Announcement from Editorial Board

The following article has been retracted due to the investigation of complaints received against it.

Title: A Note on the Classification of Linking Pairings on 2-Groups
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The scientific community takes a very strong view on this matter and we treat all unethical behavior such as plagiarism seriously. This paper published in Vol.3 No.1, 06-13, 2012, has been removed from this site.

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A Note on the Classification of Linking Pairings on 2-Groups

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ABSTRACT

It has been shown that for a linking pairing (G,ϕ) on a finite abelian group G there is a closed, connected, oriented 3-manifold, M, with first homology group $H_1(M)=G$ having linking form $\lambda_M \cong \phi$. A refinement of this result, where the manifold M is a Seifert manifold which is a rational homology sphere, was conjectured and proved in the case where the abelian group G has no 2-torsion. In this paper we deal with the case when the group G is actually a 2-group.

Keywords: Symmetric Bilinear Form; Linking Pairing; Seifert Manifolds

1. Introduction

Let G be a finite abelian group. A symmetric bilinear form on G is a map $\phi: G\times G \to \mathbb{Q}/\mathbb{Z}$ such that $\phi(x,y)=\phi(y,x)$, where $\phi(x,-)$ is a group homomorphism from G to $\mathbb{Q}/\mathbb{Z}$ for every x and $y\in G$. If $\phi(x,-)$ is not the trivial homomorphism for $x\neq 0$, we say that the form ϕ is nondegenerate. A linking pairing (G,ϕ) on G is a symmetric nondegenerate bilinear form ϕ defined on G.

Let $\mathcal{N}$ be the monoid of isomorphism classes of linking pairings on finite abelian groups under the operation of direct sum. Clearly, G has a primary decomposition of the form

$$(G,\phi)=\left(\oplus_p G_p, \oplus \phi|_{G_p}\right),$$

where for each prime p, G_p is the p-primary group and $\phi|_{G_p}$ is the restriction of ϕ to G_p. Consequently, $\mathcal{N}=\oplus_p \mathcal{N}_p$ is a corresponding decomposition of the monoid $\mathcal{N}$ such that $\mathcal{N}_p$ represents the isomorphism classes of linking pairings on G_p. The problem of classifying the isomorphism classes $\left[(G,\phi)\right]$ of linking pairings is then dependent only on the classification of $\left[\left(G_p,\phi|_{G_p}\right)\right]$, for all primes p.

It was proved in [1] that given a linking pairing (G,ϕ), there exists a closed, connected, oriented 3-manifold M with first integral homology group isomorphic to G, *i.e.*, $H_1(M)\cong G$, whose linking form

$$\lambda: \mathrm{Tors}H^2(M)\otimes \mathrm{Tors}H^2(M)\to \mathbb{Q}/\mathbb{Z}$$

is isomorphic to ϕ. Recall that for an oriented 3-manifold M, the (usual) linking form λ is defined by

$$\lambda(x,y)=\left\langle x\cup B^{-1}y,[M]\right\rangle,$$

for $x,y\in \mathrm{Tors}H^2(M)$, where $\beta: H^1(M;\mathbb{Q}/\mathbb{Z})\to H^2(M)$ denotes the $\mathbb{Q}/\mathbb{Z}$-Bockstein. Equivalently, if $Nx=Ny=0$ in $\mathrm{Tors}H^2(M)$ for some integer $N>1$, then

$$\lambda(x,y)=\frac{1}{N}\left\langle x\cup B_N^{-1}y,[M]\right\rangle,$$

where $B_N: H^1(M;Z/N)\to H^2(M)$ is the mod N Bockstein. It is worth mentioning that in [1], the 3-manifold M, corresponding to (G,ϕ), is a connected sum of the following three types of irreducible 3-manifolds, viz; lens spaces, 3-manifolds for which there is a PL embedding into S^4, and fibres of fibred 2-knots that are embedded in S^4.

It was shown in [2] that an arbitrary linking pairing can be realized as the linking form of an irreducible 3 manifold. Indeed, it was proven that all isomorphism classes of linking pairings of finite abelian groups can be realized as the linking form of a Seifert manifold which is a rational homology sphere.

Since such Seifert manifolds are irreducible, this result would imply that any linking pairing is isomorphic to the linking form of an irreducible 3-manifold. Thus, the linking form of any closed, connected, oriented 3-manifold would be isomorphic to the linking form of a Seifert manifold that is a rational homology sphere.

In the sequel, $M=\left(O,o,0\,|\,e,(a_1,b_1),\cdots,(a_n,b_n)\right)$ will denote a Seifert manifold with oriented orbit surface with genus $g = 0$. In this notation for *M*, *e* denotes the Euler number, *m* the number of singular fibres, and for each $i, (a_i, b_i)$ is a pair of relatively prime integers that characterize the twisting of the *i*-th singular fibre. Although we will follow the notation in [2] closely, we repeat the details here for the sake of clarity.

