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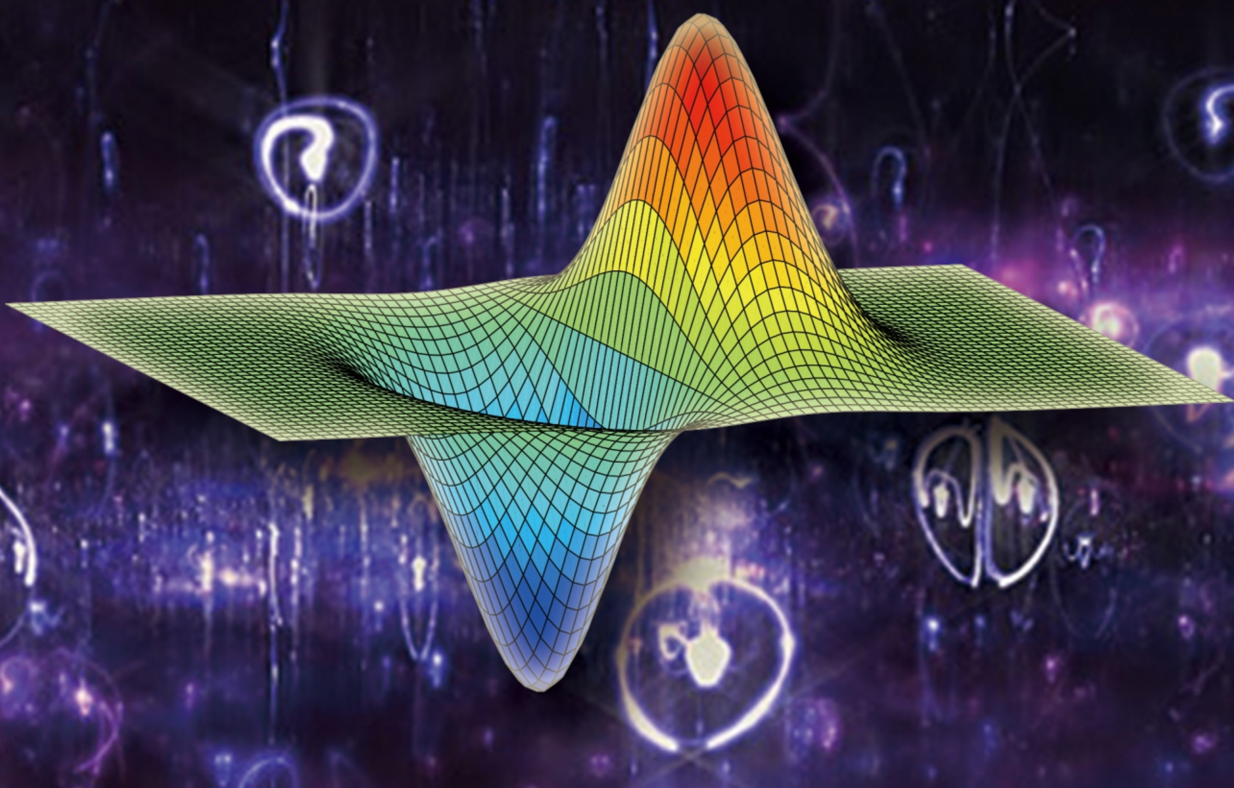

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An Algebraic Proof of the Associative Law of Elliptic Curves

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Abstract

In this paper we revisit the addition of elliptic curves and give an algebraic proof to the associative law by use of MATHEMATICA. The existing proofs of the associative law are rather complicated and hard to understand for beginners. An "elementary" proof to it based on algebra has not been given as far as we know. Undergraduates or non-experts can master the addition of elliptic curves through this paper. After mastering it they should challenge the elliptic curve cryptography.

Keywords

Elliptic Curve, Addition, Associative Law, MATHEMATICA, Elliptic Curve Cryptography

1. Introduction

Ciphering is essential for the security of internet. The RSA cryptography [1] [2] [3] is now commonly used. However, in the very near future the RSA cryptography will be replaced by the elliptic curve cryptography because of its efficiency; the RSA system is based on 2048 bits, while the elliptic system is based on 224 bits (2016, [4]).

The target reader of this note is undergraduates or non-experts. Those who are interested in cryptography are strongly encouraged to master the theory of elliptic curve cryptography as soon as possible. For this purpose they must study an additional structure of elliptic curves. However, it is not so hard except for **the associative law**.

As far as we know an algebraic proof to it has not yet been given[1]. Therefore, we give an "elementary" proof by use of MATHEMATICA for them.

[1]We don't admit usual geometric proofs in standard textbooks of elliptic curves.

2. Addition of Points of an Elliptic Curve

Let us start by recalling the definition of an elliptic curve [5] [6]

$$y^2 = x^3 + ax + b \tag{1}$$

where a and b are some real constants. In the following we consider only real category. The discriminant of the cubic equation

$$x^3 + ax + b = 0$$

is given by

$$D = -4a^3 - 27b^2 \tag{2}$$

(see for example [5]) and we assume $D < 0$ in the following, so the point crossing the real axis is just one.

For the graph of the elliptic curve (1)

$$E = \left\{ (x, y) \in \mathbf{R}^2 \mid y^2 = x^3 + ax + b \right\} \tag{3}$$

we want to introduce an addition, which is essential in the elliptic curve cryptography. For the purpose we must add the infinity point $O = (\infty, \infty)$ to (3). As a result, our space is not $\mathbf{R}^2$ but a two dimensional sphere $\mathbf{R}^2 \cup O = \mathbf{S}^2$. Later it turns out that O is the identity element of the addition, see (10), (11). This justifies the notation O for the infinity point.

Here we note

$$P = (x, y) \in E \Rightarrow -P = (x, -y) \in E \tag{4}$$

where we have adopted the notation $-P$ for the mirror image of P with respect to the real axis, see (11).

Let us introduce the addition in E. For two points $P_1, P_2 \in E$ we associate another point $P_3 \in E$. Consider the straight line passing through P_1 and P_2. We set R the crossing point of the line and the elliptic curve.

A simple-minded candidate of the addition is

$$P_1 \oplus P_2 = R$$

Unfortunately, this is not good because the associative law does not hold. Instead, we take the reflection point of R

$$P_1 \oplus P_2 = -R \equiv P_3. \tag{5}$$

This is correct as shown in the paper. See the following **Figure 1**.

Next, we want to express the addition above by use of the coordinate system. For the purpose we set

$$P_1 = (x_1, y_1),\ P_2 = (x_2, y_2) \text{ and } P_3 = (x_3, y_3).$$

Formula The addition formula

$$(x_1, y_1) \oplus (x_2, y_2) = (x_3, y_3)$$

is given by

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 - (x_1 + x_2),$$

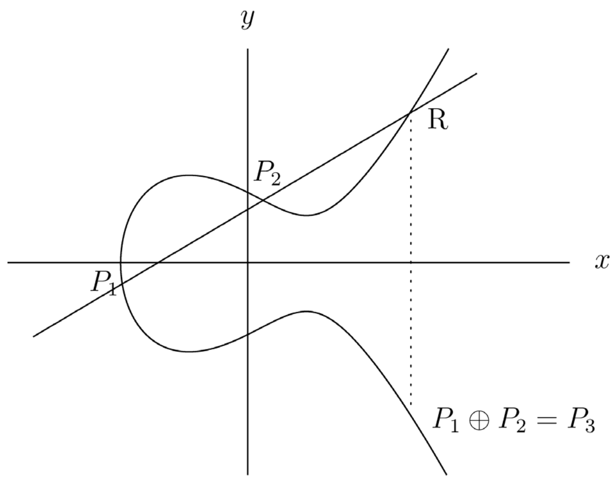

Figure 1. Addition $P_1 \neq P_2$.

$$y_3 = -\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^3 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(2x_1 + x_2) - y_1. \tag{6}$$

Proof To give an elementary proof for undergraduates or non-experts is educational.

First of all we set the coordinate of the point $R = (x_r, y_r)$ and look for x_r and y_r. The straight line passing through P_1 and P_2 is given by

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.$$

By taking $x - x_1$ into consideration we have

$$\begin{aligned} y^2 &= x^3 + ax + b \\ &= (x - x_1 + x_1)^3 + a(x - x_1 + x_1) + b \\ &= (x - x_1)^3 + 3(x - x_1)^2 x_1 + 3(x - x_1)x_1^2 + a(x - x_1) + x_1^3 + ax_1 + b \\ &= (x - x_1)^3 + 3(x - x_1)^2 x_1 + 3(x - x_1)x_1^2 + a(x - x_1) + y_1^2. \end{aligned}$$

We substitute the straight line for the equation above

$$\begin{aligned} &\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 (x - x_1)^2 + 2\frac{y_2 - y_1}{x_2 - x_1}(x - x_1)y_1 + y_1^2 \\ &= (x - x_1)^3 + 3(x - x_1)^2 x_1 + 3(x - x_1)x_1^2 + a(x - x_1) + y_1^2. \end{aligned}$$

A short calculation gives

$$\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 (x - x_1) + 2\frac{y_2 - y_1}{x_2 - x_1}y_1 = (x - x_1)^2 + 3x_1(x - x_1) + 3x_1^2 + a$$

and

$$(x - x_1)^2 - \left\{\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - 3x_1\right\}(x - x_1) + 3x_1^2 - 2\frac{y_2 - y_1}{x_2 - x_1}y_1 + a = 0.$$

This is a quadratic equation and it is easy to solve

$$x-x_1=\frac{1}{2}\left\{\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-3x_1\pm\sqrt{\left\{\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-3x_1\right\}^2-4\left(3x_1^2-2\frac{y_2-y_1}{x_2-x_1}y_1+a\right)}\right\}.$$

Here we set

$$(\#)=\left\{\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-3x_1\right\}^2-4\left(3x_1^2-2\frac{y_2-y_1}{x_2-x_1}y_1+a\right).$$

By expanding and arranging $(\#)$ we have

$$(\#)=\left(\frac{y_2-y_1}{x_2-x_1}\right)^4-6x_1\left(\frac{y_2-y_1}{x_2-x_1}\right)^2+8\frac{y_2-y_1}{x_2-x_1}y_1-3x_1^2-4a.$$

Some calculation (this is a key point) gives

$$\begin{aligned}
(\#)&=\left(\frac{y_2-y_1}{x_2-x_1}\right)^4-6x_1\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-4\frac{(y_2-y_1)^2}{x_2-x_1}\\
&\quad+4\frac{(y_2-y_1)^2}{x_2-x_1}+8\frac{y_2-y_1}{x_2-x_1}y_1-3x_1^2-4a\\
&=\left(\frac{y_2-y_1}{x_2-x_1}\right)^4-\left\{6x_1+4(x_2-x_1)\right\}\left(\frac{y_2-y_1}{x_2-x_1}\right)^2\\
&\quad+4\frac{(y_2-y_1)\left\{(y_2-y_1)+2y_1\right\}}{x_2-x_1}-3x_1^2-4a\\
&=\left(\frac{y_2-y_1}{x_2-x_1}\right)^4-2(2x_2+x_1)\left(\frac{y_2-y_1}{x_2-x_1}\right)^2+4\frac{y_2^2-y_1^2}{x_2-x_1}-3x_1^2-4a\\
&=\left(\frac{y_2-y_1}{x_2-x_1}\right)^4-2(2x_2+x_1)\left(\frac{y_2-y_1}{x_2-x_1}\right)^2+4\left(x_2^2+x_2x_1+x_1^2+a\right)-3x_1^2-4a\\
&=\left(\frac{y_2-y_1}{x_2-x_1}\right)^4-2(2x_2+x_1)\left(\frac{y_2-y_1}{x_2-x_1}\right)^2+4x_2^2+4x_2x_1+x_1^2\\
&=\left(\frac{y_2-y_1}{x_2-x_1}\right)^4-2(2x_2+x_1)\left(\frac{y_2-y_1}{x_2-x_1}\right)^2+(2x_2+x_1)^2\\
&=\left\{\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-2x_2-x_1\right\}^2
\end{aligned}$$

where in the process we have used the equation

$$\begin{aligned}
y_2^2-y_1^2&=\left(x_2^3+ax_2+b\right)-\left(x_1^3+ax_1+b\right)\\
&=(x_2-x_1)\left(x_2^2+x_2x_1+x_1^2+a\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
x-x_1&=\frac{1}{2}\left\{\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-3x_1+\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-2x_2-x_1\right\}\\
&=\frac{1}{2}\left\{2\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-4x_1-2x_2\right\}=\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-(2x_1+x_2)
\end{aligned}$$

and we finally obtain

$$x_r = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - (x_1 + x_2),$$

which is symmetric in 1 and 2. Another solution is $x = x_2$ (check this).

This gives

$$\begin{aligned} y_r &= \frac{y_2 - y_1}{x_2 - x_1}(x_r - x_1) + y_1 \\ &= \frac{y_2 - y_1}{x_2 - x_1}\left\{\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - (2x_1 + x_2)\right\} + y_1 \\ &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^3 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(2x_1 + x_2) + y_1. \end{aligned}$$

As a result we have

$$(x_3, y_3) = (x_r, -y_r)$$

and this gives the Formula (6).

Comment From the geometric definition of the addition (5) it is easy to see the commutativity

$$P_1 \oplus P_2 = P_2 \oplus P_1.$$

However, it is not clear to see this from the Formula (6). Then, a small change of y_3 in (6) gives

$$y_3 = -\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^3 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 + x_2) + \frac{y_2 x_1 - y_1 x_2}{x_2 - x_1}, \tag{7}$$

which is anti-symmetric in 1 and 2. The commutativity is very clear. In our opinion this formula is best.

Next, we must define the addition $P \oplus P$ of the same point P. The definition is usually performed by differential. By noting

$$\lim_{2 \to 1} \frac{y_2 - y_1}{x_2 - x_1} = y_1'$$

the differential of $y^2 = x^3 + ax + b$ at (x_1, y_1) gives

$$2y_1 y_1' = 3x_1^2 + a \Rightarrow y_1' = \frac{3x_1^2 + a}{2y_1}.$$

If we set for $P(x, y)$

$$P \oplus P = P_3 \ \text{ or } \ (x, y) \oplus (x, y) = (x_3, y_3) \tag{8}$$

then we obtain

$$x_3 = \left(\frac{3x^2 + a}{2y}\right)^2 - 2x,$$

$$y_3 = -\left(\frac{3x^2 + a}{2y}\right)^3 + \left(\frac{3x^2 + a}{2y}\right)3x - y \tag{9}$$

by applying the argument above to (6). See the following **Figure 2**.

There are tasks left behind. Our tasks are to show

$$P \oplus O = O \oplus P = P \tag{10}$$

and

$$P \oplus (-P) = (-P) \oplus P = O. \tag{11}$$

Exercise Consider a proof with the geometric method.

Last, we must prove the associative law

$$(P_1 \oplus P_2) \oplus P_3 = P_1 \oplus (P_2 \oplus P_3), \tag{12}$$

which is very hard for undergraduates (hard even for experts).

The geometric method usually goes like **Figure 3** ($P_1 = P$, $P_2 = Q$ and $P_3 = R$ in this figure)

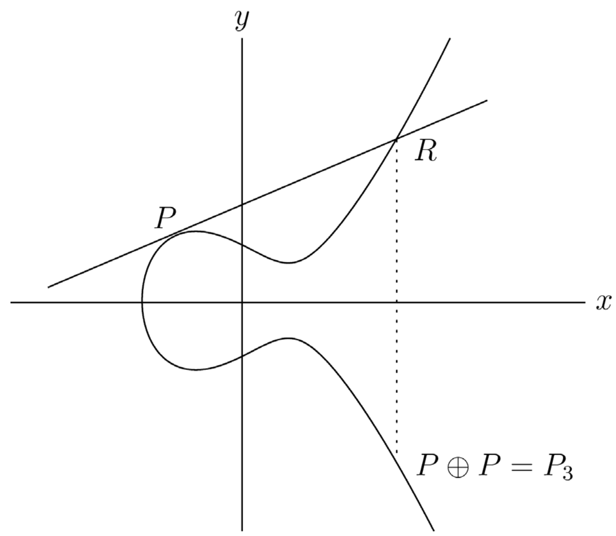

Figure 2. Addition $P_1 = P_2 = P$.

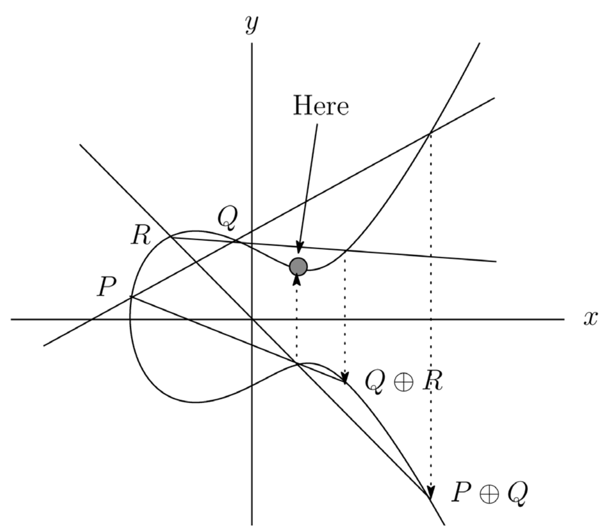

Figure 3. Associativity $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$.

However, this is not a proof but a circumstantial evidence. Therefore, we give an algebraic proof by use of MATHEMATICA[2].

