Scientific
Research

Solvability of Inverse Eigenvalue Problem for Dense Singular Symmetric Matrices

Anthony Y. Aidoo[1*], Kwasi Baah Gyamfi[2], Joseph Ackora-Prah[2], Francis T. Oduro[2]

[1]Department of Mathematics and Computer Science, Eastern Connecticut State University, Willimantic, USA
[2]Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana
Email: *aidooa@easternct.edu

ABSTRACT

Given a list of real numbers $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$, we determine the conditions under which Λ will form the spectrum of a dense $n \times n$ singular symmetric matrix. Based on a solvability lemma, an algorithm to compute the elements of the matrix is derived for a given list Λ and dependency parameters. Explicit computations are performed for $n \leq 5$ and $r \leq 4$ to illustrate the result.

Keywords: Inverse Eigenvalue Problem; Dense; Nonnegative; Singular; Symmetric

1. Introduction

The inverse eigenvalue problem (IEP) involves the reconstruction of a matrix from prescribed spectral data. In order to limit the number of usually infinitely many solutions that are usually possible, if a solution does exists, it is usually required that the matrix constructed preserves a specific structure in addition to the spectral requirement. The nonnegative inverse eigenvalue problem (NIEP) is a special case of the IEP first posed by Kolmogorov in 1937. It involves the determination of the existence of entrywise nonnegative matrices when given a set of complex numbers that make up the spectrum of the yet to be determined matrix. The extensions and related conditions have been solved in several different ways. We give a brief review of results for the general inverse eigenvalue problem (IEP) and NIEP, as related to our work. A detailed treatment can be found in [1]. We begin with a realization criterion provided by Soto [2] for the NIEP. This result is based on conditions derived by Fiedler [3] and Borobia [4]. A list of real numbers $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$ is said to be realizable if it forms the spectrum of an entry wise nonnegative matrix.

Theorem 1. *Let* $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$ *be a set of real numbers such that*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \cdots \geq \lambda_n.$$

If there exits a partition $\Lambda = \Lambda_1 \bigcup \Lambda_2 \bigcup \cdots \bigcup \Lambda_s$ *with*

$$\Lambda_k = \left\{\lambda_{k_1}, \lambda_{k_2} \cdots \lambda_{k_{p_k}}\right\}, \quad k = 1, 2, \cdots, s, \quad \lambda_{1_1} = \lambda_1,$$

$$\lambda_{k_1} \geq 0, \quad \lambda_{k_1} \geq \lambda_{k_2} \geq \cdots \geq \lambda_{k_{pk}},$$

$$S_{k_j} = \lambda_{k_j} + \lambda_{k_{pk} - j + 1}, j = 2, 3, \cdots, \left[\frac{k_{pk}}{2}\right],$$

and for k_{p_k} *odd*

$$S_{\frac{k_{pk+1}}{2}} = \min\left\{\lambda_{\frac{k_{pk+1}}{2}}, 0\right\}, k = 1, 2, \cdots, s,$$

$$T_k = \lambda_{k_1} + \lambda_{k_{pk}} + \sum_{S_{k_j} < 0} S_{k_j}, k = 1, 2, \cdots, s,$$

and

$$L = \max\left\{-\lambda_{1_{p1}} - \sum_{S_{1_j} < 0} S_{1_j}; \max_{2 \leq k \leq s}\left\{\lambda_{k_1}\right\}\right\}$$

satisfying

$$\lambda_1 \geq L - \sum_{T_k < 0, k=2}^{s} T_k,$$

then Λ *is realizable* (*by a nonnegative matrix with constant row sums*).

It has been proved that the above theorem is sufficient for the existence of an $n \times n$ symmetric nonnegative matrix with real spectrum $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, see [2].

Recently Wu studied the IEP and gave the solvability criterion for its solution given a spectrum of complex numbers in the following two theorems [5].

Theorem 2. *For a given list of complex numbers,* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *if it has closed property under com-*

plex conjugation, that is Λ *contains an element and its complex conjugation, then there must be at least one real matrix A with spectrum* Λ.