For any prime *p*, let $\nu_p(B)$ denote the largest power of *p* that divides *B i.e.*, the *p*-valuation of *B*, and set $\nu_p(0) = \infty$. Suppose *s* is the maximal *p*-valuation of the Seifert invariants $a_1, \cdots, a_m$ and *t* is a non negative integer with $0 \le t \le s$. Then for each *t*, let $a_{t,1}, \cdots, a_{t,r_t}$ denote the Seifert invariants satisfying the condition $\nu_p(at, i) = t$, for $1 \le i \le r_t$. This imposes an ordering ordering on the Seifert invariants since $\nu_p(a_{t,i}) < \nu_p(a_{t,l})$ when $t < l$. Hence the invariants and their *p*-valuations can be listed as follows:

$$\begin{array}{cccc} a_{s,1} \cdots a_{s,r_s} & \nu_p(a_{s,i}) = s & \cdots & 1 \le i \le r_s \\ \vdots & \vdots & \ddots & \vdots \\ a_{t,1} \cdots a_{t,r_t} & \nu_p(a_{t,i}) = t & \cdots & 1 \le i \le r_t \\ \vdots & \vdots & \ddots & \vdots \\ a_{s,1} \cdots a_{0,r_0} & \nu_p(a_{0,i}) = 0 & \cdots & 1 \le i \le r_0 \end{array}$$

Now with $n = \sum_{i=1}^{s} r_s$, it is possible to reorder the invariants $a_1, \cdots, a_m$ such that

$$0 \ne \nu_p(a_1) \le \nu_p(a_2) \le \cdots \le \nu_p(a_n)$$

and

$$\nu_p(a_{n+1}) = \nu_p(a_{n+2}) = \cdots = \nu_p(a_m) = 0.$$

Finally, set $A = \prod_{i-1}^{n} a_i, A_j = a_j / A \in \mathbb{Z}$ and $C = \sum b_i A_i$.

For an oriented Seifert manifold

$$M \cong (O, o; 0 | e : (a_1, b_1), \cdots, (a_n, b_n)),$$

abelianization of the fundamental group gives the presentation:

$$H_1(M) \approx \langle s_j, h | a_j s_j + b_j h = 0, \\ \text{for } j = 1, \cdots, n; \Sigma s_j - eh = 0 \rangle.$$

It then follows as in [6] that

$$H_1(M) \cong \begin{cases} \mathbb{Z} \oplus \mathrm{Tors} H_1(M), & \text{if } Ae + C = 0, \\ \mathrm{Tors} H_1(M), & \text{if } Ae + C \ne 0. \end{cases}$$

Thus when $Ae + C \ne 0$, *M* is a rational homology sphere and $H_1(M) \cong H^2(M)$ is a torsion group. Moreover, the universal coefficient theorems now imply that for some integer *q*

$$\mathrm{Tors}_p H_1(M) \cong H_1(M) \otimes Z/p^q \cong H^1(M; Z/p^q).$$

Since there is an orthogonal decomposition of the linking form λ over the p-components of $H^2(M)$ it is now clear that λ has an orthogonal decomposition over the p-torsion groups $H^1(M; Z/p^q)$.

Our main objective in the present note is to consider isomorphism classes of linking pairings on 2-groups topologically, by realizing them as the linking forms on Seifert manifolds that are rational homology spheres. To do this it suffices to show that any linking pairing ϕ on a 2-group has a block sum diagonal form in which the diagonal blocks correspond to the generators of the monoid $\mathcal{N}_2$ (cf. [1]) (see §2 below). It is known [3] and [1] that the linking pairings $E_0(k) = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix}$ and $E_1(k) = \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix}$ of $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ and $(n2^{-k})$, for $k \ge 1$, which is a linking pairing on $Z/2^k$, generate the monoid $\mathcal{N}_2$ completely.

When *G* is a 2-group, the diagonal blocks of ϕ are the generators of the monoid $\mathcal{N}_2$ (see §2 below), and we also give Seifert presentations for all of these generators there. In §3, we compute the block sum decomposition of some classes of linking forms into diagonal blocks consisting of elements of $\mathcal{N}$. The original motivation for this work is related to the abelian WRT-type invariants constructed in [4].