For the purpose let us calculate the difference

$$\left(P_1 \oplus P_2\right) \oplus P_3 - P_1 \oplus \left(P_2 \oplus P_3\right) \tag{13}$$

by MATHEMATICA. In the following program we set

$$\left(P_1 \oplus P_2\right) \oplus P_3 - P_1 \oplus \left(P_2 \oplus P_3\right) = \left(CC - FF, DD - GG\right). \tag{14}$$

and use the Formula (7) because of its high symmetry. Associativity holds when the right hand side vanishes.

Beginning of MATHEMATICA

Readers must input and execute the following program in standard form of MATHEMATICA.

We set

$$s = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - \left(x_1 + x_2\right);$$

$$t = -\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^3 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(x_1 + x_2\right) + \frac{\mathrm{Det}\left[\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\right]}{x_2 - x_1};$$

and

$$CC = \left(\frac{y_3 - t}{x_3 - s}\right)^2 - \left(s + x_3\right);$$

$$DD = -\left(\frac{y_3 - t}{x_3 - s}\right)^3 + \left(\frac{y_3 - t}{x_3 - s}\right)\left(s + x_3\right) + \frac{\mathrm{Det}\left[\begin{pmatrix} s & x_3 \\ t & y_3 \end{pmatrix}\right]}{x_3 - s};$$

and also set

$$u = \left(\frac{y_3 - y_2}{x_3 - x_2}\right)^2 - \left(x_2 + x_3\right);$$

$$v = -\left(\frac{y_3 - y_2}{x_3 - x_2}\right)^3 + \left(\frac{y_3 - y_2}{x_3 - x_2}\right)\left(x_2 + x_3\right) + \frac{\mathrm{Det}\left[\begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix}\right]}{x_3 - x_2};$$

and

$$FF = \left(\frac{v - y_1}{u - x_1}\right)^2 - \left(x_1 + u\right);$$

$$GG = -\left(\frac{v - y_1}{u - x_1}\right)^3 + \left(\frac{v - y_1}{u - x_1}\right)\left(x_1 + u\right) + \frac{\mathrm{Det}\left[\begin{pmatrix} x_1 & u \\ y_1 & v \end{pmatrix}\right]}{u - x_1}.$$

[2]We expect that undergraduates in the world can use MATHEMATICA or MAPLE, etc.

Moreover, we set

$$P = (y_1 - y_2)^2 - (x_1 - x_2)^2 (x_1 + x_2 + x_3);$$

$$Q = (y_2 - y_3)^2 - (x_2 - x_3)^2 (x_1 + x_2 + x_3);$$

$$\begin{aligned} R = {} & (x_2 - x_3) y_1^2 + (x_3 - x_1) y_2^2 + (x_1 - x_2) y_3^2 \\ & + (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)(x_1 + x_2 + x_3). \end{aligned}$$

Here, P^2 (Q^2) appears in the denominator of CC (FF) and P^3 (Q^3) in the denominator of DD (GG). The homogeneous polynomials P and Q are invariant under the permutation of $1,2,3$, whereas R changes sign.

For

$$AA = \frac{P^2 Q^2 (CC - FF)}{R}; \quad BB = \frac{P^3 Q^3 (DD - GG)}{R};$$

execute the following

$$\text{Factor}[AA]$$

$$\text{Factor}[BB]$$

Ending of MATHEMATICA

It takes about several seconds for a standard present day PC before MATHEMATICA outputs two huge *homogeneous polynomials* in x_1, x_2, x_3, y_1, y_2 and y_3 of *integer coefficients*. The "degrees" of AA and BB are 9 and 31/2, respectively, when "degree" 1 is assigned to x_1, x_2, x_3 and 3/2 for y_1, y_2 and y_3, see the curve Equation (1). In other words, AA and BB are universal polynomials of elliptic curves which are independent of the parameters *a* and *b*. More than 10 pages are required to write down the outputs. As we will see their explicit forms are irrelevant for the discussion of the associativity, we do not display them here. These polynomials have many interesting features.

From the program we have

$$CC - FF = \frac{AA}{P^2 Q^2} R, \quad DD - GG = \frac{BB}{P^3 Q^3} R. \tag{15}$$

It is very interesting and important that both have a common factor R. Note that we have not imposed the equations

$$\begin{cases} y_1^2 = x_1^3 + a x_1 + b \\ y_2^2 = x_2^3 + a x_2 + b \\ y_3^2 = x_3^3 + a x_3 + b \end{cases} \tag{16}$$

up to this point.

Last, we show

$$R = 0 \tag{17}$$

under the condition (16), which finishes the proof of associativity (14).

Here, let us give an educational proof for undergraduates. We treat the following determinant :

$$X=\begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1^2 & y_2^2 & y_3^2\end{vmatrix} \tag{18}$$

Direct calculation gives

$$\begin{aligned} X &= x_2y_3^2+x_3y_1^2+x_1y_2^2-x_2y_1^2-x_1y_3^2-x_3y_2^2 \\ &= -\left\{(x_2-x_3)y_1^2+(x_3-x_1)y_2^2+(x_1-x_2)y_3^2\right\}. \end{aligned} \tag{19}$$

On the other hand, from (16) we have

$$\begin{aligned} X &= \begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ x_1^3+ax_1+b & x_2^3+ax_2+b & x_3^3+ax_3+b\end{vmatrix} \\ &= \begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ x_1^3+ax_1 & x_2^3+ax_2 & x_3^3+ax_3\end{vmatrix} \\ &= \begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ x_1^3 & x_2^3 & x_3^3\end{vmatrix} \end{aligned}$$

by some fundamental operations.

Moreover, we have

$$\begin{aligned} X &= \begin{vmatrix}1 & 0 & 0\\ x_1 & x_2-x_1 & x_3-x_1\\ x_1^3 & x_2^3-x_1^3 & x_3^3-x_1^3\end{vmatrix} \\ &= (x_2-x_1)(x_3-x_1)\begin{vmatrix}1 & 0 & 0\\ x_1 & 1 & 1\\ x_1^3 & x_2^2+x_2x_1+x_1^2 & x_3^2+x_3x_1+x_1^2\end{vmatrix} \\ &= (x_2-x_1)(x_3-x_1)\begin{vmatrix}1 & 0 & 0\\ x_1 & 1 & 0\\ x_1^3 & x_2^2+x_2x_1+x_1^2 & (x_3-x_2)(x_3+x_2+x_1)\end{vmatrix} \\ &= (x_2-x_1)(x_3-x_1)(x_3-x_2)(x_3+x_2+x_1) \\ &= (x_1-x_2)(x_2-x_3)(x_3-x_1)(x_1+x_2+x_3) \end{aligned} \tag{20}$$

by some fundamental operations. As a result, we obtain

$$\begin{aligned} R &= (x_2-x_3)y_1^2+(x_3-x_1)y_2^2+(x_1-x_2)y_3^2 \\ &\quad +(x_1-x_2)(x_2-x_3)(x_3-x_1)(x_1+x_2+x_3) \\ &= -X+X=0 \end{aligned}$$

by (19) and (20).

As shown in the paper the elementary proof of the associative law of the points of an elliptic curve is not easy. However, it is not necessarily a bad thing for the encryption system.

In this section we reproved the following

Theorem The system $\{E,\oplus\}$ becomes an additive (abelian) group.

3. Concluding Remarks

We conclude the paper by making some comments on the elliptic curve cryptography [7] [8].

Let p be a huge prime number and $\mathbf{F}_p$ be the finite field

$$\mathbf{F}_p = \{0,1,2,\cdots,p-1\},$$

see for example [5].

Our target is an elliptic curve on $\mathbf{F}_p$

$$E_p = \left\{(x,y) \mid y^2 = x^3 + ax + b \pmod{p}\right\}.$$

For this case E_p becomes a finite set. We assume that P and $Q \in E_p$ satisfy the relation

$$Q = n_{\oplus} P \pmod{p}$$

where

$$n_{\oplus} P = P \oplus P \oplus \cdots \oplus P \ (n\text{-times}).$$

Problem For given P and Q is it possible to find n in polynomial time?

This is called the **discrete logarithm problem** and it is known as a very hard one to solve [9]. The security of the elliptic curve cryptography (which is worth studying for undergraduates or non-experts) is based on this hard problem.

Acknowledgements

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Successive Approximation of p-Class Towers

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Abstract

Let F be a number field and p be a prime. In the successive approximation theorem, we prove that, for each integer $n \geq 1$, finitely many candidates for the Galois group $\mathrm{G}_p^n F$ of the nth stage $F_p^{(n)}$ of the p-class tower $F_p^{(\infty)}$ over F are determined by abelian type invariants of p-class groups $\mathrm{Cl}_p E$ of unramified extensions E/F with degree $[E:F] = p^{n-1}$. Illustrated by the most extensive numerical results available currently, the transfer kernels $\ker(T_{F,E})$ of the p-class extensions $T_{F,E}: \mathrm{Cl}_p F \to \mathrm{Cl}_p E$ from F to unramified cyclic degree-p extensions E/F are shown to be capable of narrowing down the number of contestants significantly. By determining the isomorphism type of the maximal subgroups $S < G$ of all 3-groups G with coclass $\mathrm{cc}(G) = 1$, and establishing a general theorem on the connection between the p-class towers of a number field F and of an unramified abelian p-extension E/F, we are able to provide a theoretical proof of the realization of certain 3-groups S with maximal class by 3-tower groups $\mathrm{G}_3^\infty E$ of dihedral fields E with degree 6, which could not be realized up to now.

Keywords

p-Class Towers, Galois Groups, Second p-Class Groups, Abelian Type Invariants of p-Class Groups, p-Transfer Kernel Types, Artin Limit Pattern, Quadratic Fields, Unramified Cyclic Extensions of Degree p, Dihedral Fields of Degree $2p$, Finite p-Groups, Maximal Nilpotency Class, Maximal Subgroups, Polycyclic Pc-Presentations, Commutator Calculus, Central Series

1. Introduction

For a prime number p and an algebraic number field F, let $F_p^{(\infty)}$ be the *p-class tower*, more precisely the unramified Hilbert p-class field tower, that is the maximal unramified pro-p extension, of F. The individual stages $F_p^{(n)}$ and the

Galois groups $\mathrm{Gal}\left(F_p^{(n)}/F\right)$ of the tower

$$F = F_p^{(0)} \le F_p^{(1)} \le F_p^{(2)} \le \cdots \le F_p^{(n)} \le \cdots \le F_p^{(\infty)}$$

are described by the derived quotients $\mathfrak{G}/\mathfrak{G}^{(n)} \simeq \mathrm{G}_p^n F := \mathrm{Gal}\left(F_p^{(n)}/F\right)$ with $n \ge 1$, of the *p-class tower group* $\mathfrak{G} := \mathrm{G}_p^\infty F := \mathrm{Gal}\left(F_p^{(\infty)}/F\right)$. The purpose of this paper is to report on the most up-to-date theoretical view of *p*-class towers and the state of the art of actual numerical investigations. After a summary of algebraic and arithmetic foundations in §2, four crucial concepts will illuminate recent innovation and progress in a very ostensive way:

- the *Artin limit pattern* $\left(\tau^{(\infty)}F, \varkappa^{(\infty)}F\right)$ of the *p*-class tower $F_p^{(\infty)}$ in §3,
- *successive approximation* and the current status of computational perspectives in §4,
- *maximal subgroups* of 3-class tower groups with coclass one in §5, and
- the realization of new 3-class tower groups over *dihedral fields* in §6.

2. Algebraic and Arithmetic Foundations

2.1. Abelian Type Invariants

First, we recall the concepts of abelian type invariants and abelian quotient invariants in the context of finite *p*-groups and infinite pro-*p* groups, and we specify our conventions in their notation.

Let $p \ge 2$ be a prime number. It is well known that a finite abelian group A with order $|A|$ a power of *p* possesses a *unique* representation

$$A \simeq \oplus_{i=1}^{s}\left(\mathbb{Z}/p^{e_i}\mathbb{Z}\right)^{r_i} \tag{2.1}$$

as a direct sum with integers $s \ge 0$, $r_i \ge 1$ for $1 \le i \le s$, and *strictly decreasing* $e_1 > \cdots > e_s \ge 1$.

Definition 2.1 *The abelian type invariants of* A *are given either in power form,*

$$\mathrm{ATI}(A) := \left[\overbrace{p^{e_1}, \cdots, p^{e_1}}^{r_1 \text{ times}}, \cdots, \overbrace{p^{e_s}, \cdots, p^{e_s}}^{r_s \text{ times}}\right], \tag{2.2}$$

or in *logarithmic form* with formal exponents indicating iteration,

$$\mathrm{ATI}(A) := \left[e_1^{r_1}, \cdots, e_s^{r_s}\right]. \tag{2.3}$$

Let G be a pro-*p* group with commutator subgroup G' and *finite* abelianization $G^{ab} := G/G'$.

Definition 2.2 *The abelian quotient invariants of G are the abelian type invariants of the biggest abelian quotient of G*

$$\mathrm{AQI}(G) := \mathrm{ATI}\left(G^{ab}\right). \tag{2.4}$$

2.1.1. Higher Abelian Quotient Invariants of a Pro-*p* Group

Within the frame of group theory, abelian quotient invariants of *higher order* are defined recursively in the following manner.

Definition 2.3 *The set of all maximal subgroups of G which contain the commutator subgroup,*

$$\mathrm{Lyr}_1 G := \{ S \lhd G \mid G' \le S, (G:S) = p \}, \tag{2.5}$$

is called the *first layer* of subgroups of *G*. For any positive integer $n \ge 1$, *abelian quotient invariants of nth order* of *G* are defined recursively by

$$\tau^{(1)} G := \mathrm{AQI}(G) \text{ and } \tau^{(n)} G := \left(\tau^{(1)} G; \left(\tau^{(n-1)} S \right)_{S \in \mathrm{Lyr}_1 G} \right) \text{ for } n \ge 2. \tag{2.6}$$

2.1.2. Higher Abelian Type Invariants of a Number Field

Within the frame of algebraic number theory, abelian type invariants of *higher order* are defined recursively in the following way.

Let *F* be an algebraic number field, denote by $\mathrm{Cl}_p F$ the *p*-class group of *F*, and by $F_p^{(1)}$ the first Hilbert *p*-class field of *F*, that is, the maximal abelian unramified *p*-extension of *F*.

Definition 2.4 *The set of all unramified cyclic extensions* E/F *of degree p which are contained in the p-class field,*

$$\mathrm{Lyr}_1 F := \{ E > F \mid E \le F_p^{(1)}, [E:F] = p \} \tag{2.7}$$

is called the *first layer* of extension fields of *F*. For any positive integer $n \ge 1$, *abelian type invariants of nth order* of *F* are defined recursively by

$$\tau^{(1)} F := \mathrm{ATI}(\mathrm{Cl}_p F) \text{ and } \tau^{(n)} F := \left(\tau^{(1)} F; \left(\tau^{(n-1)} E \right)_{E \in \mathrm{Lyr}_1 F} \right) \text{ for } n \ge 2. \tag{2.8}$$

2.2. Transfer Kernel Type

Next, we explain the concept of transfer kernel type of finite *p*-groups and infinite pro-*p* groups.

2.2.1. Transfer Kernel Type of a Pro-*p* Group

Denote by $p \ge 2$ a prime number. Let *G* be a pro-*p* group with commutator subgroup G' and *finite* abelianization $G^{ab} = G/G'$.

Definition 2.5 *By the transfer kernel type of* G, *we understand the finite family of kernels,*

$$\varkappa(G) := \left(\ker(T_{G,S}) \right)_{S \in \mathrm{Lyr}_1 G}, \tag{2.9}$$

where $T_{G,S} : G/G' \to S/S'$ denotes the transfer homomorphism from *G* to the normal subgroup *S* of finite index $(G:S) = p$, as given in Formula (3.1).

More specifically, suppose that $G^{ab} \simeq C_p \times C_p$ is elementary abelian of rank two. Then $\mathrm{Lyr}_1 G$ has $p+1$ elements $S_1, \cdots, S_{p+1}$, the transfer kernel type of *G* is described briefly by a family of non-negative integers

$\varkappa(G) = (\varkappa_i)_{1 \le i \le p+1} \in [0, p+1]^{p+1}$ such that

$$\varkappa_i := \begin{cases} 0 & \text{if } \ker(T_{G,S_i}) = G/G', \\ j & \text{if } \ker(T_{G,S_i}) = S_j/G' \text{ for some } 1 \le j \le p+1, \end{cases} \tag{2.10}$$

and the symmetric group S_{p+1} of degree $p+1$ acts on $[0,p+1]^{p+1}$ via $\varkappa \mapsto \varkappa^{\pi} := \pi_0^{-1} \circ \varkappa \circ \pi$, for each $\pi \in S_{p+1}$, where the extension π_0 of π to $[0,p+1]$ fixes the zero.

Definition 2.6 *The orbit* $\varkappa(G)^{S_{p+1}}$ *is called the invariant type of G, but it is actually given by one of the orbit representatives* $(\varkappa_i)_{1\le i\le p+1}$. *Any two distinct orbit representatives* $\lambda_1,\lambda_2 \in \varkappa(G)^{S_{p+1}}$ *are called equivalent, denoted by the symbol* $\lambda_1 \sim \lambda_2$.

2.2.2. Transfer Kernel Type of a Number Field

Let F be an algebraic number field, and denote by $\mathrm{Cl}_p F$ the p-class group of F.