Proof. By the closed property of Λ under complex conjugation, construct the polynomial:

$$f(x) = \prod_{i=1}^{n}(x - \lambda_i). \tag{1}$$

By multiplying out, combining similar terms, and simplifying, the above equation can be written as:

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

where $a_1, a_2, \cdots, a_n$ are real numbers. It follows that $a_1, a_2, \cdots, a_n$ are real numbers and the polynomial (1) has zeros $\lambda_1, \lambda_2, \cdots, \lambda_n$. Thus, using $a_1, a_2, \cdots, a_n$, the matrix A can be constructed as:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}.$$

It is straightforward to show that A has spectrum $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. □

Theorem 3. *Given a list of real numbers* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *the sufficient condition that there exists at least one nonnegative matrix A with spectrum* Λ *is that* Λ *has the following properties*

$$\sum_{i=1}^{n} \lambda_i \geq 0,$$

$$\sum_{i,j=1_{i<j}}^{n} \lambda_i \lambda_j \leq 0,$$

$$\sum_{i,j,k=1_{i<j<k}}^{n} \lambda_i \lambda_j \lambda_k \geq 0,$$

$$\cdots$$

$$\prod_{i=1}^{n} \lambda_i \begin{cases} \geq 0, & n \text{ is an odd number} \\ \leq 0, & n \text{ is an even number.} \end{cases}$$

Several other results have been obtained for the inverse eigenvalue problem for Hermitian matrices (see for example [6]). It is obvious from the above theorems and the references that the solution of the IEP or even the NIEP is not unique. In addition, most of the solutions provided in the literature yield only sparse matrices. In this paper, we provide conditions for the solvability of the IEP for singular symmetric matrices. We show that unique solutions exist for matrices of rank one and give the solutions for dense matrices of rank greater than one.

2. Preliminaries

In what follows, we denote the $n \times n$ singular symmet-

ric matrix (a_{ij}) such that $a_{ij} = a_{ji}, i, j = 1, \cdots, n$ by A_n. Without loss of generality, singularity is achieved by multiplying the first row by prescribed scalars. We denote a singular symmetric matrix of rank r by $A_{(n,r)}$. In this case we write $R_{i+1} = kR_i$ to denote that the ith row is k times the first row, where $k \in \mathbb{R}$. Finally, we denote the spectrum of A_n by $\Lambda_n = \{\lambda_1, \cdots, \lambda_n\}$. If the rank of A is r, then we assume $\lambda_i \neq 0$ for $i = 1, \cdots, r$, but $\lambda_i = 0$, $i = r+1, \cdots, n$.

Lemma 1. *Let A be a singular symmetric matrix of rank r. Then there exits an isomorphism between the elements of A and its distinct non-zero eigenvalues if and only if rank* $(A) = 1$.

Corollary: The inverse eigenvalue problem has a unique solution for singular symmetric matrices of rank 1 and any prescribed row multiples.

We begin by considering $A_{(2,1)}$. By definition, $A_{(2,1)}$ is of the form:

$$A_{(2,1)} = \begin{pmatrix} a_{11} & ka_{11} \\ ka_{11} & k^2 a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}.$$

Let $\Lambda_2 = \{\lambda_1, \lambda_2\}$. Since $A_{(2,1)}$ is singular of rank 1, it follows that $\lambda_2 = 0$. We have:

$$Tr(A_{(2,1)}) = \lambda = a_{11}(1 + k^2). \text{ Therefore } a_{11} = \frac{\lambda}{1 + k^2}.$$

Hence:

$$A_{(2,1)} = \frac{\lambda}{1 + k^2} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}. \tag{2}$$

Thus $A_{(2,1)}$ has been reconstructed for given λ and prescribed scalar k.

We see from this formula that for any given λ and parameter k, we can generate any 2×2 singular symmetric matrix of rank one. For example if $k = 2$, $\lambda = 5$, we have

$$A_{(2,1)} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

3. IEP for Dense Singular Symmetric Matrices

We generalize the method above in the following two theorems, first for an $n \times n$ singular symmetric matrix of rank 1 and then of rank r, where $1 \leq r < n$.