2. Seifert Presentations for the Generators of $\mathcal{N}_2$

For a Seifert manifold

$$M = (O, o, 0 | e, (a_1, b_1), \cdots, (a_n, b_n))$$

that is a rational homology sphere satisfying the condition $Ae + C \ne 0$, there is an integer *c* such that

$$\mathrm{Tors}_p H_1(M) \cong H_1(M) \otimes \mathbb{Z}/p^s \cong H^1(M; \mathbb{Z}/p^c)$$

[2]. The fact that *M* is a rational homology sphere implies that $H_1(M)$ is a torsion group and therefore

$$H_1(M) \cong \oplus_p H^1(M; \mathbb{Z}/p^c) \cong H^2(M).$$

Furthermore, the linking form

$$\lambda : \mathrm{Tors} H^2(M) \otimes \mathrm{Tors} H^2(M) \to \mathbb{Q}/\mathbb{Z}$$

of a closed, connected, oriented 3-manifold *M* was not studied directly. Instead a linking pairing

$$\hat{\lambda}_M^p : H^1(M; \mathbb{Z}/p^c) \otimes H^1(M; \mathbb{Z}/p^c) \to \mathbb{Q}/\mathbb{Z}$$

was defined on $H^1(M; \mathbb{Z}/p^c)$, for every prime *p*, using the product structures in cohomology, by

$$\hat{\lambda}_M^p(x,y) = \frac{1}{p^c}\left\langle x \cup B_{p^c}(y), [M] \right\rangle,$$

for $x, y \in H^1(M; \mathbb{Z}/p^c)$. A linking pairing

$$\hat{\lambda}_M : H^2(M) \otimes H^2(M) \to \mathbb{Q}/\mathbb{Z}$$

was then defined on $H^2(M) \cong H_1(M)$ in terms of the pairings $\hat{\lambda}_M^p$ by setting $\hat{\lambda}_M := \oplus_p \hat{\lambda}_M^p$.

In [2] we proved that, for $p > 2$,, λ_M^p is an arbitrary linking pairing on $\mathrm{Tors}_p H_1(M)$ for arbitrary M. If λ_M^2 can be shown to be an arbitrary linking pairing on 2-groups, then all isomorphism classes of linking pairings on finite abelian groups could be realized by the pairing $\hat{\lambda}_M := \oplus_p \hat{\lambda}_M^p$, for some M. As a consequence, the linking form of any closed, connected, oriented 3-manifold must belong to one of these isomorphism classes.

For any prime p the matrix of the linking pairing $\hat{\lambda}_M^p$ on $H^1(M; \mathbb{Z}/p^c)$ for a Seifert manifold

$$M = (O, o, 0 | e, (a_1, b_1), \cdots, (a_n, b_n))$$

is given strictly in terms of the Seifert invariants. Let Λ^p be the matrix of the linking pairing $\hat{\lambda}_M^p$ with respect to the basis given in Theorem 2 [2]. Then Λ^p has the following form:

$$\Lambda^p = \begin{pmatrix} \Lambda_{1,1} & \Lambda_{1,2} & \cdots & \Lambda_{1,s} \\ \Lambda_{2,1} & \Lambda_{2,2} & \cdots & \Lambda_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{s,1} & \Lambda_{s,2} & \cdots & \Lambda_{s,s} \end{pmatrix}$$

where each $\Lambda_{l,t}$ is an $r_l \times r_t$ matrix defined below, except in the cases when $l = s$ or $t = s$. In these cases it is an $r_s - 1 \times r_t$ or $r_l \times r_s - 1$-matrix respectively. (This follows because there are only $r_s - 1$ generators that arise from level s).

1) When $l \neq t$,

$$\Lambda_{l,t} = \frac{1}{p^{t+l}} \begin{pmatrix} a_{s,1}c_{s,1} & a_{s,1}c_{s,1} & \cdots & a_{s,1}c_{s,1} \\ a_{s,1}c_{s,1} & a_{s,1}c_{s,1} & \cdots & a_{s,1}c_{s,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s,1}c_{s,1} & a_{s,1}c_{s,1} & \cdots & a_{s,1}c_{s,1} \end{pmatrix}$$

2) When $l = t \neq s$,

$$\Lambda_{t,t} = \frac{1}{p^{2t}} \begin{pmatrix} a_{s,1}c_{s,1} + a_{t,1}c_{t,1} & a_{s,1}c_{s,1} & \cdots & a_{s,1}c_{s,1} \\ a_{s,1}c_{s,1} & a_{s,1}c_{s,1} + a_{t,2}c_{t,2} & \cdots & a_{s,1}c_{s,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s,1}c_{s,1} & a_{s,1}c_{s,1} & \cdots & a_{s,1}c_{s,1} + a_{t,r_t}c_{t,r_t} \end{pmatrix}$$

3) When $l = t = s$,

$$\Lambda_{s,s} = \frac{1}{p^{2s}} \begin{pmatrix} a_{s,1}c_{s,1} + a_{s,2}c_{s,2} & a_{s,1}c_{s,1} & \cdots & a_{s,1}c_{s,1} \\ a_{s,1}c_{s,1} & a_{s,1}c_{s,1} + a_{s,3}c_{s,3} & \cdots & a_{s,1}c_{s,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s,1}c_{s,1} & a_{s,1}c_{s,1} & \cdots & a_{s,1}c_{s,1} + a_{s,r_s}c_{s,r_s} \end{pmatrix}$$