Definition 2.7 *By the transfer kernel type of F, we understand the finite family of kernels,*

$$\varkappa(F) := \left(\ker\left(T_{F,E}\right)\right)_{E\in \mathrm{Lyr}_1 F}, \tag{2.11}$$

where $T_{F,E} : \mathrm{Cl}_p F \to \mathrm{Cl}_p E$ denotes the transfer of p-classes from F to the unramified cyclic extension E of degree $[E:F]=p$, which is also known as the p-class extension homomorphism.

More specifically, suppose that $\mathrm{Cl}_p F \simeq C_p \times C_p$ is elementary abelian of rank two. Then $\mathrm{Lyr}_1 F$ has $p+1$ elements $E_1,\cdots,E_{p+1}$, the transfer kernel type of F is described briefly by a family of non-negative integers $\varkappa(F) = (\varkappa_i)_{1\le i\le p+1} \in [0,p+1]^{p+1}$ such that

$$\varkappa_i := \begin{cases} 0 & \text{if } \ker\left(T_{F,E_i}\right) = \mathrm{Cl}_p F, \\ j & \text{if } \ker\left(T_{F,E_i}\right) = \mathrm{Norm}_{E_j/F}\left(\mathrm{Cl}_p E_j\right) \text{ for some } 1\le j\le p+1, \end{cases} \tag{2.12}$$

and the symmetric group S_{p+1} of degree $p+1$ acts on $[0,p+1]^{p+1}$ via $\varkappa \mapsto \varkappa^{\pi} := \pi_0^{-1} \circ \varkappa \circ \pi$, for each $\pi \in S_{p+1}$, where the extension π_0 of π to $[0,p+1]$ fixes the zero.

Definition 2.8 *The orbit* $\varkappa(F)^{S_{p+1}}$ *is called the invariant type of F, but it is actually given by one of the orbit representatives* $(\varkappa_i)_{1\le i\le p+1}$. *Any two distinct orbit representatives* $\lambda_1,\lambda_2 \in \varkappa(F)^{S_{p+1}}$ *are called equivalent, denoted by the symbol* $\lambda_1 \sim \lambda_2$.

3. The Artin Limit Pattern

Let p be a prime number. For the recursive construction of the Artin limit pattern of a pro-p group G with commutator subgroup G' and *finite* abelianization $G^{ab} = G/G'$, we need the following considerations.

3.1. Mappings of the Artin Limit Pattern

Due to our assumptions, the first layer $\mathrm{Lyr}_1 G$ of subgroups of G is a finite set consisting of maximal normal subgroups S of G with abelian quotients G/S. Consequently, the *Artin transfer homomorphism* from G to $S \in \mathrm{Lyr}_1 G$ is distinguished by a very simple mapping law:

$$T_{G,S}: G/G' \to S/S',\ g \cdot G' \mapsto \begin{cases} g^p \cdot S' & \text{if } g \in (G/G') \setminus (S/G'), \\ g^{1+h+h^2+\cdots+h^{p-1}} \cdot S' & \text{if } g \in S/G', \end{cases} \tag{3.1}$$

where h denotes an arbitrary element in $G \setminus S$ ([1], 4.1, p. 76).

The Artin limit pattern encapsulates particular group theoretic information (connected with Artin transfers) about the lattice of subgroups of G, where each element U has at least one predecessor, except the root G itself. We select a unique predecessor in the following way: for $U \in \mathrm{Lyr}_1 S$ we put $\pi(U) := S$, and we add the formal definition $\pi(G) := G$. This enables a recursive construction, as follows:

Definition 3.1 *The collection of Artin transfers up to order n of G is defined recursively by*

$$\alpha^{(1)}G := T_{\pi(G),G} \text{ and } \alpha^{(n)}G := \left[\alpha^{(1)}G; \left(\alpha^{(n-1)}S\right)_{S \in \mathrm{Lyr}_1 G}\right] \text{ for } n \ge 2. \tag{3.2}$$

The limit of this infinite recursive nesting process is denoted by

$$\alpha^{(\infty)}G := \lim_{n \to \infty} \alpha^{(n)}G \tag{3.3}$$

and is called the *Artin transfer collection* of *G*.

Remark 3.1 *By means of the collection of Artin transfers up to order three*,

$$\alpha^{(3)}G = \left[T_{G,G}; \left(\alpha^{(2)}S\right)_{S \in \mathrm{Lyr}_1 G}\right] = \left[T_{G,G}; \left(\left[T_{G,S}; \left(T_{S,U}\right)_{U \in \mathrm{Lyr}_1 S}\right]\right)_{S \in \mathrm{Lyr}_1 G}\right],$$

it should be emphasized that our definition of stepwise relative mappings $T_{G,S}$ and $T_{S,U}$ admits finer information than the corresponding absolute mappings $T_{G,U} = T_{S,U} \circ T_{G,S}$ ([1], Thm. 3.3, p. 72), since in general the kernel of $T_{S,U}$ cannot be reconstructed from $T_{G,U}$ and $T_{G,S}$.

3.2. Objects of the Artin Limit Pattern

The infinite collection of mappings $\alpha^{(\infty)}G$ is only the foundation for the objects $\tau^{(\infty)}G$ and $\varkappa^{(\infty)}G$ we are really interested in.

Definition 3.2 *The iterated abelian quotient invariants up to order n of G are defined recursively by*

$$\tau^{(1)}G := \mathrm{AQI}(G) \text{ and } \tau^{(n)}G := \left[\tau^{(1)}G; \left(\tau^{(n-1)}S\right)_{S \in \mathrm{Lyr}_1 G}\right] \text{ for } n \ge 2. \tag{3.4}$$

Similarly, the *iterated transfer kernels up to order n* of G are defined recursively by

$$\varkappa^{(1)}G := \ker\left(T_{\pi(G),G}\right) \text{ and } \varkappa^{(n)}G := \left[\varkappa^{(1)}G; \left(\varkappa^{(n-1)}S\right)_{S \in \mathrm{Lyr}_1 G}\right] \text{ for } n \ge 2. \tag{3.5}$$

Both are collected in the *n*th *order Artin pattern* $\mathrm{AP}^{(n)}G := \left(\tau^{(n)}G, \varkappa^{(n)}G\right)$ of *G*. The limits of these infinite recursive nesting processes are called the *abelian invariant collection* of *G*,

$$\tau^{(\infty)}G := \lim_{n \to \infty} \tau^{(n)}G, \tag{3.6}$$

and the *transfer kernel collection* of *G*,

$$\varkappa^{(\infty)}G := \lim_{n\to\infty} \varkappa^{(n)}G. \tag{3.7}$$

Finally, the pair $\mathrm{ALP}(G) := \left(\tau^{(\infty)}G, \varkappa^{(\infty)}G\right)$ is called the *Artin limit pattern* of *G*.

Remark 3.2 *For a finite p-group G, the recursive nesting processes in the definition of the Artin limit pattern are actually finite.*

The abelian quotient invariants are a *unary* concept, since $\tau^{(1)}G = \mathrm{AQI}(G) = \mathrm{ATI}(G/G')$ depends on *G* only. The first order abelian quotient invariants $\tau^{(1)}G$ already contain non-trivial information on the abelianization of *G*.

The transfer kernels are a *binary* concept for $S < G$, since $\varkappa^{(1)}S = \ker\left(T_{\pi(S),S}\right)$ depends on $\pi(S)$ and *S*. The first order transfer kernel of *G* is trivial: $\varkappa^{(1)}G = \ker\left(T_{\pi(G),G}\right) = \ker\left(T_{G,G}\right) = \ker\left(\mathrm{id}_{G/G'}\right) = 1$, and non-trivial information starts with the transfer kernels of second order $\varkappa^{(1)}S = \ker\left(T_{\pi(S),S}\right) = \ker\left(T_{G,S}\right)$ for $S \in \mathrm{Lyr}_1 G$ which are members of $\varkappa^{(2)}G$.

The analogous constructions for a *number field F* instead of a pro-*p* group *G*, along the lines of §§2.1.2 and 2.2.2, lead to the *Artin limit pattern* $\mathrm{ALP}(F) := \left(\tau^{(\infty)}F, \varkappa^{(\infty)}F\right)$ of *F*.

3.3. Connection between Pro-*p* Groups and Number Fields

Let $F_p^{(\infty)}$ be the Hilbert *p*-class tower of the number field *F*, that is, the maximal unramified pro-*p* extension of *F*, and denote by $\mathrm{G}_p^{\infty}F = \mathrm{Gal}\left(F_p^{(\infty)}/F\right)$ its Galois group, which is briefly called the *p-tower group* of *F*. Now we are going to employ the abelian type invariant collection $\tau^{(\infty)}F$ of *F*, and the abelian quotient invariant collection $\tau^{(\infty)}\left(\mathrm{G}_p^{\infty}F\right)$ of $\mathrm{G}_p^{\infty}F$, *i.e.*, the first component of the respective Artin limit pattern. The transfer kernel collections $\varkappa^{(\infty)}$ will be considered further in §5.

Theorem 3.1 *For each integer* $n \ge 1$, *the abelian quotient invariants of nth order of the p-tower group* $\mathrm{G}_p^{\infty}F$ *of F are equal to the abelian type invariants of nth order of the number field F*

$$(\forall n \ge 1) \quad \tau^{(n)}\left(\mathrm{G}_p^{\infty}F\right) = \tau^{(n)}F \text{ and thus } \tau^{(\infty)}\left(\mathrm{G}_p^{\infty}F\right) = \tau^{(\infty)}F. \tag{3.8}$$

The invariant type of the *p*-tower group $\mathrm{G}_p^{\infty}F$ of *F* coincides with the invariant type of the number field *F*

$$\varkappa\left(\mathrm{G}_p^{\infty}F\right)^{S_{p+1}} = \varkappa(F)^{S_{p+1}}. \tag{3.9}$$

Even the orbit representatives of the transfer kernel types of $\mathrm{G}_p^{\infty}F$ and *F* coincide,

$$\varkappa\left(\mathrm{G}_p^{\infty}F\right) = \left(\ker\left(T_{\mathrm{G}_p^{\infty}F,U_i}\right)\right)_{1\le i\le p+1} = \left(\ker\left(T_{F,E_i}\right)\right)_{1\le i\le p+1} = \varkappa(F), \tag{3.10}$$

provided that the $U_i \in \mathrm{Lyr}_1\left(\mathrm{G}_p^{\infty}F\right)$ and the $E_i \in \mathrm{Lyr}_1 F$ are connected by $U_i = \mathrm{Gal}\left(F_p^{(\infty)}/E_i\right)$, for each $1 \le i \le p+1$. Otherwise, we only have equivalence $\varkappa\left(\mathrm{G}_p^{\infty}F\right) \sim \varkappa(F)$.

Proof. The claims are well-known consequences of the Artin reciprocity law of class field theory [2] [3].

In contrast to the full p-tower group $\mathfrak{G} = \mathrm{G}_p^\infty F$, the Galois groups $\mathrm{G}_p^m F := \mathrm{Gal}\left(F_p^{(m)}/F\right) \simeq \mathfrak{G}/\mathfrak{G}^{(m)}$ of the finite stages $F_p^{(m)}$ of the p-class tower of F, that is, of the higher Hilbert p-class fields of the number field F, in general fail to reveal the abelian type invariants of nth order of the number field F. More precisely, there is a strict upper bound on the order n of the ATI of F which coincide with the AQI of order n of the mth p-class group $\mathrm{G}_p^m F$ of F with a fixed integer $m \ge 0$, namely the bound $n \le m$.

Theorem 3.2 (***Successive Approximation Theorem.***)

Let F be a number field, p a prime, and m,n integers. The abelian invariant collection $\tau^{(\infty)}F$ of F is approximated successively by the iterated AQI of sufficiently high p-class groups of F:

$$(\forall n \ge 1) \quad (\forall m \ge n) \quad \tau^{(n)}\left(\mathrm{G}_p^m F\right) = \tau^{(n)} F. \tag{3.11}$$

However, the transfer kernel type is a phenomenon of second order:

$$(\forall m \ge 2) \quad \varkappa\left(\mathrm{G}_p^m F\right) \sim \varkappa(F), \tag{3.12}$$

in particular, the metabelian second p-class group $\mathfrak{M} := \mathrm{G}_p^2 F \simeq \mathfrak{G}/\mathfrak{G}''$ of F is sufficient for determining the transfer kernel type of F.

Proof. This is one of the main results in ([4], Thm. 1.19, p. 78) and ([5], p. 13).

In general, the upper bound on the order n of the ATI of F in Theorem 3.2 seems to be sharp, in the following sense, where $m = n-1$.

Conjecture 3.1 (***Stage Separation Criterion.***)

Denote by $\ell_p F$ the length of the p-class tower of F, that is the derived length $\mathrm{dl}\left(\mathrm{G}_p^\infty F\right)$ of the p-tower group of F. It is determined in terms of iterated AQI of higher p-class groups of F by the following condition:

$$(\forall n \ge 1) \quad \ell_p F \ge n \Leftrightarrow \tau^{(n)}\left(\mathrm{G}_p^{n-1} F\right) < \tau^{(n)} F. \tag{3.13}$$

The sufficiency of the condition in Conjecture 3.1 is a proven theorem ([5], p. 13).

4. Successive Approximation of the *p*-Class Tower

4.1. Computational Perspectives

Our first attempt to find sound asymptotic tendencies in the distribution of higher non-abelian p-class groups $\mathrm{G}_p^n F = \mathrm{Gal}\left(F_p^{(n)}/F\right)$, with $n \ge 2$, among the finite p-groups was planned in 1991 already ([6], 3, Remark, p. 77). However, the insurmountable obstacles in the required computations limited the progress for twenty years. In 2012, we finally succeeded in the significant break-through of computing the second 3-class groups $\mathfrak{M} = \mathrm{G}_3^2 F$, that is, the metabelianizations $\mathfrak{G}/\mathfrak{G}^{(2)}$ of the 3-class tower groups $\mathfrak{G} = \mathrm{Gal}\left(F_3^{(\infty)}/F\right)$ of all 4596 quadratic fields $F = \mathbb{Q}\left(\sqrt{d}\right)$ with fundamental discriminants in the remarkable range $-10^6 < d < 10^7$ and elementary bicyclic 3-class group

$\mathrm{Cl}_3F \simeq C_3 \times C_3$ of rank two ([7], 6, pp. 495-499). The underlying computational techniques were based on the *principalization algorithm via class group structure* which we had invented in 2009 and implemented by means of the number theoretic computer algebra system PARI/GP [8] in 2010, as described in ([9], 5-6, pp. 446-455).

Throughout this paper, isomorphism classes of finite groups G are characterized uniquely by their identifier in the SmallGroups Database [10] [11], which is denoted by a pair $\langle o,i\rangle$ consisting of the order $o = \mathrm{ord}(G)$ and a positive integer *i*, delimited with angle brackets. The counter $1 \le i \le N(o)$ is unique for a fixed value of the order *o*. In the computational algebra system MAGMA [12] [13] [14], the upper bound $N(o)$ can be obtained as return value of the function NumberOfSmallGroups(*o*), provided that IsInSmallGroupDatabase(*o*) returns true. The identifier of a given finite group G can be retrieved as return value of the function IdentifyGroup(*G*), provided that Can IdentifyGroup(*o*) returns true.

4.2. Trivial Towers with $\ell_p F = 0$

For the decision if the *p*-class tower of a number field F is trivial with length $\ell_p F = 0$ it suffices to compute the class number $h(F)$ of the field.

Theorem 4.1 (*Trivial p-class tower.*)

The *p*-class tower of a number field F is trivial, $F_p^{(\infty)} = F$, with length $\ell_p F = 0$, if and only if the class number $h(F) = \#\mathrm{Cl}(F)$ is not divisible by *p*, i. e., the *p*-class number is $h_p F = 1$.

Proof. The proof consists of a sequence of equivalent statements: The class number satisfies $p \nmid h(F)$. $\Leftrightarrow$ The *p*-valuation of $h(F)$ is $v_p(h(F)) = 0$. $\Leftrightarrow$ The *p*-class number is $\#\mathrm{Cl}_p F = h_p F = p^{v_p(h(F))} = 1$. $\Leftrightarrow$ The *p*-class group $\mathrm{Cl}_p F = 1$ is trivial. $\Leftrightarrow$ The *p*-class rank $\rho_p = \dim_{\mathbb{F}_p}\left(\mathrm{Cl}(F)/\mathrm{Cl}(F)^p\right)$ is equal to zero. $\Leftrightarrow$ The number of unramified cyclic extensions E/F of degree *p* is $\frac{p^{\rho_p}-1}{p-1} = \frac{p^0-1}{p-1} = \frac{1-1}{p-1} = 0$. $\Leftrightarrow$ The maximal unramified *p*-extension $F_p^{(\infty)}$ of F coincides with *F*. $\Leftrightarrow$ The Galois group $\mathrm{G}_p^\infty F = \mathrm{Gal}\left(F_p^{(\infty)}/F\right) = \mathrm{Gal}(F/F) = 1$ is trivial. $\Leftrightarrow$ The length of the *p*-class tower is $\ell_p F = \mathrm{dl}\left(\mathrm{G}_p^\infty F\right) = \mathrm{dl}(1) = 0$.

Already C. F. Gauss was able to compute class numbers $h(F)$ of quadratic fields $F = \mathbb{Q}\left(\sqrt{d}\right)$, at a time when the concept of class field theory was not yet coined. Nowadays, there exist extensive tables of quadratic class numbers which even contain the structures of the associated class groups $\mathrm{Cl}(F)$. In 1998, Jacobson [15] covered all real quadratic fields with positive discriminants in the range $0 < d < 10^9$, and in 2016, Mosunov and Jacobson [16] investigated all imaginary quadratic fields with negative discriminants $-10^{12} < d < 0$. Now we apply these results to class field theory.