Theorem 4. *Given the spectrum and the row dependence relations* $R_{i+1} = k_i R_i$, $i = 1, \cdots, n-1$, *where the* k_i's *are nonzero real numbers, the inverse eigenvalue problem for a* $n \times n$ *singular symmetric matrix of rank 1 is solvable.*

Proof. Given the spectrum $\Lambda_n = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, since rank $A_n = 1$, it follows from our notation above that $\lambda_1 \neq 0$ and $\lambda_i = 0$, $i = 2, \cdots, n$. Let k_i, $i = 1, \cdots, k_{n-1}$ be the row multiples. Letting

$$A_{(n,1)} = \begin{pmatrix} a_{11} & a_{11}k_1 & a_{11}k_1k_2 & \cdots & a_{11}k_1k_2\cdots k_{n-1} \\ a_{11}k_1 & a_{11}k_1^2 & a_{11}k_1^2k_2 & \cdots & a_{11}k_1^2k_2\cdots k_{n-1} \\ a_{11}k_1k_2 & a_{11}k_1^2k_2 & a_{11}k_1^2k_2^2k_3 & \cdots & a_{11}k_1^2k_2^2k_3\cdots \dot{k}_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{11}k_1k_2\cdots k_{i-1} & a_{11}k_1^2k_2\cdots k_{i-1} & a_{11}k_1^2k_2^2k_3\cdots k_{i-1} & \cdots & a_{11}k_1^2k_2^2k_3\cdots k_{n-1}^2 \end{pmatrix}.$$

Then

$$Tr\left(A_{(n,1)}\right)$$
$$= \lambda = a_{11}\left(1 + k_1^2 + k_1^2k_2^2 + k_1^2k_2^2k_3^2 + \cdots + k_1^2k_2^2\cdots k_{n-1}^2\right).$$

Hence

$$a_{11} = \frac{\lambda}{1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2\cdots k_{n-1}^2}.$$

We note that for $n = 2$, Equation (2) shows that $\lambda = \frac{\lambda}{1+k^2}$. Thus by writing the matrix $A_{(n+1,1)}$, the result follows by induction on n. □

We state the following theorem for the general case where A_n has rank r.

Theorem 5. *The inverse eigenvalue problem for an n × n singular symmetric matrix of rank r is solvable provided that n − r arbitrary parameters are prescribed.*

Proof. Let $A_{(n,r)} = \left[a_{ij}\right]$, where $n \geq 2$. It is obvious that,

$$a_{ij} = a_{11}\begin{cases} \prod_{s=0}^{j-1} k_s, & i < j \\ \left(\prod_{s=0}^{i-1} k_s\right)^2, & i = j \\ \prod_{s=0}^{i-1} k_s, & i > j \end{cases}$$

where

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2\cdots k_{n-r}^2}$$

and

$$\begin{aligned} a_{ij} &= \lambda_2, \quad \text{where } i = j = n - r + 2 \\ \vdots &= \vdots \\ a_{nn} &= \lambda_r \end{aligned} \tag{3}$$

$\lambda_1, \lambda_2, \cdots, \lambda_r \in \mathbb{R}$ such that $R_i = k_{i-1}R_{i-1}$ where $k_i \in R$, $i = 2, \cdots, n-r+1$ and R_i is the ith row of A.

$Tr(A) = 1 + \sum_{s=1}^{n}\left(\prod_{s=1}^{i-1} k_s\right)^2$. We see that for $r = 1$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2\cdots k_{n-1}^2}.$$

$r = 2$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2\cdots k_{n-2}^2}.$$

$r = 3$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2\cdots k_{n-3}^2}.$$

$r = 4$:

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2\cdots k_{n-4}^2}.$$

The result follows for any rank $1 < r < n$. □

4. Numerical Examples

In this section, we illustrate the results above with small matrices of size $3 \leq n \leq 5$ of rank $1 \leq r \leq 4$. We begin with the singular 3×3 symmetric matrix of rank 1, $A_{(3,1)}$, which is of the form:

$$A_{(3,1)} = a_{11}\begin{pmatrix} 1 & k_1 & k_1k_2 \\ k_1 & k_1^2 & k_1^2k_2 \\ k_1k_2 & k_1^2k_2 & k_1^2k_2^2 \end{pmatrix}, a_{11} = \frac{\lambda}{1 + k_1^2 + k_1^2k_2^2}.$$