This gives the linking matrix for the *p*-component of $H_1(M)$ regardless of the prime p. Thus, it suffices to show that by varying the Seifert invariants of $M = (O, o, 0 | e, (a_1, b_1), \cdots, (a_n, b_n))$, the linking pairing

$$\hat{\lambda}_M^2 : H^1(M; \mathbb{Z}/2^c) \otimes H^1(M; \mathbb{Z}/2^c) \to \mathbb{Q}/\mathbb{Z}$$

defined by $\hat{\lambda}_M^2(x,y) = \frac{1}{2^c}\left\langle x \cup B_{2^c}(y), [M] \right\rangle$ becomes an arbitrary linking pairing. The first step towards this goal is to find Seifert presentations for the generators $(\pm 52^k)$, $E_0(k)$ and $E_1(k)$ of the semi-group $\mathcal{N}_2$.

Remark 1. *In order to find a Seifert presentation for a given linking pairing, we first use the description of the p-torsion of the first integral homology given in* [5] *to find a Seifert manifold whose first integral homology is isomorphic to the underlying group of the linking pairing. Using the cohomology ring structure of the manifold, described in* [6], *we alter the Seifert invariants so that the homology of the manifold remains fixed, so that the resulting linking matrix* (*given above*) *has specific characteristics. The techniques developed in* [7,8] *are then used to find the block sum diagonal form of the linking pairing.*

We now proceed to prove

Theorem 1. *The linking pairings*

$$E_0(k) = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix} : (\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k) \times (\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k) \to \mathbb{Q}/\mathbb{Z},$$

arise from the Seifert presentation

$$M=\left(O,o,0\mid 1,\left(2^k,2^k-1\right),\left(2^k,1\right),\left(2^k,1\right)\right)$$

for all k.

Proof. A special case of the Main Theorem in [5] (which gives a presentation for the Serre p-component of the first integral homology of any Seifert manifold) shows that for the Seifert manfiold

$$M=\left(O,o,0\mid e,\left(a_1,b_1\right),\cdots,\left(a_n,b_n\right)\right)$$

$$\operatorname{Tors}_p H_1(M)=\mathbb{Z}/p^c\oplus\mathbb{Z}/p^{\nu_p(a_1)}\oplus\cdots\oplus\mathbb{Z}/p^{\nu_p(a_{n-2})}.$$

Now reorder the Seifert invariants so that their p-valuations are in ascending order, that is, $\nu_p(a_1)\le\nu_p(a_2)\le\cdots\le\nu_p(a_n)$ and the number c is defined as

$$c=\nu_p(Ae+C)-\nu_p(A)+\nu_p(a_{n-1})+\nu_p(a_n).$$

Applying this theorem to the Seifert manifold

$$M=\left(O,o,0\mid 1,\left(2^k,2^k-1\right),\left(2^k,1\right),\left(2^k,1\right)\right)$$

with $\nu_2(a_1)=\nu_2\left(2^k\right)=k$ and

$$\begin{aligned}c&=\nu_2(Ae+C)-\nu_2(A)+\nu_2(a_2)+\nu_2(a_3)\\&=2k-3k+k+k=k\end{aligned}$$

shows that $\operatorname{Tors}_2 H_1(M)\cong H_1(M)\cong\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k$ as required. The linking matrix can now be computed directly from [2] as follows:

$$\begin{aligned}\Lambda^2&=\frac{1}{2^{2k}}\begin{pmatrix}2^k\left(-2^k+1\right)+2^k(-1) & 2^k\left(-2^k+1\right)\\ 2^k\left(-2^k+1\right) & 2^k\left(-2^k+1\right)+2^k(-1)\end{pmatrix}\\&=\frac{1}{2^{2k}}\begin{pmatrix}2^k\left(-2^k\right) & 2^k\left(-2^k+1\right)\\ 2^k\left(-2^k+1\right) & 2^k\left(2^k\right)\end{pmatrix}=\frac{1}{2^k}\begin{pmatrix}0&1\\1&0\end{pmatrix}=E_0(k).\end{aligned}$$

Theorem 2. *The linking pairing*

$$E_1(k)=\begin{pmatrix}2^{1-k}&2^{-k}\\2^{-k}&2^{1-k}\end{pmatrix}:\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\times\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\to\mathbb{Q}/\mathbb{Z},$$

has Seifert presentation

$$M=\left(O,o,0\mid 1,\left(2^k,2^k-1\right),\left(2^k,2^k-1\right),\left(2^k,2^k-1\right)\right)$$