Corollary 4.1 (*Statistics for* $p = 3$.) *The asymptotic proportion of imaginary quadratic fields* $F = \mathbb{Q}\left(\sqrt{d}\right)$, *with negative discriminants* $d < 0$, *whose class number* $h(F)$ *is, respectively is not, divisible by* $p=3$ *is given as* 43.99%, *respectively* 56.01%, *by the heuristics of Cohen, Lenstra and Martinet. In* **Table 1**, *the approximations of these theoretical limits by relative frequencies in various ranges* $L<d<0$ *are shown.*

Proof. The heuristic asymptotic limits are given in ([17], 2, (1.1.c), p. 126). Their approximation by discriminants $L<d<0$ with $L=-10^6$ in ([18], Example, p. 843) and ([6], 2, Remark, and 3, Remark, p. 77), where $118455+3190=121645$, is still rather far away from the limits. In contrast, the approximations associated with the bounds $L=-10^{11}$ and $L=-10^{12}$ in ([16], p. 2001) are very close already.

4.3. Abelian Single-Stage Towers with $\ell_p F=1$

The first stage $F_p^{(1)}$ of the *p*-class tower of a number field *F* is determined by the structure of the *p*-class group $\mathrm{Cl}_p F$ of *F* as a finite abelian *p*-group. This is exactly the first order Artin pattern

$$\mathrm{AP}^{(1)}F=\left(\tau^{(1)}F,\varkappa^{(1)}F\right)=\left(\mathrm{ATI}\left(\mathrm{Cl}_pF\right),\ker\left(T_{F,F}\right)\right), \quad (4.1)$$

since the trivial $\ker\left(T_{F,F}\right)=1$ does not contain information. However, only in the case of *p*-class rank one, $\rho_p=\dim_{\mathbb{F}_p}\left(\mathrm{Cl}(F)/\mathrm{Cl}(F)^p\right)=1$, it is warranted that the exact length of the tower is $\ell_p F=1$. A statistical example ([6], 2, Remark, p. 77) is shown in **Table 2**.

Theorem 4.2 *A number field F with non-trivial cyclic p-class group* Cl_pF *has an abelian p-class tower of exact length* $\ell_pF=1$, *in fact, the Galois group* $\mathrm{G}_p^\infty F\simeq \mathrm{G}_p^1F\simeq\mathrm{Cl}_pF$ *is cyclic.*

Proof. Suppose that $\mathrm{Cl}_pF>1$ is non-trivial and cyclic. If the *p*-class tower had a length $\ell_pF\geq 2$, the second *p*-class group $\mathfrak{M}=\mathrm{G}_p^2F$ would be a

Table 1. Imaginary quadratic fields *F* with non-trivial, resp. trivial, 3-class tower.

L	$\#(3\mid h(F))$	rel. fr.	$\#(3\nmid h(F))$	rel. fr.	w. r. t. #total
-10^6	121645	40.02%	182323	59.98%	303968
-10^{11}	13206088529	43.45%	17190266523	56.55%	30396355052
-10^{12}	132584350621	43.62%	171379200091	56.38%	303963550712

Table 2. Imaginary quadratic fields *F* with cyclic 3-class tower for $-10^6<d<0$.

$\mathrm{Cl}_3F\simeq$	abs. fr.	rel. fr.	w. r. t. $\#(\rho_3=1)$
C_3	80115	67.63%	118455
C_9	26458	22.34%	118455
C_{27}	8974	7.58%	118455
C_{81}	2472	2.09%	118455
C_{243}	393	0.33%	118455
C_{729}	43	0.04%	118455

non-abelian finite p-group with cyclic abelianization $\mathfrak{M}/\mathfrak{M}' \simeq \mathrm{Cl}_p F$. However, it is well known that a nilpotent group with cyclic abelianization is abelian, which contradicts the assumption of a length $\ell_p F \geq 2$.

Remark 4.1 *We interpret the computation of abelian type invariants* $\tau^{(1)}F$ *of the Sylow* 3*-subgroup* $\mathrm{Cl}_3 F$ *of the ideal class group* $\mathrm{Cl}(F)$ *of a quadratic field* $F = \mathbb{Q}(\sqrt{d})$ *as the determination of the single-stage approximation* $\mathfrak{G}/\mathfrak{G}' \simeq \mathrm{G}_3^1 F \simeq \mathrm{Cl}_3 F$ *of the* 3*-class tower group* $\mathfrak{G} = \mathrm{G}_3^\infty F$ *of F. This step yields complete information about the lattice of all unramified abelian* 3*-extensions* E/F *within the Hilbert* 3*-class field* $F_3^{(1)}$ *of F.*

4.4. Metabelian Two-Stage Towers with $\ell_p F = 2$

According to the Successive Approximation Theorem 3.2, the second stage $F_p^{(2)}$ of the p-class tower of a number field F is determined by the second order Artin pattern

$$\begin{aligned}&\mathrm{AP}^{(2)}F = \left(\tau^{(2)}F, \varkappa^{(2)}F\right)\\ &= \left(\left[\mathrm{ATI}\left(\mathrm{Cl}_p F\right); \left(\mathrm{ATI}\left(\mathrm{Cl}_p E\right)\right)_{E \in \mathrm{Lyr}_1 F}\right], \left[\ker\left(T_{F,F}\right); \left(\ker\left(T_{F,E}\right)\right)_{E \in \mathrm{Lyr}_1 F}\right]\right).\end{aligned} \tag{4.2}$$

The determination of $\mathrm{AP}^{(2)}F$ for a quadratic field F with 3-class rank $\rho_3 = 2$ requires the computation of four 3-class groups $\mathrm{Cl}_3 E_i$ of unramified cyclic cubic extensions $E_1, \cdots, E_4$ and of four transfer kernels $\ker\left(T_{F,E_i}\right)$.

Whereas Mosunov and Jacobson [16] were able to determine the class groups $\mathrm{Cl}(F)$ of more than 300 billion, precisely 303963550712, imaginary quadratic fields F with discriminants $-10^{12} < d < 0$ by parallel processes on multiple cores of a supercomputer in several years of total CPU time, it is currently definitely out of scope to compute the class groups $\mathrm{Cl}(E_i), 1 \leq i \leq 4$, for the 22757307168 unramified cyclic cubic extensions E_i/F, of absolute degree six, of the 5689326792 imaginary quadratic fields F with discriminants $-10^{12} < d < 0$ and 3-class rank $\rho_3 = 2$.

Therefore, it must not be underestimated that Boston, Bush and Hajir [19] succeeded in completing this task for the smaller range $-10^8 < d < 0$ with 461925 imaginary quadratic fields F having 3-class rank $\rho_3 = 2$, and 1847700 associated *totally complex dihedral fields* E_i of degree six ([7], Prp. 4.1, p. 482). For this purpose the authors used the computational algebra system MAGMA [12] [13] [14] in a distributed process involving several processors with multiple cores. 276375 of these quadratic fields F have a 3-class group $\mathrm{Cl}_3 F \simeq C_3 \times C_3$.

Imaginary quadratic fields $F = \mathbb{Q}(\sqrt{d})$ with negative discriminants $d < 0$ are the simplest number fields with respect to their unit group U_F, which is a finite torsion group of Dirichlet unit rank zero. This fact has considerable consequences for their p-class tower groups, according to the Shafarevich theorem [20], corrected in ([21], Thm. 5.1, p. 28), [22].

Theorem 4.3 *Among the finite* 3*-groups* G *with elementary bicyclic abelianization* $G/G' \simeq C_3 \times C_3$ *of rank two, there exist only two metabelian* *groups with GI-action* (*generator inverting action*). *and relation rank* $d_2G = 2$ (*so-called Schur σ-groups* [23] [19]), *namely* $\langle 243,5\rangle$ *and* $\langle 243,7\rangle$.

1) These are the groups of smallest order which are admissible as 3-class tower groups $G \simeq \mathrm{G}_3^\infty F$ of imaginary quadratic fields F with 3-class group $\mathrm{Cl}_3F \simeq C_3 \times C_3$.

2) Generally, for any number field F, these groups are determined uniquely by the second order Artin pattern.

(a) If $\mathrm{AP}^{(2)}F = \left(\left[1^2;\left(21,21,1^3,21\right)\right],\left[1;\left(2241\right)\right]\right)$ then $\mathrm{G}_3^\infty F \simeq \langle 243,5\rangle$.

(b) If $\mathrm{AP}^{(2)}F = \left(\left[1^2;\left(1^3,21,1^3,21\right)\right],\left[1;\left(4224\right)\right]\right)$ then $\mathrm{G}_3^\infty F \simeq \langle 243,7\rangle$.

3) The actual distribution of these 3-class tower groups G among the 276375 imaginary quadratic fields $F = \mathbb{Q}\left(\sqrt{d}\right)$ with 3-class group $\mathrm{Cl}_3F \simeq C_3 \times C_3$ and discriminants $-10^8 < d < 0$ is presented in Table 3.

Proof. All finite 3-groups G with abelianization $G/G' \simeq C_3 \times C_3$ are vertices of the descendant tree $\mathcal{T}(R)$ with abelian root $R = \langle 9,2\rangle \simeq C_3 \times C_3$. A search for metabelian vertices with relation rank $d_2G = 2$ in this tree yields three hits $\langle 27,4\rangle$, $\langle 243,5\rangle$, and $\langle 243,7\rangle$, but only the latter two of them possess a GI-action.

The abelianization G/G' of a finite 3-group G which is realized as the 3-class tower group $\mathrm{G}_p^\infty F$ of an algebraic number field F is isomorphic to the 3-class group Cl_3F of F. When F is imaginary quadratic, it possesses signature $(r_1,r_2) = (0,1)$ and torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1 = 0$. If $G/G' \simeq \mathrm{Cl}_3F \simeq C_3 \times C_3$, then the generator rank of G is $d_1G = 2$ and the Shafarevich theorem implies bounds for the relation rank $2 = d_1G \le d_2G \le d_1G + r = 2$.

The entries of Table 3 have been taken from [19].

More recently, Boston, Bush and Hajir [24] used MAGMA [14] for computing the class groups of the 481756 real quadratic fields F having 3-class rank $\rho_3 = 2$ and discriminants in the range $0 < d < 10^9$, and the class groups of the 1927024 associated *totally real dihedral fields* E_i of degree six, arising from unramified cyclic cubic extensions E_i/F ([7], Prp. 4.1, p. 482). 415698 of these quadratic fields F have a 3-class group $\mathrm{Cl}_3F \simeq C_3 \times C_3$ (415699 according to([15], Tbl. 7)).

Real quadratic fields $F = \mathbb{Q}\left(\sqrt{d}\right)$ with positive discriminants $d > 0$ are the second simplest number fields with respect to their unit group U_F, which is an infinite group of torsionfree Dirichlet unit rank one. Again, there are remarkable consequences for their p-tower groups, by the Shafarevich theorem ([21], Thm. 5.1, p. 28).

Theorem 4.4 *Among the finite* 3*-groups* G *with elementary bicyclic abelianization* $G/G' \simeq C_3 \times C_3$ *of rank two, there exist infinitely many*

Table 3. Frequencies of metabelian 3-class tower groups G for $-10^8 < d < 0$.

$G \simeq$	abs. fr.	rel. fr.	w. r. t.	rel. fr.	w. r. t.	measure [19]	$\lvert d\rvert_{\min}$
$\langle 243,5\rangle$	83353	30.16%	276375	18.04%	461925	$128/729 \approx 17.56\%$	4027
$\langle 243,7\rangle$	41398	14.98%	276375	8.96%	461925	$64/729 \approx 8.78\%$	12131

metabelian groups with RI-action and relation rank $d_2G=3$ (*so-called Schur + 1 σ-groups* [24]), *but only three of minimal order* 3^4, *namely* $\langle 81,7\rangle$, $\langle 81,8\rangle$ *and* $\langle 81,10\rangle$.

1) These are the groups of smallest order which are admissible as 3-class tower groups $G\simeq \mathrm{G}_3^{\infty}F$ of real quadratic fields F with 3-class group $\mathrm{Cl}_3F\simeq C_3\times C_3$.

2) Generally, for any number field F, these groups are determined uniquely by the second order Artin pattern.

(a) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(1^3,1^2,1^2,1^2\right)\right],\left[1;(2000)\right]\right)$ then $\mathrm{G}_3^{\infty}F\simeq\langle 81,7\rangle$.

(b) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(21,1^2,1^2,1^2\right)\right],\left[1;(2000)\right]\right)$ then $\mathrm{G}_3^{\infty}F\simeq\langle 81,8\rangle$.

(c) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(21,1^2,1^2,1^2\right)\right],\left[1;(1000)\right]\right)$ then $\mathrm{G}_3^{\infty}F\simeq\langle 81,10\rangle$.

3) The actual distribution of these 3-class tower groups G among the 415698 real quadratic fields $F=\mathbb{Q}\left(\sqrt{d}\right)$ with 3-class group $\mathrm{Cl}_3F\simeq C_3\times C_3$ and discriminants $0<d<10^9$ is presented in **Table 4**. Additionally, the frequencies of the groups $\langle 243,5\rangle$ and $\langle 243,7\rangle$ in Theorem 4.3 are given.

Proof. A search for metabelian vertices G of minimal order with relation rank $d_2G=3$ in the descendant tree $\mathcal{T}(R)$ with abelian root $R=\langle 9,2\rangle\simeq C_3\times C_3$ yields three hits $\langle 81,7\rangle$, $\langle 81,8\rangle$, and $\langle 81,10\rangle$. All of them possess a RI-action.

The abelianization G/G' of a finite 3-group G which is realized as the 3-class tower group $\mathrm{G}_p^{\infty}F$ of an algebraic number field F is isomorphic to the 3-class group Cl_3F of F. When F is real quadratic, it possesses signature $(r_1,r_2)=(2,0)$ and torsionfree Dirichlet unit rank $r=r_1+r_2-1=1$. If $G/G'\simeq\mathrm{Cl}_3F\simeq C_3\times C_3$, then the generator rank of G is $d_1G=2$ and the Shafarevich theorem implies bounds for the relation rank $2=d_1G\le d_2G\le d_1G+r=3$.

The entries of **Table 4** have been taken from [24].

In [24], Boston, Bush and Hajir only computed the first component of the second order Artin pattern $\mathrm{AP}^{(2)}F=\left(\tau^{(2)}F,\varkappa^{(2)}F\right)$ in Formula (4.2), that is, the abelian type invariants $\tau^{(2)}F$ of second order of real quadratic fields F with discriminants $0<d<10^9$. Determining the second component $\varkappa^{(2)}F$, the transfer kernel type of F, is considerably harder with respect to the computational expense. Consequently, the most extensive numerical results on transfer kernels available currently, have been computed by ourselves for the smaller ranges $0<d<10^8$ in [25] [26], and, even computing third order Artin

Table 4. Frequencies of metabelian 3-class tower groups G for $0<d<10^9$.

$G\simeq$	abs. fr.	rel. fr.	w. r. t.	rel. fr.	w. r. t.	measure [24]	$d_{\min}$
$\langle 81,7\rangle$	122955	29.58%	415698	25.52%	481756	$1664/6561\approx 25.36\%$	142097
$\langle 81,8\rangle$ or $\langle 81,10\rangle$	208236	50.09%	415698	43.22%	481756	$8320/19683\approx 42.27\%$	32009
$\langle 243,5\rangle$	13712	3.30%	415698	2.85%	481756	$1664/59049\approx 2.82\%$	422573
$\langle 243,7\rangle$	6691	1.61%	415698	1.39%	481756	$832/59049\approx 1.41\%$	631769

patterns, for $0<d<10^7$ in [27] [28]. With the aid of these results, we now illustrate that the transfer kernels $\ker\left(T_{F,E}\right)$ of 3-class extensions $T_{F,E}:\mathrm{Cl}_3F\to\mathrm{Cl}_3E$ from real quadratic fields F to unramified cyclic cubic extensions E/F are capable of narrowing down the number of contestants for the 3-tower group $\mathrm{G}_3^\infty F$ significantly, and thus of refining the statistics in [24].

Corollary 4.2

1) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(32,1^2,1^2,1^2\right)\right],\left[1;\left(1000\right)\right]\right)$ then $\mathrm{G}_3^\infty F\simeq\langle 729,96\rangle$.

2) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(32,1^2,1^2,1^2\right)\right],\left[1;\left(2000\right)\right]\right)$ then $\mathrm{G}_3^\infty F\simeq\langle 729,i\rangle$ with $i\in\{97,98\}$.

3) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(2^2,1^2,1^2,1^2\right)\right],\left[1;\left(0000\right)\right]\right)$ then $\mathrm{G}_3^\infty F\simeq\langle 729,i\rangle$ with $i\in\{99,100,101\}$.

The actual distribution of these 3-class tower groups G among the 34631, respectively 2576, real quadratic fields $F=\mathbb{Q}\left(\sqrt{d}\right)$ with 3-class group $\mathrm{Cl}_3F\simeq C_3\times C_3$ and discriminants $0<d<10^8$, respectively $0<d<10^7$, is presented in Table 5.