For $k_1 = -1$, $k_2 = 2$, $\lambda = 6$, we have:

$$A_{(3,1)} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}.$$

Similarly, given k_1, k_2 and k_3, we obtain the following singular symmetric matrix:

$$A_{(4,1)} = a_{11}\begin{pmatrix} 1 & k_1 & k_1k_2 & k_1k_2k_3 \\ k_1 & k_1^2 & k_1^2k_2 & k_1^2k_2k_3 \\ k_1k_2 & k_1^2k_2 & k_1^2k_2^2 & k_1^2k_2^2k_3 \\ k_1k_2k_3 & k_1^2k_2k_3 & k_1^2k_2^2k_3 & k_1^2k_2^2k_3^2 \end{pmatrix}$$

In this case, $a_{11} = \frac{\lambda}{1 + k_1^2 + k_1^2k_2^2 + k_1^2k_2^2k_3^2}$. When $\lambda = 2$, $k_1 = 3$, $k_2 = 2$ and $k_3 = 4$, we obtain the following 4×4 singular symmetric matrix of rank one:

$$A_{(4,1)} = \begin{pmatrix} \dfrac{1}{311} & \dfrac{3}{311} & \dfrac{6}{311} & \dfrac{24}{311} \\ \dfrac{3}{311} & \dfrac{9}{311} & \dfrac{18}{311} & \dfrac{72}{311} \\ \dfrac{6}{311} & \dfrac{18}{311} & \dfrac{36}{311} & \dfrac{144}{311} \\ \dfrac{24}{311} & \dfrac{72}{311} & \dfrac{144}{311} & \dfrac{576}{311} \end{pmatrix}.$$

Obviously, the two examples given above can easily be verified for density, symmetry, rank, and that they satisfy the prescribed spectrum. Only the nonzero eigenvalue and the k's have to be known for the solution of this type of IEP. As such, for given nonzero eigenvalue and k's, the solution is unique.

The examples below clearly illustrate Theorem 5. For an $n \times n$ singular symmetric matrix of rank r, $n - r$ arbitrary values must be prescribed in addition to the nonzero spectrum for the IEP to be solvable. This clearly leads to infinitely many solutions. We illustrate this with the following examples beginning with the IEP for $n \times n$ singular symmetric matrices of rank 2. $A_{(3,2)}$ is of the form:

$$A_{(3,2)} = \begin{pmatrix} a_{11} & ka_{11} & a_{13} \\ ka_{11} & k^2a_{11} & ka_{13} \\ a_{13} & ka_{13} & a_{33} \end{pmatrix}$$

Here, $Tr(A) = \lambda_1 + \lambda_2 = a_{11}(1+k^2) + a_{33}$ and $\lambda_1 \lambda_2 = a_{11}(1+k^2)a_{33}$. Hence $a_{33} = \dfrac{\lambda_1 \lambda_2}{a_{11}(1+k^2)}$. Thus

$$a_{11}^2(1+k^2)^2 - a_{11}(1+k^2)(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 = 0.$$

which yields $a_{11} = \dfrac{\lambda_1}{1+k^2}$ and $\lambda_2 = a_{33}$. Therefore a_{13} becomes a free variable. When $\lambda_1 = 2$, $\lambda_2 = 3$, $k = 4$ and $a_{13} = 5$, for example, we obtain the following singular symmetric matrix:

$$A_{(3,2)} = \begin{pmatrix} \dfrac{2}{17} & \dfrac{8}{17} & 5 \\ \dfrac{8}{17} & \dfrac{32}{17} & 20 \\ 5 & 20 & 3 \end{pmatrix}.$$

In general, the solution of the IEP for $A_{n,r}$ leads to the solution of an rth degree polynomial equation in a_{11} of the form:

$$\begin{aligned} 0 = {} & a_{11}^r(1+k_1^2+\cdots+k_{n-r}^2) \\ & - \left(\sum_{i=1}^r \lambda_i\right)(1+k_1^2+\cdots+k_{n-r}^2)a_{11}^{r-1} \\ & + \sum_{k=1}^r \left(\prod_{i=k}^{k+1} \lambda_i\right)(1+k_1^2+\cdots+k_{n-r}^2)a_{11}^{r-2} \\ & - \sum_{k=1}^r \left(\prod_{i=k}^{k+2} \lambda_i\right)(1+k_1^2+\cdots+k_{n-r}^2)a_{11}^{r-3}+\cdots-\left(\prod_{i=1}^r \lambda_i\right). \end{aligned} \tag{4}$$