Proof. Applying Theorem 1 [5] to the Seifert manifold

$$M=\left(O,o,0\mid 1,\left(2^k,2^k-1\right),\left(2^k,2^k-1\right),\left(2^k,2^k-1\right)\right)$$

with $\nu_2(a_1)=\nu_2\left(2^k\right)=k$ and

$$\begin{aligned}c&=\nu_2(Ae+C)-\nu_2(A)+\nu_2(a_2)+\nu_2(a_3)\\&=2k-3k+k+k=k\end{aligned}$$

Gives $\operatorname{Tors}_2 H_1(M)\cong H_1(M)\cong\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k$. Again, the linking matrix can be computed from [2] to give:

$$\begin{aligned}\Lambda^2&=\frac{1}{2^{2k}}\begin{pmatrix}2^k\left(-2^k+1\right)+2^k\left(-2^k+1\right) & 2^k\left(-2^k+1\right)\\ 2^k\left(-2^k+1\right) & 2^k\left(-2^k+1\right)+2^k\left(-2^k+1\right)\end{pmatrix}\\&=\frac{1}{2^{2k}}\begin{pmatrix}2^k(2) & 2^k\left(-2^k+1\right)\\ 2^k\left(-2^k+1\right) & 2^k(2)\end{pmatrix}=\frac{1}{2^k}\begin{pmatrix}2&1\\1&2\end{pmatrix}=E_1(k)\end{aligned}$$

Remark 2. The Seifert presentations given in Theorems 1 and 2 give homeomorphic Seifert manifolds. However, the twisting of the solid tori associated to the singular fibres is not the same and gives rise to different linking pairings.

These theorems allow us to find Seifert presentations for other linking forms.

Example 1. *The Seifert presentation for the linking pairing* $E_0(3)\oplus E_1(3)$ *on* $\left(\mathbb{Z}/8\oplus\mathbb{Z}/8\right)^2$ *is given by*:

$$M=\left(O,o,0\mid 1,\left(2^3,7\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right)\right)$$

Observe that $\nu_2(a_i)=\nu_2\left(2^3\right)=3$ and

$$\begin{aligned}c&=\nu_2(Ae+C)-\nu_2(A)+\nu_2(a_4)+\nu_2(a_5)\\&=12-15+3+3=3,\end{aligned}$$

so by Theorem 1 [5]

$$\operatorname{Tors}_2 H_1(M)\cong H_1(M)\cong\left(\mathbb{Z}/2^3\oplus\mathbb{Z}/2^3\right)^2.$$

The linking matrix in this case is,

$$\Lambda^2=\frac{1}{2^6}\begin{pmatrix}2^3(-7)+2^3(-1) & 2^3(-7) & 2^3(-7) & 2^3(-7)\\ 2^3(-7) & 2^3(-7)+2^3(-1) & 2^3(-7) & 2^3(-7)\\ 2^3(-7) & 2^3(-7) & 2^3(-7)+2^3(-1) & 2^3(-7)\\ 2^3(-7) & 2^3(-7) & 2^3(-7) & 2^3(-7)+2^3(-1)\end{pmatrix}=\frac{1}{8}\begin{pmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{pmatrix}$$

Performing simultaneous row and column operations on Λ^2 shows that

$$\Lambda^2\cong\frac{1}{8}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&-2&1\\0&0&1&-2\end{pmatrix}\cong E_0(3)\oplus E_1(3).$$

Theorem 3. *The linking pairings*

$$\left(\frac{-5}{2^k}\right):\mathbb{Z}/2^k\times\mathbb{Z}/2^k\to\mathbb{Q}/\mathbb{Z},$$

and

$$\left(\frac{5}{2^k}\right):\mathbb{Z}/2^k\times\mathbb{Z}/2^k\to\mathbb{Q}/\mathbb{Z}$$

arise from the Seifert presentations

$$M=\left(O,o,0\,|\,1,\left(2^{k+2},1\right),\left(2^k,1\right)\right)$$

and

$$M=\left(O,o,0\,|\,1,\left(2^{k+2},-1\right),\left(2^k,-1\right)\right)$$

respectively.

Proof. Theorem 1 [5] shows that for each Seifert presentation, $H_1(M)\cong\mathbb{Z}/2^k$. When

$$M=\left(O,o,0\,|\,1,\left(2^{2k},1\right),\left(2^k,1\right)\right),$$

$$\Lambda^2=12^{2k}\left(2^{k+2}(-1)+2^k(-1)\right)$$

$$=12^{2k}\left(2^k(-4-1)\right)=\left(\frac{-5}{2^k}\right)$$

Similarly, when $M=\left(O,o,0\,|\,1,\left(2^{2k},1\right),\left(2^k,1\right)\right)$,

$$\Lambda^2=12^{2k}\left(2^{k+2}(1)+2^k(1)\right)$$

$$=12^{2k}\left(2^k(4+1)\right)=\left(\frac{5}{2^k}\right)$$

□

Remark 3. We can use this result to find other linking pairings. For instance, the linking pairings