4.5. Non-Metabelian Three-Stage Towers with $\ell_pF=3$

According to the Successive Approximation Theorem 3.2, the third stage $F_p^{(3)}$ of the p-class tower of a number field F is usually determined by the third order Artin pattern

$$\mathrm{AP}^{(3)}F=\left(\tau^{(3)}F,\varkappa^{(3)}F\right)=\left(\left[\tau^{(1)}F;\left(\tau^{(2)}E\right)_{E\in\mathrm{Lyr}_1F}\right],\left[\varkappa^{(1)}F;\left(\varkappa^{(2)}E\right)_{E\in\mathrm{Lyr}_1F}\right]\right). \quad (4.3)$$

It is interesting, however, that there are extensive collections of quadratic fields F with 3-class towers of exact length $\ell_3F=3$, which can be characterized by the second order Artin pattern already. We begin with imaginary quadratic fields $F=\mathbb{Q}\left(\sqrt{d}\right)$ with discriminants $d<0$.

Theorem 4.5 *Among the finite 3-groups G with elementary bicyclic abelianization $G/G'\simeq C_3\times C_3$ of rank* two, *there exist infinitely many non-*

Table 5. Frequencies of metabelian 3-class tower groups G for $0<d<10^8$, resp. 10^7.

$G\simeq$	abs. fr.	rel. fr.	w. r. t.	$d_{\min}$
$\langle 81,7\rangle$	10244	29.58%	34631	142097
$\langle 81,8\rangle$	10514	30.36%	34631	32009
$\langle 81,10\rangle$	7104	20.51%	34631	72329
$\langle 729,96\rangle$	242	0.70%	34631	790085
$\langle 729,97\rangle$ or $\langle 729,98\rangle$	713	2.06%	34631	494236
$\langle 729,99\rangle$	66	2.56%	2576	62501
$\langle 729,100\rangle$	42	1.63%	2576	152949
$\langle 729,101\rangle$	42	1.63%	2576	252977

metabelian groups with GI-action and relation rank $d_2G = 2$ (*so-called Schur σ-groups* [19] [23]), *but only seven of minimal order* 3^8, *namely* $\langle 6561, i\rangle$ *with* $i \in \{606, 616, 617, 618, 620, 622, 624\}$.

1) These are the groups of smallest order which are admissible as non-metabelian 3-class tower groups $G \simeq \mathrm{G}_3^{\infty}F$ of imaginary quadratic fields F with 3-class group $\mathrm{Cl}_3F \simeq C_3 \times C_3$.

2) Exceptionally, for an imaginary quadratic field F, the trailing six of these groups are determined by the second order Artin pattern already.

(a) If $\mathrm{AP}^{(2)}F = \left(\left[1^2;\left(32,21,1^3,21\right)\right],\left[1;(1313)\right]\right)$ then $\mathrm{G}_3^{\infty}F \simeq \langle 6561, 616\rangle$.

(b) If $\mathrm{AP}^{(2)}F = \left(\left[1^2;\left(32,21,1^3,21\right)\right],\left[1;(2313)\right]\right)$ then $\mathrm{G}_3^{\infty}F \simeq \langle 6561, i\rangle$ with $i \in \{617, 618\}$.

(c) If $\mathrm{AP}^{(2)}F = \left(\left[1^2;(32,21,21,21)\right],\left[1;(1231)\right]\right)$ then $\mathrm{G}_3^{\infty}F \simeq \langle 6561, 622\rangle$.

(d) If $\mathrm{AP}^{(2)}F = \left(\left[1^2;(32,21,21,21)\right],\left[1;(2231)\right]\right)$ then $\mathrm{G}_3^{\infty}F \simeq \langle 6561, i\rangle$ with $i \in \{620, 624\}$.

3) The actual distribution of these 3-class tower groups G among the 24476 imaginary quadratic fields $F = \mathbb{Q}\left(\sqrt{d}\right)$ with 3-class group $\mathrm{Cl}_3F \simeq C_3 \times C_3$ and discriminants $-10^7 < d < 0$ is presented in **Table 6**.

Proof. By a similar but more extensive search than in the proof of Theorem 4.3. Data for **Table 6** has been computed by ourselves in June 2016 using MAGMA [14].

Remark 4.2 *It should be pointed out that items* (1) *and* (2) *of Theorem* 4.5 *are not valid for real quadratic fields, as documented in* ([29], *Thm.* 7.8, *p.* 162, *and Thm.* 7.12, *p.* 165).

The group $\langle 6561, 606\rangle$ belongs to the infinite Shafarevich cover of the metabelian group $\langle 729, 45\rangle$ with respect to imaginary quadratic fields ([30], Cor. 6.2, p. 301), [31]. It shares a common second order Artin pattern with all other elements of the Shafarevich cover. Third order Artin patterns must be used for its identification, as shown in ([29], Thm. 7.14, p. 168).

Now we turn to real quadratic fields $F = \mathbb{Q}\left(\sqrt{d}\right)$ with discriminants $d > 0$.

Theorem 4.6 *Among the finite* 3*-groups* G *with elementary bicyclic abelianization* $G/G' \simeq C_3 \times C_3$ *of rank two, there exist infinitely many non-*

Table 6. Frequencies of non-metabelian 3-class tower groups G for $-10^7 < d < 0$.

$G \simeq$	abs. fr.	rel. fr.	w. r. t.	type	$\varkappa$	$\lvert d\rvert_{\min}$
$\langle 6561, 616\rangle$	760	3.11%	24476	E.6	(1313)	15544
$\langle 6561, 617\rangle$ or $\langle 6561, 618\rangle$	1572	6.42%	24476	E.14	(2313)	16627
$\langle 6561, 622\rangle$	798	3.26%	24476	E.8	(1231)	34867
$\langle 6561, 620\rangle$ or $\langle 6561, 624\rangle$	1583	6.47%	24476	E.9	(2231)	9748

metabelian groups with RI-action and relation rank $d_2G=3$ (*so-called Schur* + 1 *σ-groups* [24]), *but only nine of minimal order* 3^7, *namely* $\langle 2187,i\rangle$ *with* $i\in\{270,271,272,273,284,291,307,308,311\}$.

1) These are the groups of smallest order which are admissible as non-metabelian 3-class tower groups $G\simeq \mathrm{G}_3^\infty F$ of real quadratic fields F with 3-class group $\mathrm{Cl}_3F\simeq C_3\times C_3$.

2) Exceptionally, for a real quadratic field F, four of these groups are determined by the second order Artin pattern already.

(a) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(2^2,21,1^3,21\right)\right],\left[1;\left(0313\right)\right]\right)$ then $\mathrm{G}_3^\infty F\simeq\langle 2187,i\rangle$ with $i\in\{284,291\}$.

(b) If $\mathrm{AP}^{(2)}F=\left(\left[1^2;\left(2^2,21,21,21\right)\right],\left[1;\left(0231\right)\right]\right)$ then $\mathrm{G}_3^\infty F\simeq\langle 2187,i\rangle$ with $i\in\{307,308\}$.

3) The actual distribution of these 3-class tower groups G among the 415698 real quadratic fields $F=\mathbb{Q}\left(\sqrt{d}\right)$ with 3-class group $\mathrm{Cl}_3F\simeq C_3\times C_3$ and discriminants $1<d<10^9$ is presented in **Table 7**.

Proof. The claims for transfer kernel type c.18, $\varkappa(F)\sim(0313)$, are a consequence of ([21], Prp. 7.1, p. 32, Thm. 7.1, p. 33, and Rmk. 7.1, p. 35), those for type c.21, $\varkappa(F)\sim(0231)$, have been proved in ([21], Prp. 8.1, p. 42, Thm. 8.1, p. 44, and Rmk. 8.2, p. 45). A slightly stronger result is the Main Theorem ([21], Thm. 2.1, p. 22).

Remark 4.3 *The groups* $\langle 2187,i\rangle$ *with* $i\in\{270,271,272,273\}$ *are elements of the infinite Shafarevich cover of the metabelian group* $\langle 729,45\rangle$ *with respect to real quadratic fields.*

The group $\langle 2187,311\rangle$ belongs to the infinite Shafarevich cover of the metabelian group $\langle 729,57\rangle$ with respect to real quadratic fields.

These five groups share a common second order Artin pattern with all other elements of the relevant Shafarevich cover. Third order Artin patterns must be employed for their identification, as shown in ([29], Thm. 7.13, p. 167, and Thm. 7.15, p. 169).

5. Maximal Subgroups of 3-Groups of Coclass One

Let $\left(\gamma_i(G)\right)_{i\ge 1}$ be the descending lower central series of the group G, defined recursively by $\gamma_1(G):=G$ and $\gamma_i(G):=\left[\gamma_{i-1}(G),G\right]$ for $i\ge 2$, in particular, $\gamma_2(G)=G'$ is the commutator subgroup of G. A finite p-group G is nilpotent with $\gamma_1(G)>\gamma_2(G)>\cdots>\gamma_c(G)>\gamma_{c+1}(G)=1$ for some integer $c\ge 1$, which is

Table 7. Frequencies of non-metabelian 3-class tower groups G for $0<d<10^9$.

$G\simeq$	abs. fr.	rel. fr.	w. r. t.	type	$\varkappa$	$d_{\min}$
$\langle 2187,284\rangle$ or $\langle 2187,291\rangle$	4318	1.04%	415698	c.18	(0313)	534824
$\langle 2187,307\rangle$ or $\langle 2187,308\rangle$	4377	1.05%	415698	c.21	(0231)	540365

called the *nilpotency class* $\mathrm{cl}(G)=c$ of *G*. When *G* is of order p^n, for some integer $n\ge 1$, the *coclass* of *G* is defined by $\mathrm{cc}(G):=n-c$ and $\mathrm{lo}(G):=n$ is called the *logarithmic order* of *G*.

Finite 3-groups *G* with coclass $\mathrm{cc}(G)=1$ were investigated by N. Blackburn [32] in 1958. All of these CF-groups, which exclusively have *cyclic factors* $\gamma_i(G)/\gamma_{i+1}(G)$ of their descending central series for $i\ge 2$, are necessarily metabelian with second derived subgroup $G''=1$ and abelian commutator subgroup G' and possess abelianization $G/G'\simeq C_3\times C_3$, according to Blackburn [33].

For the statement of Theorem 5.1, we need a precise ordering of the four maximal subgroups $H_1,\cdots,H_4$ of the group $G=\langle x,y\rangle$, which can be generated by two elements x,y, according to the Burnside basis theorem. For this purpose, we select the generators x,y such that

$$H_1=\langle y,G'\rangle,\quad H_2=\langle x,G'\rangle,\quad H_3=\langle xy,G'\rangle,\quad H_4=\langle xy^2,G'\rangle,\tag{5.1}$$

and $H_1=\chi_2(G)$, provided that *G* is of nilpotency class $\mathrm{cl}(G)\ge 3$. Here we denote by

$$\chi_2(G):=\left\{g\in G\mid\left(\forall h\in\gamma_2(G)\right)[g,h]\in\gamma_4(G)\right\}\tag{5.2}$$

the *two-step centralizer* of G' in *G*.

Parametrized Presentations of Metabelian 3-Groups

The identification of the groups will be achieved with the aid of parametrized polycyclic power-commutator presentations, as given by Blackburn [32], Miech [34], and Nebelung [35]:

$$\begin{aligned}G_a^n(z,w):=\big\langle x,y,s_2,\cdots,s_{n-1}\mid s_2=[y,x],\left(\forall_{i=3}^{n}\right)s_i=[s_{i-1},x],\ s_n=1,[y,s_2]=s_{n-1}^a,\\ \left(\forall_{i=3}^{n-1}\right)[y,s_i]=1,\ x^3=s_{n-1}^w,\ y^3s_2^3s_3=s_{n-1}^z,\left(\forall_{i=2}^{n-3}\right)s_i^3s_{i+1}^3s_{i+2}=1,\ s_{n-2}^3=s_{n-1}^3=1\big\rangle,\end{aligned}\tag{5.3}$$

where $a\in\{0,1\}$ and $w,z\in\{-1,0,1\}$ are bounded parameters, and the *index of nilpotency* $n=\mathrm{cl}(G)+1=\mathrm{cl}(G)+\mathrm{cc}(G)=\log_3(\mathrm{ord}(G))=:\mathrm{lo}(G)$ is an unbounded parameter.

The following lemma generalizes relations for second and third powers of generators in ([27], Lem. 3.1), [28].

Lemma 5.1 *Let* $G=\langle x,y\rangle$ *be a finite* 3*-group with two generators* $x,y\in G$. *Denote by* $s_2:=[y,x]$ *the main commutator, and by* $s_3:=[s_2,x]$ *and* $t_3:=[s_2,y]$ *the two iterated commutators. Then the second and third power of the element* xy, *respectively* xy^2, *are given by*

$$\begin{aligned}(xy)^2=x^2y^2s_2t_3 \text{ and } (xy)^3=x^3y^3s_2^3s_3t_3^2, \text{ respectively}\\ (xy^2)^2=x^2y^4s_2^2t_3^2 \text{ and } (xy^2)^3=x^3y^6s_2^6s_3^2t_3^2,\end{aligned}\tag{5.4}$$

provided that $t_3\in\zeta(G)$ is central, $t_3^3=1$, and $[s_3,y]=1$.

Proof. We begin by preparing three commutator relations:

$$yx=xy[y,x]=xys_2,\quad s_2x=xs_2[s_2,x]=xs_2s_3 \text{ and } s_2y=ys_2[s_2,y]=ys_2t_3.\tag{5.5}$$

Now we prove the power relations by expanding the power expressions by iterated substitution of the commutator relations in Formula (5.5), always observing that t_3 belongs to the centre, $t_3^3 = 1$, and $s_3 y = y s_3$ commute:

$$(xy)^2 = xyxy = xxys_2y = x^2yys_2t_3 = x^2y^2s_2t_3, \text{ and thus}$$

$$\begin{aligned}(xy)^3 &= (xy)^2xy = x^2y^2s_2t_3xy = x^2y^2s_2xyt_3 = x^2yyxs_2s_3yt_3 = x^2yxys_2s_2ys_3t_3 \\ &= x^2xys_2ys_2ys_2t_3s_3t_3 = x^3yys_2t_3ys_2t_3s_2s_3t_3^2 = x^3y^2s_2ys_2s_2s_3t_3^4 \\ &= x^3y^2ys_2t_3s_2^2s_3t_3 = x^3y^3s_2^3s_3t_3^2, \text{ respectively}\end{aligned}$$

$$\begin{aligned}(xy^2)^2 &= xyyxyy = xyxys_2yy = xxys_2yys_2t_3y = x^2yys_2t_3ys_2yt_3 = x^2y^2s_2yys_2t_3t_3^2 \\ &= x^2y^2ys_2t_3ys_2t_3^3 = x^2y^3s_2ys_2t_3 = x^2y^3ys_2t_3s_2t_3 = x^2y^4s_2^2t_3^2, \text{ and thus}\end{aligned}$$

$$\begin{aligned}(xy^2)^3 &= (xy^2)^2xy^2 = x^2y^4s_2^2t_3^2xy^2 = x^2y^4s_2s_2xyyt_3^2 = x^2y^4s_2xs_2s_3yyt_3^2 \\ &= x^2yyyyxs_2s_3s_2yys_3t_3^2 = x^2yyyxys_2s_2ys_2t_3ys_3^2t_3^2 = x^2yyxys_2ys_2ys_2t_3s_2ys_3^2t_3^3 \\ &= x^2yxys_2yys_2t_3ys_2t_3s_2ys_2t_3s_3^2t_3^4 = x^2xys_2yys_2t_3ys_2ys_2t_3^2ys_2t_3s_2s_3^2t_3^2 \\ &= x^3yys_2t_3ys_2yys_2t_3s_2t_3^3ys_2^2s_3^2t_3^3 = x^3y^2s_2ys_2yys_2t_3^2s_2ys_2^2s_3^2 \\ &= x^3y^2ys_2t_3ys_2t_3ys_2t_3^2ys_2t_3s_2^2s_3^2 = x^3y^3s_2ys_2t_3^2ys_2yt_3^3s_2^3s_3^2 \\ &= x^3y^3ys_2t_3s_2t_3^2yys_2t_3s_2^3s_3^2 = x^3y^4s_2s_2yys_2t_3^4s_2^3s_3^2 = x^3y^4s_2s_2yys_2^4s_3^2t_3 \\ &= x^3y^4s_2ys_2t_3ys_2^4s_3^2t_3 = x^3y^4s_2ys_2ys_2^4s_3^2t_3^2 = x^3y^4ys_2t_3ys_2t_3s_2^4s_3^2t_3^2 \\ &= x^3y^5s_2ys_2t_3^2s_2^4s_3^2t_3^2 = x^3y^5ys_2t_3s_2^5s_3^2t_3^4 = x^3y^6s_2^6s_3^2t_3^2.\end{aligned}$$

Theorem 5.1 *Let* $G = \langle x, y\rangle \simeq G_a^n(z, w)$ *be a finite* 3*-group of coclass* $\mathrm{cc}(G) = 1$ *and order* $|G| = 3^n$ *with generators* x, y *such that* $y \in \chi_2(G)$ *is contained in the two-step centralizer of G, whereas* $x \in G \setminus \chi_2(G)$, *given by a polycyclic power commutator presentation with parameters* $a \in \{0,1\}$, $w, z \in \{-1, 0, 1\}$, *and index of nilpotency* $n \ge 4$.

Then three of the four maximal subgroups, $H_i = \langle xy^{i-2}, G'\rangle < G$, $2 \le i \le 4$, are non-abelian 3-groups of coclass $\mathrm{cc}(H_i) = 1$, as listed in **Table 8** in dependence on the parameters n, a, z, w.

The supplementary **Table 9** shows the abelian maximal subgroups of the remaining two extra special 3-group of coclass $\mathrm{cc}(G) = 1$ and order $|G| = 3^3$.