To solve the case for $n = 4$ and $r = 2$, we deduce from the general polynomial equation above that the following quadratic in a_{11} holds:

$$a_{11}^2\left(1+k_1^2+k_1^2k_2^2\right)^2$$
$$-\left(\lambda_1+\lambda_2\right)\left(1+k_1^2+k_1^2k_2^2\right)a_{11}+\lambda_1\lambda_2 = 0.$$

This yields: $a_{11} = \dfrac{\lambda_1}{1+k_1^2+k_1^2k_2^2}$, $a_{44} = \lambda_2$ and a_{14} becomes a free variable. For $\lambda_1 = 6$, $\lambda_2 = 5$, $k_1 = 4$, $k_2 = 3$ and $a_{14} = 2$, we obtain a singular symmetric matrix below:

$$A_{(4,2)} = \begin{pmatrix} \dfrac{6}{161} & \dfrac{24}{161} & \dfrac{72}{161} & 2 \\ \dfrac{24}{161} & \dfrac{96}{161} & \dfrac{288}{161} & 8 \\ \dfrac{72}{161} & \dfrac{288}{161} & \dfrac{864}{161} & 24 \\ 2 & 8 & 24 & 5 \end{pmatrix}.$$

Similarly, for $A_{(5,2)}$, we obtain the following quadratic in a_{11}:

$$a_{11}^2\left(1+k_1^2+k_1^2k_2^2+k_1^2k_2^2k_3^2\right)^2$$
$$-\left(\lambda_1+\lambda_2\right)\left(1+k_1^2+k_1^2k_2^2+k_1^2k_2^2k_3^2\right)a_{11}\lambda_1\lambda_2 = 0.$$

The solution gives: $a_{11} = \dfrac{\lambda_1}{1+k_1^2+k_1^2k_2^2+k_1^2k_2^2k_3^2}$ and $\lambda_2 = a_{55}$ where a_{15} is a free variable. When $\lambda_1 = 2$, $\lambda_2 = 5$, $k_1 = 3$, $k_2 = 5$, $k_3 = 7$ and $a_{15} = 4$, we obtain a singular symmetric matrix below:

$$A_{(5,2)} = \begin{pmatrix} \dfrac{2}{11260} & \dfrac{6}{11260} & \dfrac{30}{11260} & \dfrac{210}{11260} & 4 \\ \dfrac{6}{11260} & \dfrac{18}{11260} & \dfrac{90}{11260} & \dfrac{630}{11260} & 12 \\ \dfrac{30}{11260} & \dfrac{90}{11260} & \dfrac{450}{11260} & \dfrac{3150}{11260} & 60 \\ \dfrac{210}{11260} & \dfrac{630}{11260} & \dfrac{3150}{11260} & \dfrac{22050}{11260} & 420 \\ 4 & 12 & 60 & 420 & 5 \end{pmatrix}.$$

By the same method, $A_{(4,3)}$ leads to the following cubic equation:

$$a_{11}^3\left(1+k^2\right)^3 - \left(\lambda_1+\lambda_2+\lambda_3\right)a_{11}^2\left(1+k^2\right)^2$$
$$+ a_{11}\left(1+k^2\right)\left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\right)-\lambda_1\lambda_2\lambda_3 = 0.$$

Solving the above cubic equation we obtain the following roots. $\lambda_1 = a_{11}\left(1+k^2\right) \Rightarrow a_{11} = \dfrac{\lambda_1}{1+k^2}$, $\lambda_2 = a_{33}$ and $\lambda_3 = a_{44}$, where a_{13}, a_{14} and a_{34} are free variables.

Finally, we want to consider 5×5 singular symmetric matrix of rank 4. Using Equation (2), we obtain the fol-

lowing quartic equation in a_{11} where λ_1, λ_2, λ_3 and λ_4 are the nonzero members of the spectrum.

$$a_{11}^4\left(1+k^2\right)^4 - \left(\lambda_1+\lambda_2+\lambda_3+\lambda_4\right)a_{11}^3\left(1+k^2\right)^3$$

$$\cdot\left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3\lambda_2\lambda_4+\lambda_3\lambda_4\right)a_{11}^2\left(1+k^2\right)^2$$

$$-\left(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4\right)a_{11}\left(1+k^2\right)$$

$$+\lambda_1\lambda_2\lambda_3\lambda_4 = 0.$$

Factoring the above quartic equation, we obtain the following results: $a_{11} = \dfrac{\lambda_1}{1+k^2}$, $\lambda_2 = a_{33}$, $\lambda_3 = a_{44}$ and $\lambda_4 = a_{55}$. The free variables are a_{13}, a_{14}, a_{15}, a_{35}, a_{34} and a_{45}.