$$\left(\frac{-5}{2^k}\right)\oplus\left(\frac{-5}{2^k}\right):\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\times\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\to\mathbb{Q}/\mathbb{Z}$$

$$\left(\frac{5}{2^k}\right)\oplus\left(\frac{5}{2^k}\right):\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\times\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\to\mathbb{Q}/\mathbb{Z}$$

$$\left(\frac{-5}{2^k}\right)\oplus\left(\frac{5}{2^k}\right):\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\times\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)\to\mathbb{Q}/\mathbb{Z}$$

have Seifert presentations

$$M=\left(O,o,0\,|\,1,\left(2^{2k},1\right),\left(2^k,1\right),\left(2^k,1\right)\right),$$

$$M=\left(O,o,0\,|\,1,\left(2^{2k},1\right),\left(2^k,-1\right),\left(2^k,-1\right)\right),$$

$$M=\left(O,o,0\,|\,1,\left(2^{2k},1\right),\left(2^k,1\right),\left(2^k,-1\right)\right),$$

respectively.

For example, the linking matrix on $\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k$ corresponding to the Seifert presentation

$$M=\left(O,o,0\,|\,1,\left(2^{2k},1\right),\left(2^k,1\right),\left(2^k,1\right)\right)$$

is

$$\Lambda^2=\frac{1}{2^{2k}}\begin{pmatrix}2^{2k}(-1)+2^k(-1) & 2^{2k}(-1)\\ 2^{2k}(-1) & 2^{2k}(-1)+2^k(-1)\end{pmatrix}$$

$$=\frac{1}{2^k}\begin{pmatrix}2^k(-1) & 0\\ 0 & 2^k(-1)\end{pmatrix}\cong\frac{1}{2^k}\begin{pmatrix}-5&0\\0&-5\end{pmatrix}$$

$$=\left(\frac{-5}{2^k}\right)\oplus\left(\frac{-5}{2^k}\right)$$

Theorem 4. *The linking pairing*

$$\underbrace{\left(\frac{-5}{2^k}\right)\oplus\cdots\oplus\left(\frac{-5}{2^k}\right)}\oplus\underbrace{\left(\frac{5}{2^k}\right)\oplus\cdots\oplus\left(\frac{5}{2^k}\right)}$$

on $\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)^{s+t}$ *arises from the Seifert presentation*

$$M=\left(O,o,0\,|\,1,\left(2^{2k},1\right),\underbrace{\left(2^k,1\right)\cdots\left(2^k,1\right)}\underbrace{\left(2^k,-1\right)\cdots\left(2^k,-1\right)}\right)$$

Proof. As before Theorem 1 [5] shows that

$$\mathrm{Tors}_2H_1(M)\cong H_1(M)\cong\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)^{s+t}.$$

In this case the linking matrix is,

$$\Lambda^2 = \frac{1}{2^{2k}} \begin{pmatrix} 2^{2k}(-1)+2^k(-1) & 2^{2k}(-1) & \cdots & 2^{2k}(-1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) \\ 2^{2k}(-1) & 2^{2k}(-1)+2^k(-1) & \cdots & 2^{2k}(-1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2^{2k}(-1) & \cdots & 2^{2k}(-1) & 2^{2k}(-1)+2^k(+1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{2k}(-1) & \cdots & 2^{2k}(-1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) & 2^{2k}(-1)+2^k(+1) \end{pmatrix}$$

$$= \frac{1}{2^{2k}} \begin{pmatrix} 2^k(-2^k-1) & 2^{2k}(-1) & \cdots & 2^{2k}(-1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) \\ 2^{2k}(-1) & 2^k(-2^k-1) & \cdots & 2^{2k}(-1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2^{2k}(-1) & \cdots & 2^{2k}(-1) & 2^k(-2^k+1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{2k}(-1) & \cdots & 2^{2k}(-1) & 2^{2k}(+1) & \cdots & 2^{2k}(+1) & 2^k(-2^k+1) \end{pmatrix}$$

$$= \frac{1}{2^k} \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & +1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & +1 \end{pmatrix}$$

which is congruent to the desired result. □

3. Computer-Aided Computations

A complete additive system of invariants for linking pairings on 2-groups was given in [1]. This system of invariants was later described in a slightly modified form in [8] and will be used here.

Let λ denote a linking pairing on a 2-group and let $q_\lambda : G \to \mathbb{Q}/\mathbb{Z}$ denote the quadratic form over λ defined by $q_\lambda(x) = \lambda(x,x)$. Define the Gauss sum associated to q_λ as

$$\Gamma(G, q_\lambda) = |G|^{-12} \sum_{x \in G} e^{2\pi i q_\lambda(x)}.$$

To describe the complete system of invariants (r_2^k, σ_2^k) given in [1] let r_2^k denote the rank of the 2-group G and set

$$\sigma_2^k = \begin{cases} \operatorname{Arg}(\tau_2^k(\lambda)), & \text{if } \tau_2^k(\lambda) \neq 0, \\ \infty, & \text{if } \tau_2^k(\lambda) = 0 \end{cases} \quad (1.3)$$

where $\tau_2^k(\lambda)$ is described in terms of the Gauss sum $\tau_2^k(\lambda) = \Gamma(G, 2^{k-1} q_\lambda)$.

