (0,1), & (1-\beta-\alpha\rho+\alpha,\alpha p), & \left[(1-\lambda,\mu)\right]^{r}, & (1-\rho,p);\end{matrix}\right.\right).\end{aligned}\tag{6.2.1}$$

6.3. Relationship with Wright Function

If $\alpha,\beta,\gamma,\lambda,\rho\in\mathbb{C}$, $\mathrm{Re}(\alpha,\beta,\gamma,\lambda,\rho)>0$, $\delta,\mu,p>0$ (1.2) can be written as

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(z;s,r)=\frac{[\Gamma(\lambda)]^r\Gamma(\rho)}{[\Gamma(\gamma)]^s}\sum_{n=0}^{\infty}\frac{[\Gamma(\gamma+\delta n)]^s z^{pn+\rho-1}}{\Gamma(\alpha(pn+\rho-1)+\beta)[(\lambda+\mu n)]^r\Gamma(\rho+pn)}, \tag{6.3.1}$$

from (1.9) for (6.3.1), we get

$$=\frac{[\Gamma(\lambda)]^r\Gamma(\rho)z^{\rho-1}}{[\Gamma(\gamma)]^s}\,{}_{s+1}\Psi_{r+2}\left[\begin{matrix}[(\gamma,\delta)]^s, & (1,1); & z^p\\ (\beta+\alpha\rho-\alpha,\alpha p), & [(\lambda,\mu)]^r, & (\rho,p);\end{matrix}\right]$$

7. Summary

In Section 1, an extended version of Mittag-Leffler function of 10 indices established as an Equation (1.2) including with some necessary information of Bessel function, some well-known integral transforms and generalized hypergeometric functions with their family. Results obtained in Sections 2 to 6 are interesting generalizations of (Shukla and Prajapati [1]) and stimulate the scope of further research in the field of generalization Mittag-Leffler function.

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