As an example, if we let $\lambda_2 = 4$, $\lambda_3 = 7$, $\lambda_4 = 11$, $k = 2$, $a_{13} = 8$, $a_{14} = -4$, $a_{15} = 5$, $a_{35} = 14$, $a_{34} = 6$ and $a_{45} = 17$

$$A_{(5,4)} = \begin{pmatrix} 1.3 & 2.7 & 8 & -4 & 5 \\ 2.6 & 5.4 & 16 & -8 & 10 \\ 8 & 16 & 4 & 6 & 14 \\ -4 & -8 & 6 & 7 & 17 \\ 5 & 10 & 14 & 17 & 12 \end{pmatrix}.$$

We provide in the appendix, a program that generates any dense $n \times n$ singular symmetric matrix of rank 1 for given row multipliers. The program could be easily modified for rank $1 < r < n$.

5. Summary and Discussion

The IEP has a variety of applications which include control design, system identification, principal component analysis, geophysics and mechanical systems. Several interpretations and applications of the IEP have been considered. However, the direct IEP involving full matrices has received relatively less attention owing to the intractability of the problem for even small sized matrices. In this paper, we focused on the IEP for dense singular symmetric matrices. We showed that while unique solutions exists for an $n \times n$ matrices of rank 1, infinitely many solution exist for $n \times n$ singular symmetric matrices of rank r, where $1 < r < n$. With our method, an extension to nonsingular matrices for small n, could become feasible via a numerical analytic interpretation of the IEP.

6. Acknowledgements

REFERENCES

[1] M. T. Chu and G. H. Golub, "Inverse Eigenvalue Problem: Theory, Algorithms and Applications," Oxford University Press, Oxford, 2005. doi:10.1093/acprof:oso/9780198566649.001.0001

[2] R. L. Soto, "Realizability Criterion for Symmetric Nonnegative Inverse Eigenvalue Problem," *Linear Algebra and Its Applications*, Vol. 416, No. 2-3, 2006, pp. 783-794. doi:10.1016/j.laa.2005.12.021

[3] M. Fiedler, "Eigenvalues of Nonnegative Symmetric Matrices," *Linear Algebra and Its Applications*, Vol. 9, No. 1, 1974, pp. 119-142. doi:10.1016/0024-3795(74)90031-7

[4] A. Borobia, "On Nonnegative Inverse Eigenvalue Problem," *Linear Algebra and Its Applications*, Vol. 223/224, 1995, pp. 131-140. doi:10.1016/0024-3795(94)00343-C

[5] J. Wu, "On Open Problems of Nonnegative Inverse Eigenvalues Problem," *Advances in Pure Mathematics*, Vol. 1, No. 4, 2011, pp. 128-132. doi:10.4236/apm.2011.14025

[6] R. Fernandes and C. M. da Focenca, "The Inverse Eigenvalue Problem for Hermitian Matrices," *Linear Multilinear Algebra*, Vol. 57, No. 7, 2009, pp. 673-682. doi:10.1080/03081080802187870

Appendix

The following program generates an $n \times n$ singular symmetric matrix of *rank* 1

```
function ()
%UNTITLED2 Summary of function goes here
%   Detailed explanation goes here
lambda=input('Enter a positive number as trace of the
        generalized matrix')
k = input('1.Enter m positive numbers that characterize
    the generalided matrix')
m = length(k);h(1)=1;ss=[];
for i=1:m
    h(i+1)=h(i)*k(i);
end
h;kk=1;
```

```
for i=m:-1:1
    kk=kk*(k(i)^2)+1;
end
kk;
format short eng
a=lambda/kk
ss=a*h;
mm(1,:)=ss;
for i=2:(m+1);
    mm(i,:)=k(i-1)*mm((i-1),:);
end
    mm
    %fprintf('The required singular matrix of rank 1 is:\n
%d \n',mm)
tr=trace(mm)
```