Proposition 1. (Kawauchi-Kojima, [1]) *The series* $\{(r_2^k, \sigma_2^k)\}$, *where k runs over all positive integers, is a complete, minimal, additive system of invariants of linking pairings on 2-groups.*

We now describe how to identify a linking summand on a 2-group. The combinatorial device referred to as an admissible table is introduced in [7] to deal with the fact that the decomposition of a linking pairing on a 2-group is not unique.

A table is a function $T : I \to \mathcal{M}$, which maps an interval, that is, a sequence of consecutive integers, into a monoid $\mathcal{M}$. We will examine tables of the form $T : m \in I \to (r_2(m), \sigma_2(m)) \in \mathbb{N} \times \overline{\mathbb{Z}/8}$. A hole in a table T is an element $m \in I$ such that $r_2(m) = 0$. The set of all holes of I is denoted I^0. The set of all elements $m \in I$ satisfying $r_2(m) \neq 0$ and $\sigma_2(m) \neq \infty$ is denoted I_8. An element of $I^0 \cup I_8$ is called a blank. A table T is called admissible if there is a linking pairing λ on a finite 2-group such that $T(m) = (r_2^k(m), \sigma_2^k(m))$ for all $m \in I$ (cf. [7]).

Observe that the set of tables $\mathcal{T} = \{T : \mathbb{N} \to \mathbb{N} \times \overline{\mathbb{Z}/8}\}$ is a monoid under the obvious operation of addition on tables.

Proposition 2. (Deloup, [7]) *The monoid* $\mathcal{N}_2$ *of isomorphism classes of linking pairings on 2-groups is isomorphic to the monoid* $\mathcal{T}$ *of admissible tables.*

Let λ and λ' be two linking pairings defined on 2-groups.

Proposition 3. (Deloup, [7]) *The linking pairing* λ *has an orthogonal summand* λ' *of order* 2^k *if and only if* $\tau_2^k(\lambda) = 0$.

If λ' is an orthogonal summand of λ, it is clear that

$$r_2^k(\lambda) \geq r_2^k(\lambda') \text{ for all } k \in \mathbb{N}.$$

Furthermore, because of Proposition 2, it is also clear that if $k \in \mathbb{N}$ is a blank for λ, then for λ' to be an orthogonal summand of λ, k must be a blank for λ'.

Assuming that (λ, λ') satisfy condition (1) define $S_{\lambda,\lambda'}$ to be the set of tables

$$\left\{T_S : m \in \mathbb{N}^* \to \left(r_2(k), \sigma_2\right) \in \mathbb{N} \times \overline{\mathbb{Z}/8}\right\}$$

where for each $m \in \mathbb{N}^*$,

$$r_2(m) = r_2^k \lambda - r_2^k(\lambda'),$$

$$\sigma_2(m) = \begin{cases} \sigma_2^k(\lambda) - \sigma_2^k(\lambda'), & \text{if } k \text{ is a blank for } \lambda', \\ \text{any element of } \overline{\mathbb{Z}/8}, & \text{otherwise.} \end{cases}$$

Proposition 4. (Deloup, [7]) *Given two linking pairings λ and λ' defined on 2-groups, λ' is an orthogonal summand of λ if and only if conditions* (1) *and* (2) *hold and there is an admissible table $T_S \in S_{\lambda,\lambda'}$.*

Proposition 4 gives a procedure for determining the block sum diagonal form of an arbitrary linking pairing λ. Firstly, determine the possible combinations of orthogonal sums involving $E_0(k)$ and $E_1(k)$ that can occur in the block sum diagonal form of the linking matrix. Next, use the complete system of invariants $\left\{\left(r_2^k, \sigma_2^k\right)\right\}$ to determine which of these orthogonal sums is isomorphic to λ.

In the following examples we use the procedure given above for determining the block diagonal form of a linking matrix. However, we will only provide the final tables since they give the complete system of invariants for the given linking pairing and therefore determine whether or not two linking pairings are isomorphic.