Proof. For an index of nilpotency $n \ge 4$, the first maximal subgroup

Table 8. Non-abelian maximal subgroups $H_i < G$ of 3-groups G of coclass 1.

$G \simeq$	n	a	z	w	$H_2 = \langle x, G'\rangle$	$H_3 = \langle xy, G'\rangle$	$H_4 = \langle xy^2, G'\rangle$
$G_0^n(0,0)$	≥4	0	0	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,0)$
$G_0^n(0,1)$	≥4	0	0	1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_0^n(1,0)$	≥4	0	1	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_0^n(-1,0)$	≥4	0	−1	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_1^n(0,-1)$	≥5	1	0	−1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,0)$
$G_1^n(0,0)$	≥5	1	0	0	$\simeq G_0^{n-1}(0,0)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$
$G_1^n(0,1)$	≥5	1	0	1	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$	$\simeq G_0^{n-1}(0,1)$

Table 9. Abelian maximal subgroups $H_i < G$ of extra special 3-groups G.

$G \simeq$	n	a	z	w	$H_1 = \langle y, G' \rangle$	$H_2 = \langle x, G' \rangle$	$H_3 = \langle xy, G' \rangle$	$H_4 = \langle xy^2, G' \rangle$
$G_0^3(0,0)$	3	0	0	0	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$	$\simeq C_3 \times C_3$
$G_0^3(0,1)$	3	0	0	1	$\simeq C_3 \times C_3$	$\simeq C_9$	$\simeq C_9$	$\simeq C_9$

$H_1 = \langle y, G' \rangle$ of G coincides with the two-step centralizer $\chi_2(G)$ of G, which is a *nearly homocyclic abelian* 3-group $A(3, n-1)$ of order 3^{n-1}, when $a = 0$. For $a = 1$, we have $H_1/H_1' \simeq A(3, n-1)$.

We transform all relations of the group $G \simeq G_a^n(z,w)$ into relations of the remaining three maximal subgroups $H \simeq G_\alpha^{n-1}(\zeta, \omega)$ of G.

The *polycyclic commutator relations* $s_2 = [y, x]$, $s_i = [s_{i-1}, x]$ for $3 \le i \le n$, and the *nilpotency relation* $s_n = 1$ for the group $G = \langle x, y \rangle$, with lower central series $\gamma_i G = \langle s_i, \gamma_{i+1} G \rangle$ for $i \ge 2$, can be used immediately for the subgroup $H_2 = \langle x, G' \rangle = \langle x, s_2 \rangle$ with lower central series $\gamma_i H_2 = \langle t_i, \gamma_{i+1} H_2 \rangle$, where $t_i := s_{i+1}$ for $i \ge 2$, and $t_{n-1} = 1$.

For the lower central series of $H_3 = \langle xy, G' \rangle$ and $H_4 = \langle xy^2, G' \rangle$, we must employ the *main commutator relation* $[y, s_2] = s_{n-1}^a$, and $[y, s_i] = 1$ for $i \ge 3$. According to the *right product rule* for commutators, we have

$[s_{i-1}, xy] = [s_{i-1}, y] \cdot [s_{i-1}, x]^y = 1 \cdot s_i^y = s_i [s_i, y] = s_i \cdot 1 = s_i$, for $i \ge 4$, but

$[s_2, xy] = [s_2, y] \cdot [s_2, x]^y = s_{n-1}^{-a} s_3^y = s_{n-1}^{-a} s_3 [s_3, y] = s_{n-1}^{-a} s_3$, and in a similar fashion $[s_{i-1}, xy^2] = [s_{i-1}, y] \cdot [s_{i-1}, xy]^y = 1 \cdot s_i^y = s_i [s_i, y] = s_i \cdot 1 = s_i$, for $i \ge 4$, but again exceptionally $[s_2, xy^2] = [s_2, y] \cdot [s_2, xy]^y = s_{n-1}^{-a} y^{-1} s_{n-1}^{-a} s_3 y = s_{n-1}^{-2a} s_3 = s_{n-1}^{a} s_3$. For $a = 1$, the *left product rule* for commutators shows

$[s_{n-1}^{\mp 1} s_3, xy^{\pm 1}] = [s_{n-1}^{\mp 1}, xy^{\pm 1}]^{s_3} \cdot [s_3, xy^{\pm 1}] = s_4$, that is, the slight anomaly for the main commutator disappears in the next step. Thus, the lower central series is $\gamma_i H_j = \langle t_i, \gamma_{i+1} H_j \rangle$ for $i \ge 2$, $3 \le j \le 4$, where generally $t_i := s_{i+1}$ for $i \ge 3$, and $t_2 := s_3$ for $a = 0$, $t_2 := s_{n-1}^{2-j} s_3$ for $a = 1$. In particular, $H_3 = \langle xy, s_2 \rangle$ and $H_4 = \langle xy^2, s_2 \rangle$.

The main commutator relation for all three subgroups H_2, H_3, H_4 of any group $G \simeq G_a^n(z,w)$ with $n \ge 4$ is $[s_2, t_2] = 1 = t_{n-2}^\alpha$, that is $\alpha = 0$, generally, and it remains to determine ζ, ω.

For this purpose, we come to the *power relations* of G, $x^3 = s_{n-1}^w$,

$y^3 s_2^3 s_3 = s_{n-1}^z$, and $s_i^3 s_{i+1}^3 s_{i+2} = 1$ for $i \ge 2$, supplemented by (5.4):

$(xy)^3 = x^3 y^3 s_2^3 s_3 s_{n-1}^{-2a} = s_{n-1}^w s_{n-1}^z s_{n-1}^{-2a}$ and $(xy^2)^3 = x^3 (y^3 s_2^3 s_3)^2 s_{n-1}^{-2a} = s_{n-1}^w s_{n-1}^{2z} s_{n-1}^{-2a}$, and we use these relations to determine ζ, ω in dependence on w, z, a. Generally, we have $s_2^3 t_2^3 t_3 = s_2^3 s_3^3 s_4 = 1$ for $a = 0$,

$s_2^3 t_2^3 t_3 = s_2^3 s_{n-1}^{3(2-j)} s_3^3 s_4 = s_2^3 s_3^3 s_4 = 1$ for $a = 1$, and thus uniformly $\zeta = 0$.

For $G_0^n(0,0)$, we uniformly have $x^3 = (xy)^3 = (xy^2)^3 = 1$, and thus $\omega = 0$ for all three subgroups. For $G_0^n(0,1)$, we uniformly have

$x^3 = (xy)^3 = (xy^2)^3 = s_{n-1}$, and thus $\omega = 1$ for all three subgroups. For $G_0^n(\pm 1, 0)$, we have $x^3 = 1$, but $(xy)^3 = s_{n-1}^{\pm 1}$, $(xy^2)^3 = s_{n-1}^{\pm 2} = s_{n-1}^{\mp 1}$, and thus $\omega = 0$ for H_2 but $\omega = 1$ for H_3, H_4, since $G_0^n(0,-1) \simeq G_0^n(0,1)$.

For $G_1^n(0,-1)$, we have $x^3 = s_{n-1}^{-1}$, but $(xy)^3 = (xy^2)^3 = s_{n-1}^{-3} = 1$, and thus $\omega = 1$ for H_2 but $\omega = 0$ for H_3, H_4. For $G_1^n(0,0)$, we have $x^3 = 1$, but $(xy)^3 = (xy^2)^3 = s_{n-1}^{-2} = s_{n-1}$, and thus $\omega = 0$ for H_2 but $\omega = 1$ for H_3, H_4. For $G_1^n(0,1)$, we have $x^3 = s_{n-1}$, $(xy)^3 = (xy^2)^3 = s_{n-1}^{-1}$, and thus $\omega = 1$ for all three subgroups, again observing that $G_0^n(0,-1) \simeq G_0^n(0,1)$.

The only 3-groups G of coclass $\mathrm{cc}(G) = 1$ and order $|G| = 3^3$ are the two extra special groups $G_0^3(0,0)$ and $G_0^3(0,1)$. Since $t_2 = s_3 = 1$, all their four maximal subgroups, $H_1 = \langle y, s_2 \rangle$, $H_2 = \langle x, s_2 \rangle$, $H_3 = \langle xy, s_2 \rangle$, $H_4 = \langle xy^2, s_2 \rangle$, are abelian. For $w = z = 0$, s_2 is independent of the other generator, and $H_i \simeq C_3 \times C_3$ for $1 \le i \le 4$. However, for $w = 1$, $z = 0$, we have $x^3 = (xy)^3 = (xy^2)^3 = s_2$, $s_2^3 = 1$, and thus $H_2 \simeq H_3 \simeq H_4 \simeq C_9$, whereas $H_1 \simeq C_3 \times C_3$.

6. A General Theorem for Arbitrary Base Fields

Suppose that p is a prime, F is an algebraic number field with non-trivial p-class group $\mathrm{Cl}_p F > 1$, and E is one of the unramified abelian p-extensions of F. We show that, even in this general situation, a finite p-class tower of F exerts a very severe restriction on the p-class tower of E.

Theorem 6.1 *Assume that F possesses a p-class tower $F_p^{(\infty)} = F_p^{(n)}$ of exact length $\ell_p F = n$ for some integer $n \ge 1$. Then the Galois group $\mathrm{Gal}(E_p^{(\infty)}/E)$ of the p-class tower of E is a subgroup of index $E:F$ of the p-class tower group $\mathrm{Gal}(F_p^{(\infty)}/F)$ of F and the length of the p-class tower of E is bounded by $\ell_p E \le n$.*

Proof. According to the assumptions, there exists a tower of field extensions,

$$F < E \le F_p^{(1)} \le E_p^{(1)} \le F_p^{(2)} \le E_p^{(2)} \le \cdots \le F_p^{(n)} \le E_p^{(n)} \le F_p^{(n+1)},$$

where $\ell_p F = n$ enforces the coincidence $F_p^{(n)} = E_p^{(n)} = F_p^{(n+1)}$ of the trailing three fields. Since $\mathrm{Gal}(F_p^{(n)}/F) / \mathrm{Gal}(F_p^{(n)}/E) \simeq \mathrm{Gal}(E/F)$, the group index of $\mathrm{Gal}(E_p^{(n)}/E) = \mathrm{Gal}(F_p^{(n)}/E)$ in $\mathrm{Gal}(F_p^{(n)}/F)$ is equal to the field degree $[E:F]$ and $\mathrm{Gal}(E_p^{(\infty)}/E) = \mathrm{Gal}(E_p^{(n)}/E)$ is a subgroup of index $[E:F]$ of $\mathrm{Gal}(F_p^{(n)}/F) = \mathrm{Gal}(F_p^{(\infty)}/F)$. The equality $E_p^{(n)} = E_p^{(n+1)}$ implies the bound $\ell_p E \le n$.

We shall apply Theorem 6.1 to the situation where $p = 3$, $n = 2$, and E is an unramified cyclic cubic extension of F, whence $\mathrm{Gal}(E_3^{(\infty)}/E)$ is a maximal subgroup of $\mathrm{Gal}\left(F_3^{(\infty)}/F\right)$.

6.1. Application to Quadratic Base Fields

Proposition 6.1 *Let G be a finite 3-group with elementary bicyclic abelianization $G/G' \simeq C_3 \times C_3$. Then the following conditions are equivalent:*

1) The transfer kernel type of G is D.10, $\varkappa(G) \sim (2241)$.

2) The abelian quotient invariants of the four maximal subgroups $H_1, \cdots, H_4$ of G are $\tau(G) \sim (21, 21, 1^3, 21)$.

3) The isomorphism types of the four maximal subgroups of G are $H_1 \simeq H_2 \simeq H_4 \simeq \langle 3^4, 3 \rangle$ and $H_3 \simeq \langle 3^4, 13 \rangle$.

4) The group G is isomorphic to the Schur σ-group $\langle 3^5, 5 \rangle$ with relation rank $d_2 = 2$.

Proof. We put $G := \langle 243, 5 \rangle$ and use the presentation [14]

$$G = \langle x, y, s_2, s_3, t_3 \mid s_2 = [y, x], s_3 = [s_2, x], t_3 = [s_2, y], x^3 = s_3, y^3 = s_3 \rangle.$$

Then we obtain the maximal subgroups

$H_1 = \langle y, G' \rangle = \langle y, s_2, s_3 \rangle$, since $t_3 = [s_2, y]$,

$H_2 = \langle x, G' \rangle = \langle x, s_2, t_3 \rangle$, since $s_3 = [s_2, x]$,

$H_3 = \langle xy, G' \rangle = \langle xy, s_2, s_3 \rangle$, since $[s_2, xy] = s_3 t_3$,

$H_4 = \langle xy^2, G' \rangle = \langle xy^2, s_2, s_3 \rangle$, since $[s_2, xy^2] = s_3 t_3^2$.

Using Lemma 5.1, and comparing to the abstract presentations [14]

$\langle 81, 3 \rangle = \langle \xi, \upsilon, \sigma_2, \tau \mid \sigma_2 = [\upsilon, \xi], \tau = \xi^3 \rangle$ and

$\langle 81, 13 \rangle = \langle \xi, \upsilon, \zeta, \sigma_2 \mid \sigma_2 = [\upsilon, \xi], \xi^3 = \sigma_2, \upsilon^3 = \zeta^3 = 1 \rangle$,

we conclude

$H_1 = \langle y, s_2, s_3 \rangle = \langle y, s_2 \rangle \simeq \langle 81, 3 \rangle$, since $y^3 = s_3 \neq [s_2, y] = t_3$,

$H_2 = \langle x, s_2, t_3 \rangle \simeq \langle 81, 13 \rangle$, since $x^3 = s_3 = [s_2, x]$,

$H_3 = \langle xy, s_2, s_3 \rangle = \langle xy, s_2 \rangle \simeq \langle 81, 3 \rangle$, since $(xy)^3 = t_3^2 \neq [s_2, xy] = s_3 t_3$,

$H_4 = \langle xy^2, s_2, s_3 \rangle = \langle xy^2, s_2 \rangle \simeq \langle 81, 3 \rangle$, since $(xy^2)^3 = s_3^2 t_3^2 \neq [s_2, xy^2] = s_3 t_3^2$.

Theorem 6.2 *Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic field with elementary bicyclic 3-class group $\mathrm{Cl}_3 F \simeq C_3 \times C_3$. Then the following conditions are equivalent:*

1) The transfer kernel type of F is D.10, $\varkappa(F) \sim (2241)$.

2) The abelian type invariants of the 3-class groups $\mathrm{Cl}_3 E_i$ of the four unramified cyclic cubic extensions E_i/F are $\tau(F) \sim (21, 21, 1^3, 21)$.

3) The second 3-class group $\mathrm{G}_3^2 F$ of F has the maximal subgroups $H_1 \simeq H_2 \simeq H_4 \simeq \langle 3^4, 3 \rangle$ and $H_3 \simeq \langle 3^4, 13 \rangle$.

4) The 3-class tower group $\mathrm{G}_3^\infty F$ of F is the Schur σ-group $\langle 3^5, 5 \rangle$ with relation rank $d_2 = 2$.

Proof. The claims follow from Proposition 6.1 by applying the Successive Approximation Theorem 3.2 of first order.

Corollary 6.1 *Let F be a quadratic field which satisfies one of the equivalent* *conditions in Theorem* 6.2. *Then the length of the* 3*-class tower of F is* $\ell_3 F = 2$. *The four unramified cyclic cubic extensions* E_i/F *are absolutely dihedral of degree* 6, *with torsionfree Dirichlet unit rank* $r \ge 2$, *and possess* 3*-class towers of length* $\ell_3 E_i = 2$. *More precisely,* $\mathrm{Cl}_3 E_3 \simeq C_3 \times C_3 \times C_3$ *and* $\mathrm{G}_3^\infty E_3 \simeq \langle 3^4, 13\rangle$ *with relation rank* $d_2 = 5$, *but* $\mathrm{Cl}_3 E_i \simeq C_9 \times C_3$ *and* $\mathrm{G}_3^\infty E_i \simeq \langle 3^4, 3\rangle$ *with relation rank* $d_2 = 4$ *for* $i \in \{1,2,4\}$.

Proof. This is a consequence of Theorems 6.1 and 6.2, satisfying the Shafarevich theorem.

Proposition 6.2 *Let G be a finite* 3*-group with elementary bicyclic abelianization* $G/G' \simeq C_3 \times C_3$. *Then the following conditions are equivalent:*

1) The transfer kernel type of G is D.5, $\varkappa(G) \sim (4224)$.

2) The abelian quotient invariants of the four maximal subgroups $H_1, \cdots, H_4$ of G are $\tau(G) \sim (1^3, 21, 1^3, 21)$.

3) The isomorphism types of the four maximal subgroups of G are $H_1 \simeq H_3 \simeq \langle 3^4, 13\rangle$ and $H_2 \simeq H_4 \simeq \langle 3^4, 3\rangle$.

4) The group G is isomorphic to the Schur σ-group $\langle 3^5, 7\rangle$ with relation rank $d_2 = 2$.