Example 2. *Some tables of some fundamental linking pairings.*

1) $E_0(3)$

k	1	2	3	4	$\cdots$
r_2^k	0	0	2	0	$\cdots$
σ_2^k	0	0	0	0	$\cdots$

2) $E_1(3)$

k	1	2	3	4	$\cdots$
r_2^k	0	0	2	0	$\cdots$
σ_2^k	0	4	0	0	$\cdots$

3) $E_0(3) \oplus E_1(3)$

k	1	2	3	4	$\cdots$
r_2^k	0	0	4	0	$\cdots$
σ_2^k	0	4	0	0	$\cdots$

4) $E_0(3) \oplus E_0(3) \cong E_1(3) \oplus E_1(3)$

k	1	2	3	4	$\cdots$
r_2^k	0	0	4	0	$\cdots$
σ_2^k	0	0	0	0	$\cdots$

5) $E_0(3) \oplus E_0(3) \oplus E_0(3) \cong E_0(3) \oplus E_1(3) \oplus E_1(3)$

k	1	2	3	4	$\cdots$
r_2^k	0	0	6	0	$\cdots$
σ_2^k	0	0	0	0	$\cdots$

6) $E_0(3) \oplus E_0(3) \oplus E_1(3) \cong E_1(3) \oplus E_1(3) \oplus E_1(3)$

k	1	2	3	4	$\cdots$
r_2^k	0	0	6	0	$\cdots$
σ_2^k	0	4	0	0	$\cdots$

Example 3. *We determine the block sum diagonal form of the matrix of the following linking pairing.*

$$\Lambda^2 = \frac{1}{8}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Simultaneous row and column operations do not yield an immediate diagonalization as in Example 1. In fact, this matrix has three different (but isomorphic) block diagonal representations in terms of the generators $E_0(3)$ and $E_1(3)$ of $\mathcal{N}_2$. This exhibits one of the main difficulties in the algebraic classification of linking pairings on 2-groups.

Observe that the table $T_{(\Lambda^2)}$ for Λ^2 is

k	1	2	3	4	$\cdots$
r_2^k	0	0	6	0	$\cdots$
σ_2^k	0	4	0	0	$\cdots$

This table is identical to the table for $E_0(3) \oplus E_0(3) \oplus E_1(3)$ given in Example 2. Thus

$$\Lambda^2 \cong E_0(3) \oplus E_0(3) \oplus E_1(3)$$
$$\cong E_1(3) \oplus E_1(3) \oplus E_1(3).$$

Example 4. *Consider the Seifert presentation*

$$M=\left(O,o,0\,|\,1,\left(2^3,7\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right),\left(2^3,1\right)\right)$$

The corresponding linking pairing is on $\mathrm{Tors}_2 H_1(M)\cong\left(\mathbb{Z}/2^3\oplus\mathbb{Z}/2^3\right)^4$. It has linking matrix

$$\Lambda^2=\frac{1}{8}\begin{pmatrix}0&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1\\1&1&0&1&1&1&1&1\\1&1&1&0&1&1&1&1\\1&1&1&1&0&1&1&1\\1&1&1&1&1&0&1&1\\1&1&1&1&1&1&0&1\\1&1&1&1&1&1&1&0\end{pmatrix}$$

The table for this matrix is

k	1	2	3	4	$\cdots$
r_2^k	0	0	8	0	$\cdots$
σ_2^k	0	0	0	0	$\cdots$

Comparing this table to the tables given in Example 2 shows that in this case

$$\Lambda^2\cong E_0(3)\oplus E_0(3)\oplus E_1(3)\oplus E_1(3).$$

As it turns out the computation of the invariant $\tau_2^k(\lambda)$ is extremely time consuming. Computer algorithms have been developed to deal with this problem and other problems concerned with finding isomorphisms between different linking pairings. These examples and other computer calculations using more general matrices and the tables for the generators of $\mathcal{N}_2$ easily led to the following more general result.

Theorem 5. 1) *The linking pairing* $E_0(k)\oplus\cdots\oplus E_0(k)\oplus E_1(k)$ *on* $\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)^t$, *when* t *is odd, has Seifert presentation*

$$M=\left(O,o,0\,|\,1,\left(2^k,2^k-1\right),\underbrace{\left(2^k,1\right),\left(2^k,1\right),\cdots,\left(2^k,1\right)}\right).$$

2) *The linking pairing* $E_0(k)\oplus E_0(k)\oplus\cdots\oplus E_0(k)$ *on* $\left(\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\right)^t$, *when* t *is even, has Seifert presentation*

$$M=\left(O,o,0\,|\,1,\left(2^k,2^k-1\right),\underbrace{\left(2^k,1\right),\left(2^k,1\right),\cdots,\left(2^k,1\right)},\right).$$

In conclusion, we have considered isomorphism classes of linking pairings on 2-groups topologically, by realizing them as the linking forms on Seifert manifolds that are rational homology spheres. Indeed, we have presented a procedure to determine if the linking pairing ϕ on a 2-group has a block sum diagonal form in which the diagonal blocks correspond to the generators of the monoid $\mathcal{N}_2$ and as such detecting isomorphic linking forms.

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