Proof. We put $G := \langle 243, 7\rangle$ and use the presentation [14]

$$G = \langle x, y, s_2, s_3, t_3 \mid s_2 = [y, x], s_3 = [s_2, x], t_3 = [s_2, y], x^3 = s_3, y^3 = s_3^2 \rangle.$$

Similarly as in Proposition 6.1, we obtain the maximal subgroups

$$H_1 = \langle y, G'\rangle = \langle y, s_2, s_3\rangle, \quad H_2 = \langle x, G'\rangle = \langle x, s_2, t_3\rangle,$$

$$H_3 = \langle xy, G'\rangle = \langle xy, s_2, s_3\rangle, \text{ and } \quad H_4 = \langle xy^2, G'\rangle = \langle xy^2, s_2, s_3\rangle.$$

Using Lemma 5.1, and comparing to the abstract presentations

$\langle 81,3\rangle = \langle \xi, \upsilon, \sigma_2, \tau \mid \sigma_2 = [\upsilon, \xi], \tau = \xi^3\rangle$ and

$\langle 81,13\rangle = \langle \xi, \upsilon, \zeta, \sigma_2 \mid \sigma_2 = [\upsilon, \xi], \xi^3 = \sigma_2, \upsilon^3 = \zeta^3 = 1\rangle$,

we conclude

$H_1 = \langle y, s_2, s_3\rangle = \langle y, s_2\rangle \simeq \langle 81,3\rangle$, since $y^3 = s_3^2 \ne [s_2, y] = t_3$,

$H_2 = \langle x, s_2, t_3\rangle \simeq \langle 81,13\rangle$, since $x^3 = s_3 = [s_2, x]$,

$H_3 = \langle xy, s_2, s_3\rangle = \langle xy, s_2\rangle \simeq \langle 81,3\rangle$, since $(xy)^3 = s_3 t_3^2 \ne [s_2, xy] = s_3 t_3$,

$H_4 = \langle xy^2, s_2, s_3\rangle \simeq \langle 81,13\rangle$, since $(xy^2)^3 = s_3 t_3^2 = [s_2, xy^2]$.

Theorem 6.3 *Let* $F = \mathbb{Q}(\sqrt{d})$ *be a quadratic field with elementary bicyclic* 3*-class group* $\mathrm{Cl}_3 F \simeq C_3 \times C_3$. *Then the following conditions are equivalent:*

1) The transfer kernel type of F is D.5, $\varkappa(F) \sim (4224)$.

2) The abelian type invariants of the 3-class groups $\mathrm{Cl}_3 E_i$ of the four unramified cyclic cubic extensions E_i/F are $\tau(F) \sim (1^3, 21, 1^3, 21)$.

3) The second 3-class group $\mathrm{G}_3^2 F$ of F has the maximal subgroups $H_1 \simeq H_3 \simeq \langle 3^4, 13\rangle$ and $H_2 \simeq H_4 \simeq \langle 3^4, 3\rangle$.

4) The 3-class tower group $\mathrm{G}_3^\infty F$ of F is the Schur σ-group $\langle 3^5, 7\rangle$ with relation rank $d_2 = 2$.

Proof. The claims follow from Proposition 6.2 by applying the Successive

Approximation Theorem 3.2 of first order.

Corollary 6.2 *Let F be a quadratic field which satisfies one of the equivalent conditions in Theorem* 6.3. *Then the length of the* 3*-class tower of F is* $\ell_3 F = 2$. *The four unramified cyclic cubic extensions* E_i/F *are absolutely dihedral of degree* 6, *with torsionfree Dirichlet unit rank* $r \ge 2$, *and possess* 3*-class towers of length* $\ell_3 E_i = 2$. *More precisely,* $\mathrm{Cl}_3 E_i \simeq C_3 \times C_3 \times C_3$ *and* $\mathrm{G}_3^\infty E_i \simeq \langle 3^4, 13 \rangle$ *with relation rank* $d_2 = 5$ *for* $i \in \{1,3\}$, *but* $\mathrm{Cl}_3 E_i \simeq C_9 \times C_3$ *and* $\mathrm{G}_3^\infty E_i \simeq \langle 3^4, 3 \rangle$ *with relation rank* $d_2 = 4$ *for* $i \in \{2,4\}$.

Proof. This is a consequence of Theorems 6.1 and 6.3, satisfying the Shafarevich theorem.

6.2. Application to Dihedral Fields

We recall that a dihedral field E of degree 6 is an absolute Galois extension $E/\mathbb{Q}$ with group $\mathrm{Gal}(E/\mathbb{Q}) = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \sigma\tau = \tau\sigma^{-1} \rangle$. It is a cyclic cubic relative extension E/F of its unique quadratic subfield $F = E^\sigma$, and it contains three isomorphic, conjugate non-Galois cubic subfields $L = E^\tau$, L^σ, L^{σ^2}. The conductor c of E/F is a nearly squarefree positive integer with special prime factors, and the discriminants satisfy the relations $d_E = c^4 d_F^3$ and $d_L = c^2 d_F$. Here, we shall always be concerned with unramified extensions, characterized by the conductor $c = 1$, and thus $d_E = d_F^3$, a perfect cube, and equal $d_L = d_F$.

6.2.1. Totally Complex Dihedral Fields

The computational information on 3-tower groups $G := \mathrm{G}_3^\infty F$ of imaginary quadratic fields F in Table 3 admits the purely theoretical deduction of impressive statistics for 3-tower groups $S := \mathrm{G}_3^\infty E$ of totally complex dihedral fields E in Table 10 by means of the Corollaries 6.1 and 6.2. We use the crucial new insight that the groups $S \lhd G$ are maximal subgroups of G, because the extensions E/F are unramified cyclic of degree 3.

6.2.2. Totally Real Dihedral Fields

The computational information on 3-tower groups $G := \mathrm{G}_3^\infty F$ of real quadratic fields F in Table 4 admits the purely theoretical deduction of impressive statistics for 3-tower groups $S := \mathrm{G}_3^\infty E$ of totally real dihedral fields E in Table 11 by means of Theorem 5.1. Again, we use the innovative result that the groups $S \lhd G$ are maximal subgroups of G, since the extensions E/F are unramified cyclic cubic.

Table 10. Frequencies of dihedral 3-class tower groups S for $-10^{24} < d_E < 0$.

$G \simeq$	$\tau^{(1)}G$	abs. fr.	$S \simeq$	$\tau^{(1)}S$	abs. fr.	$\lvert d_E \rvert_{\min}$
$\langle 243,5 \rangle$	1^2	83353	$\langle 81,3 \rangle$	21	250059	4027^3
$\langle 243,5 \rangle$	1^2	83353	$\langle 81,13 \rangle$	1^3	83353	4027^3
$\langle 243,7 \rangle$	1^2	41398	$\langle 81,3 \rangle$	21	82796	12131^3
$\langle 243,7 \rangle$	1^2	41398	$\langle 81,13 \rangle$	1^3	82796	12131^3

Table 11. Frequencies of dihedral 3-class tower groups S for $0 < d_E < 10^{27}$.

$G \simeq$	$\tau^{(1)}G$	abs. fr.	$S \simeq$	$\tau^{(1)}S$	abs. fr.	$(d_E)_{\min}$
$\langle 81,7\rangle$	1^2	122955	$\langle 27,3\rangle$	1^2	122955	142097^3
$\langle 81,7\rangle$	1^2	122955	$\langle 27,4\rangle$	1^2	245910	142097^3
$\langle 81,7\rangle$	1^2	122955	$\langle 27,5\rangle$	1^2	122955	142097^3

The first row of **Table 11** reveals extensive realizations of the extraspecial group $S = \langle 27,3\rangle$ as 3-tower group of dihedral fields. This is the first time that $S = \langle 27,3\rangle$ occurs as a 3-tower group. It is forbidden for quadratic fields, and it did not occur for cyclic cubic fields and bicyclic biquadratic fields, up to now.

Theorem 6.4 (***A new realization as*** 3***-tower group.***) *The extraspecial* 3*-group* $S = \langle 27,3\rangle$ *of coclass* 1 *and exponent* 3 *occurs as* 3*-class tower group* $\mathrm{G}_3^\infty E$ *of totally real dihedral fields E of degree* 6.

Proof. The group $S = \langle 27,3\rangle$ possesses the relation rank $d_2 S = 4$. According to the Shafarevich Theorem, it is therefore excluded as 3-tower group $\mathrm{G}_3^\infty F$ of both, imaginary and real quadratic fields F. However, the combination of Theorem 5.1 and Theorem 6.1 proves its occurrence as 3-class tower group $\mathrm{G}_3^\infty E$ of totally real dihedral fields E of degree 6, as visualized in **Table 11**.

Theorem 6.5 (**3*-class tower groups of totally real dihedral fields.***) *Let* $F = \mathbb{Q}(\sqrt{d})$ *be a real quadratic field with* 3*-class group* $\mathrm{Cl}_3 F \simeq C_3 \times C_3$ *and fundamental discriminant* $d > 1$. *Suppose the second order Artin pattern* $\mathrm{AP}^{(2)} F = \left(\tau^{(2)}(F), \varkappa^{(2)}(F)\right)$ *is given by the abelian type invariants* $\tau^{(2)}(F) = \left[1^2;\left(2^2,1^2,1^2,1^2\right)\right]$ *and the transfer kernel type* $\varkappa^{(2)}(F) = [1;(0000)]$. *Let* E_2, E_3, E_4 *be the three unramified cyclic cubic relative extensions of F with* 3*-class group* $\mathrm{Cl}_3 E_i \simeq C_3 \times C_3$.

Then $E_i/\mathbb{Q}$ is a totally real dihedral extension of degree 6, for each $2 \le i \le 4$, and the connection between the component $\#\varkappa^{(3)}(F)_i = \#\ker\left(T_{E_i,F_3^{(1)}}\right)$ of the third order transfer kernel type $\varkappa^{(3)}(F)$ and the 3-class tower group $S_i = G_3^\infty E_i = \mathrm{Gal}\left((E_i)_3^{(\infty)}/E_i\right)$ of E_i is given in the following way:

$$\begin{aligned} \#\varkappa^{(3)}(F)_i = 3 &\Leftrightarrow S_i \simeq \langle 243,27\rangle \text{ with } \varkappa(S_i) = (1000), \\ \#\varkappa^{(3)}(F)_i = 9 &\Leftrightarrow S_i \simeq \langle 243,26\rangle \text{ with } \varkappa(S_i) = (0000). \end{aligned} \tag{6.1}$$

Proof. This theorem was expressed as a conjecture in [27] [28], and is now an immediate consequence of Theorems 6.1 and 5.1.

Remark 6.1 *Recall that each unramified cyclic cubic relative extension* E_i/F, $1 \le i \le 4$, *gives rise to a dihedral absolute extension* $E_i/\mathbb{Q}$ *of degree* 6, *that is an* S_3*-extension* ([7], *Prp.* 4.1, *p.* 482). *For the trailing three fields* E_i, $2 \le i \le 4$, *in the stable part of* $\tau^{(2)}(F) = \left[1^2;\left(2^2,1^2,1^2,1^2\right)\right]$, *i.e. with* $\mathrm{Cl}_3 E_i \simeq C_3 \times C_3$, *we have constructed the unramified cyclic cubic extensions* $\tilde{E}_{i,j}/E_i$, $1 \le j \le 4$, *and determined the Artin pattern* $\mathrm{AP}^{(2)} E_i$ *of* E_i, *in particular, the transfer kernel type of* E_i *in the fields* $\tilde{E}_{i,j}$ *of absolute degree* 18. *The dihedral fields* E_i *of* *degree* 6 *share a common polarization* $\tilde{E}_{i,1} = F_3^{(1)}$, *the Hilbert* 3*-class field of F, which is contained in the relative* 3*-genus field* $\left(E_i/F\right)^*$, *whereas the other extensions* $\tilde{E}_{i,j}$ *with* $2 \le j \le 4$ *are non-abelian over F, for each* $2 \le i \le 4$. *Our computational results underpin Theorem* 6.5 *concerning the infinite family of totally real dihedral fields* E_i *for varying real quadratic fields F.*

7. Conclusion

Guided by the Successive Approximation Theorem 3.2 in terms of the Artin limit pattern, we have given a most up-to-date survey concerning the finite 3-groups which are populated most densely by 3-class tower groups $\mathrm{G}_3^{\infty}F$ of quadratic number fields $F = \mathbb{Q}\left(\sqrt{d}\right)$ in Sections 4.2-4.5. In particular, the discovery of non-metabelian 3-class towers with exact length $\ell_3 F = 3$, which is currently the maximal proven finite length, in Theorems 4.5 and 4.6, is entirely due to our cooperation with M. R. Bush, initiated by our joint paper [36]. With Theorems 5.1 and 6.1, we have finally presented a new technique for deriving theoretical conclusions on 3-class towers of dihedral fields with degree six from corresponding results for quadratic fields.

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Study on the Existence of Sign-Changing Solutions of Case Theory Based a Class of Differential Equations Boundary-Value Problems

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Abstract

By using the fixed point theorem under the case structure, we study the existence of sign-changing solutions of A class of second-order differential equations three-point boundary-value problems, and a positive solution and a negative solution are obtained respectively, so as to popularize and improve some results that have been known.

Keywords

Case Theory, Boundary-Value Problems, Fixed Point Theorem, Sign-Changing Solutions

1. Introduction

The existence of nonlinear three-point boundary-value problems has been studied [1]-[6], and the existence of sign-changing solutions is obtained. In the past, most studies were focused on the cone fixed point index theory [7] [8] [9], just a few took use of case theory to study the topological degree of non-cone mapping and the calculation of fixed point index, and the case theory was combined with the topological degree theory to study the sign-changing solutions. Recent study Ref. [10] [11] have given the method of calculating the topological degree under the case structure, and taken use of the fixed point theorem of non-cone mapping to study the existence of nontrivial solutions for the nonlinear Sturm-Liouville problems. Relevant studies as [12] [13] [14].

Inspired by the Ref. [8]-[13] and by using the new fixed point theorem under the case structure, this paper studies three-point boundary-value problems for A class of nonlinear second-order equations

$$\begin{cases} u''(t)+f(u(t))=0, 0\le t\le 1; \\ u'(0)=0, u(1)=\alpha u(\eta), \end{cases} \tag{1}$$

Existence of the sign-changing solution, constant $0<\alpha<1, 0<\eta<1$, $f\in C(R,R)$.

Boundary-value problem (1) is equivalent to Hammerstein nonlinear integral equation hereunder

$$u(t)=\int_0^1 G(t,s)f(u(s))\mathrm{d}s, 0\le t\le 1 \tag{2}$$

Of which $G(t,s)$ is the Green function hereunder

$$G(t,s)=\frac{1}{1-\alpha}\begin{cases}(1-s)-\alpha(\eta-s), 0\le s\le\eta, 0\le t\le s; \\ (1-s), \eta\le s\le 1, 0\le t\le s; \\ (1-\alpha\eta)-t(1-\alpha), 0\le s\le\eta, s\le t\le 1; \\ (1-\alpha y)-t(1-\alpha), \eta\le s\le 1, s\le t\le 1.\end{cases}$$

Defining linear operator K as follow

$$(Ku)(t)=\int_0^1 G(t,s)u(s)\mathrm{d}s, u\in C[0,1]. \tag{3}$$

Let $Fu(t)=f(u(t))$, $t\in[0,1]$, obviously composition operator $A=KF$, *i.e.*

$$(Au)(t)=\int_0^1 G(t,s)f(u(s))\mathrm{d}s, 0\le t\le 1 \tag{4}$$

It's easy to get: $u\in C^2[0,1]$ is the solution of boundary-value problem (1), and $u\in C[0,1]$ is the solution of operator equation $u=Au$.

We note that, in Ref. [9] [10], an abstract result on the existence of sign-changing solutions can be directly applied to problem (1). After the necessary preparation, when the non-linear term f is under certain assumptions, we get the existence of sign-changing solution of such boundary-value problems. Compared with the Ref. [8], we can see that we generalize and improve the non-linear term f, and remove the conditions of strictly increasing function, and the method is different from Ref. [8].

For convenience, we give the following conditions.

(H_1) $f(u):R\to R$ continues, $f(u)u>0$, $\forall u\in R, u\ne 0$, and $f(0)=0$.

(H_2) $\lim\limits_{u\to 0}\frac{f(u)}{u}=\beta$, and $n_0\in N$, make $\lambda_{2n_0}<\beta<\lambda_{2n_0+1}$, of which $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\lambda_{n+1}<\cdots$ is the positive sequence of $\cos\sqrt{x}=\alpha\cos\eta\sqrt{x}$.

(H_3) exists $\varepsilon>0$, make $\lim\limits_{|u|\to+\infty}\sup\frac{f(u)}{u}\le\lambda_1-\varepsilon$.

2. Knowledge

Provided P is the cone of E in *Banach* space, the semi order in E is exported by cone P. If the constant $N>0$, and $\theta\le x\le y\Rightarrow\|x\|\le N\|y\|$, then P is a normal cone; if P contains internal point, *i.e.* $\text{int}\, P \neq \varnothing$, then P is a solid cone.

E becomes a case when semi order $\leq$, *i.e.* any $x, y \in E$, $\sup\{x, y\}$ and $\inf\{x, y\}$ is existed, for $x \in E$, $x^+ = \sup\{x, \theta\}$, $x^- = \sup\{-x, \theta\}$, we call positive and negative of x respectively, call $|x| = x^+ + x^-$ as the modulus of x. Obviously, $x^+ \in P$, $x^- \in (-P)$, $|x| \in P$, $x = x^+ - x^-$.

For convenience, we use the following signs: $x_+ = x^+$, $x_- = -x^-$. Such that $x = x_+ + x_-$, $|x| = x_+ - x_-$.

Provided Banach space $E = C[0,1]$, and E's norm as $\|\cdot\|$, *i.e.* $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Let $P = \{u \in E \mid u(t) \geq 0, t \in [0,1]\}$, then P is the normal cone of E, and E becomes a case under semi order $\leq$.

Now we give the definitions and theorems

Def 1 [10] provided $D \subset E, A: D \to E$ is an operator (generally a nonlinear). If $Ax = Ax_+ + Ax_-, \forall x \in E$, then A is an additive operator under case structure; if $v^* \in E$, and $Ax = Ax_+ + Ax_- + v^*, \forall x \in E$, then A is a quasi additive operator.

Def 2 provided x is a fixed point of A, if $x \in (P \setminus \{\theta\})$, then x is a positive fixed point; if $x \in ((-P) \setminus \{\theta\})$, then x is a negative fixed point; if $x \notin (P \cup (-P))$, then x is a sign-changing fixed point.

Lemma 1 [6] $G(t,s)$ is a nonnegative continuous function of $[0,1] \times [0,1]$, and when $t, s \in [0,1]$, $G(t,s) \geq \gamma G(0,s)$, of which $\gamma = \dfrac{\alpha(1-\eta)}{1-\alpha\eta}$.

Lemma 2 $K: P \to P$ is completely continuous operator, and $A: E \to E$ is completely continuous operator.

Lemma 3 A is a quasi additive operator under case structure.

Proof: Similar to the proofs in Lemma 4.3.1 in Ref. [10], get Lemma 3 works.

Lemma 4 [6] the eigenvalues of the linear operator K are $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \cdots, \dfrac{1}{\lambda_n}, \dfrac{1}{\lambda_{n+1}}, \cdots$. And the sum of algebraic multiplicity of all eigenvalues is 1, of which λ_n is defined by (H_2).

The lemmas hereunder are the main study bases.

Lemma 5 [10] provided E is *Banach* space, P is the normal cone in E, $A: E \to E$ is completely continuous operator, and quasi additive operator under case structure. Provided that

1) There exists positive bounded linear operator B_1, and B_1's $r(B_1) < 1$, and $u^* \in P, u_1 \in P$, get

$$-u^* \leq Ax \leq B_1 x + u_1, \forall x \in P;$$

2) There exists positive bounded linear operator B_2, B_2's $r(B_2) < 1$, and $u_2 \in P$, get

$$Ax \geq B_2 x - u_2, \forall x \in (-P);$$

3) $A\theta = \theta$, there exists *Frechet* derivative A'_θ of A at θ, 1 is not the eigenvalue of A'_θ, and the sum μ of algebraic multiplicity of A'_θ's all eigenvalues in the range $(1, \infty)$ is a nonzero even number,

$$A\left(P\setminus\{\theta\}\right)\subset \overset{\circ}{P},\ A\left((-P)\setminus\{\theta\}\right)\subset -\overset{\circ}{P}$$

Then A exists three nonzero fixed points at least: one positive fixed point, one negative fixed point and a sign-changing fixed point.

3. Results

Theorem provided (H_1) (H_2) (H_3) works, boundary-value problem (1) exists a sign-changing solution at least, and also a positive solution and a negative solution.

Proof provided linear operator $B=\left(\lambda_1-\frac{\varepsilon}{2}\right)K$, Lemma 2 knows $B:C[0,1]\to C[0,1]$ is a positive bounded linear operator. Lemma 4 gets K's $r(K)=\frac{1}{\lambda_1}$, so $r(B)=\left(\lambda_1-\frac{\varepsilon}{2}\right)r(K)=1-\frac{\varepsilon}{2\lambda_1}<1$.

(H_3) knows $m>0$ and gets

$$f(u)\le\left(\lambda_1-\frac{\varepsilon}{2}\right)u+m,\ \forall t\in[0,1],\ u\ge 0 \tag{5}$$

$$f(u)\ge\left(\lambda_1-\frac{\varepsilon}{2}\right)u-m,\ \forall t\in[0,1],\ u\le 0 \tag{6}$$

Let $u_0(t)=m\int_0^1 G(t,s)\mathrm{d}s$, obviously, $u_0\in P$. Such that, for any $u(t)\in P$, there

$$\begin{aligned}(Au)(t)&=\int_0^1 G(t,s)f(u(s))\mathrm{d}s\\&\le\int_0^1 G(t,s)\left(\left(\lambda_1-\frac{\varepsilon}{2}\right)u+m\right)\mathrm{d}s\\&\le\left(\lambda_1-\frac{\varepsilon}{2}\right)\int_0^1 G(t,s)u(s)\mathrm{d}s+m\int_0^1 G(t,s)\mathrm{d}s\\&=\left(\lambda_1-\frac{\varepsilon}{2}\right)Ku(t)+u_0(t)\\&=Bu(t)+u_0(t)\end{aligned}$$

And for any $u^*\in P$, from (H_1), obviously gets $(Au)(t)\ge -u^*(t)$.

For any $u(t)\in -P$, there

$$\begin{aligned}(Au)(t)&=\int_0^1 G(t,s)f(u(s))\mathrm{d}s\\&\ge\int_0^1 G(t,s)\left(\left(\lambda_1-\frac{\varepsilon}{2}\right)u-m\right)\mathrm{d}s\\&\ge\left(\lambda_1-\frac{\varepsilon}{2}\right)\int_0^1 G(t,s)u(s)\mathrm{d}s-m\int_0^1 G(t,s)\mathrm{d}s\\&=\left(\lambda_1-\frac{\varepsilon}{2}\right)Ku(t)-u_0(t)\\&=Bu(t)-u_0(t)\end{aligned}$$

Consequently (1) (2) in lemma 5 works.

We note that $f(0)=0$ can get $A\theta=\theta$, from (H_2), we know $\forall\varepsilon>0$, and $\exists r>0$ gets

$$\left|f(u)-\beta u\right|\le\varepsilon u,|u|\le r$$

Then

$$\left|(Fu)(t)-\lambda u(t)\right|=\left|f(u(t))-\beta u(t)\right|\le\varepsilon\|u\|,\forall\|u\|\le r$$
$$\|Au-A\theta-\beta Ku\|=\|K(Fu)-\beta Ku\|\le\varepsilon\|K\|\|u\|,\forall\|u\|\le r$$

Such that

$$\lim_{\|u\|\to 0}\frac{\|Au-A\theta-\beta Ku\|}{\|u\|}=0$$

i.e. $A'_\theta=\beta K$, from lemma 4 we get linear operator K's eigenvalue is $\frac{1}{\lambda_n}$, then A'_θ's eigenvalue is $\frac{\beta}{\lambda_n}$. Because $\lambda_{2n_0}<\beta<\lambda_{2n_0+1}$, let μ be the sum of algebraic multiplicity of A'_θ's all eigenvalues in the range $(1,\infty)$, then $\mu=2n_0$ is an even number.

From (H_1) $f(u)u>0$, $u\in R\setminus\{0\}$, there

$$f(u(t))>0,\forall t\in[0,1],u(t)>0,$$
$$f(u(t))<0,\forall t\in[0,1],u(t)<0.$$

Easy to get

$$F(P\setminus\{\theta\})\subset P\setminus\{\theta\},F((-P)\setminus\{\theta\})\subset(-P)\setminus\{\theta\},$$

Lemma (1) for any $u(t)\in P$,

$(Ku)(t)=\int_0^1G(t,s)u(s)\mathrm{d}s\ge\gamma\int_0^1G(0,s)u(s)\mathrm{d}s$,

consequently $K(P\setminus\{\theta\})\subset \mathring{P}$. Such that

$$A(P\setminus\{\theta\})\subset\mathring{P},A((-P)\setminus\{\theta\})\subset-\mathring{P},$$

Such that (3) in lemma 5 works. According to lemma 5, A exists three nonzero fixed points at least: one positive fixed point, one negative fixed point and one sign-changing fixed point. Which states that boundary-value problem (1) has three nonzero solutions at least: one positive solution, one negative solution and one sign-changing solution.

4. Conclusion

Provided that all conditions of the theorem are satisfied, and $f(u)$ is an odd function, then boundary-value problem (1) has four nonzero solutions at least: one positive solution, one negative solution and two sign-changing solutions.

Note

By using case theory to study the topological degree of non-cone mapping and the calculation of fixed point index, it's an attempt to combine case theory and topological degree theory, the author thinks it's an up-and-coming topic and expects to have further progress on that.

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The Analyticity for the Product of Analytic Functions on Octonions and Its Applications

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Abstract

Given two left $\mathbf{O}^c$-analytic functions f, g in some open set Ω of $\mathbf{R}^8$, we obtain some sufficient conditions for fg is also left $\mathbf{O}^c$-analytic in Ω. Moreover, we prove that $f\lambda$ is a left $\mathbf{O}^c$-analytic function for any constants $\lambda \in \mathbf{O}^c$ if and only if $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system. Some applications and connections with Cauchy-Kowalewski product are also considered.

Keywords

Octonions, $\mathbf{O}^c$-Analytic Functions, Stein-Weiss Conjugate Harmonic System, Cauchy-Kowalewski Product

1. Introduction

Let Ω be an open set of $\mathbf{R}^8$. A function f in $C^1(\Omega, \mathbf{O})$ is said to be left (right) $\mathbf{O}$-analytic in Ω when

$$Df = \sum_{i=0}^{7} e_i \frac{\partial f}{\partial x_i} = 0 \quad \left(fD = \sum_{i=0}^{7} \frac{\partial f}{\partial x_i} e_i = 0 \right),$$

where the Dirac D-operator and its adjoint $\bar{D}$ are the first-order systems of differential operators in $C^1(\Omega, \mathbf{O})$ defined by $D = \sum_0^7 e_i \frac{\partial}{\partial x_i}$ and $\bar{D} = e_0 \frac{\partial}{\partial x_0} - \sum_1^7 e_i \frac{\partial}{\partial x_i}$.

If f is a simultaneously left and right $\mathbf{O}$-analytic function, then f is called an $\mathbf{O}$-analytic function. If f is a (left) $\mathbf{O}$-analytic function in $\mathbf{R}^8$, then f is called a (left) $\mathbf{O}$-entire function.

Since octonions is non-commutative and non-associative, the product $f(x)g(x)$ of two left $\mathbf{O}$-analytic functions $f(x)$ and $g(x)$ is generally no longer a left $\mathbf{O}$-analytic function. Furthermore, if $g(x)\equiv\lambda$ becomes an octonionic constant function, the product $f(x)\lambda$ is also probably not a left $\mathbf{O}$-analytic function; that is, the collection of left $\mathbf{O}$-analytic functions is not a right module (see [1]).

The purpose of this paper is to study the analyticity for the product of two left $\mathbf{O}^c$-analytic functions in the framework of complexification of $\mathbf{O}$, $\mathbf{O}^c$. Especially, the analyticity for the product of left $\mathbf{O}^c$-analytic functions and $\mathbf{O}^c$ constants will be consider more by us.

The rest of this paper is organized as follows. Section 2 is an overview of some basic facts concerning octonions and octonionic analysis. Section 3 we give some sufficient conditions for the product $f(x)g(x)$ of two left $\mathbf{O}^c$-analytic functions $f(x)$ and $g(x)$ is also a left $\mathbf{O}^c$-analytic function. In Section 3, we prove that, $f(x)\lambda$ is a left $\mathbf{O}^c$-analytic function for any constants $\lambda\in\mathbf{O}^c$ if and only if $\overline{f(x)}$ is a complex Stein-Weiss conjugate harmonic system. This gives the solution of the problem in [2]. In the last section we give some applications for our results.

2. Preliminaries: Octonions and Octonionic Analysis

It is well known that there are only four normed division algebras [3] [4] [5]: the real numbers $\mathbf{R}$, complex numbers $\mathbf{C}$, quaternions $\mathbf{H}$ and octonions $\mathbf{O}$, with the relations $\mathbf{R}\subseteq\mathbf{C}\subseteq\mathbf{H}\subseteq\mathbf{O}$. In other words, for any $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)\in\mathbf{R}^n$, if we define a product "xy" such that $xy\in\mathbf{R}^n$ and $|x\cdot y|=|x||y|$, where $|x|=\sqrt{\sum_1^n x_i^2}$, then the only four values of n are 1,2,4,8. Quaternions $\mathbf{H}$ is not commutative and octonions $\mathbf{O}$ is neither commutative nor associative. Unlike $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, the non-associative octonions can not be embedded into the associative Clifford algebras [6].

Octonions stand at the crossroads of many interesting fields of mathematics, they have close relations with Clifford algebras, spinors, Bott periodicity, Projection and Lorentzian geometry, Jordan algebras, and exceptional Lie groups, and also, they have many applications in quantum logic, special relativity and supersymmetry [3] [4].

Denote the set $\mathscr{W}$ by

$$\mathscr{W}=\{(1,2,3),(1,4,5),(1,7,6),(2,4,6),(2,5,7),(3,4,7),(3,6,5)\}.$$

Then the multiplication rules between the basis $e_0,e_1,\cdots,e_7$ on octonions are given by [3] [7]:

$$e_0^2=e_0,\ e_ie_0=e_0e_i=e_i,\ e_i^2=-1,\ i=1,2,\cdots,7,$$

and for any triple $(\alpha,\beta,\gamma)\in\mathscr{W}$,

$$e_\alpha e_\beta=e_\gamma=-e_\beta e_\alpha,\quad e_\beta e_\gamma=e_\alpha=-e_\gamma e_\beta,\quad e_\gamma e_\alpha=e_\beta=-e_\alpha e_\gamma.$$

For each $x=\sum_0^7 x_ie_i\in\mathbf{O}(x_i\in\mathbf{R},i=0,1,\cdots,7)$, x_0 is called the scalar part of

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x and $\underline{x}=\sum_1^7 x_i e_i$ is termed its vector part. Then the norm of x is $|x|=\left(\sum_0^7 x_i^2\right)^{\frac{1}{2}}$ and its conjugate is defined by $\bar{x}=\sum_0^7 x_i\bar{e}_i=x_0-\underline{x}$. We have $x\bar{x}=\bar{x}x=\sum_0^7 x_i^2$, $\overline{xy}=\bar{y}\bar{x}\,(x,y\in\mathbf{O})$ Hence, $x^{-1}=\frac{\bar{x}}{|x|^2}$ is the inverse of $x(\neq 0)$.

Let $x=\sum_0^7 x_i e_i, y=\sum_0^7 y_i e_i\in\mathbf{O}\,(x_i,y_i\in\mathbf{R}, i=0,1,\cdots,7)$, then

$$xy=x_0y_0-\underline{x}\cdot\underline{y}+x_0\underline{y}+y_0\underline{x}+\underline{x}\times\underline{y}, \tag{2.1}$$

where $\underline{x}\cdot\underline{y}:=\sum_1^7 x_i y_i$ is the inner product of vectors $\underline{x},\underline{y}$ and

$$\begin{aligned}\underline{x}\times\underline{y}:=\,&e_1\left(A_{23}+A_{45}-A_{67}\right)+e_2\left(-A_{13}+A_{46}+A_{57}\right)+e_3\left(A_{12}+A_{47}-A_{56}\right)\\&+e_4\left(-A_{15}-A_{26}-A_{37}\right)+e_5\left(A_{14}-A_{27}+A_{36}\right)\\&+e_6\left(A_{17}+A_{24}-A_{35}\right)+e_7\left(-A_{16}+A_{25}+A_{34}\right)\end{aligned}$$

is the cross product of vectors $\underline{x},\underline{y}$, with

$$A_{ij}=\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix},\quad i,j=1,2,\cdots,7.$$

For any $x,y\in\mathbf{O}$, the inner product and cross product of their vector parts satisfy the following rules [8]:

$$\left(\underline{x}\times\underline{y}\right)\cdot\underline{x}=0,\quad \left(\underline{x}\times\underline{y}\right)\cdot\underline{y}=0,\quad \underline{x}\parallel\underline{y}\Leftrightarrow\underline{x}\times\underline{y}=0,\quad \underline{x}\times\underline{y}=-\underline{y}\times\underline{x}.$$

We usually utilize associator as an useful tool on ontonions since its non-associativity. Define the associator $[x,y,z]$ of any $x,y,z\in\mathbf{O}$ by $[x,y,z]=(xy)z-x(yz)$.

The octonions obey the following some weakened associative laws.

For any $x,y,z,u,v\in\mathbf{O}$, we have (see [7])

$$[x,y,z]=[y,z,x],\quad [x,z,y]=-[x,y,z],\quad [x,x,y]=0=[\bar{x},x,y] \tag{2.2}$$

and the so-called Moufang identities [5]

$$(uvu)x=u(v(ux)),\quad x(uvu)=((xu)v)u,\quad u(xy)u=(ux)(uy).$$

Proposition 2.1 ([7]). *For any* $i,j,k\in\{0,1,\cdots,7\}$, $\left[e_i,e_j,e_k\right]=0\Leftrightarrow ijk=0$ *or* $(i-j)(j-k)(k-i)=0$ *or* $\left(e_ie_j\right)e_k=\pm1$.

Proposition 2.2 ([7]). *Let* e_i,e_j,e_k *be three different elements of* $\{e_1,e_2,\cdots,e_7\}$ *and* $\left(e_ie_j\right)e_k\neq\pm1$. *Then* $\left(e_ie_j\right)e_k=-e_i\left(e_je_k\right)$.

Since octonions is an alternative algebra (see [3] [9] [10]), we have the following power-associativity of octonions.

Proposition 2.3. *Let* $x_1,x_2,\cdots,x_k\in\mathbf{O}$, $(l_1,\cdots,l_n)$ *be* n *elements out of* $\{1,\cdots,k\}$ *repetitions being allowed and let* $\left(x_{l_1}x_{l_2}\cdots x_{l_n}\right)_{\otimes_n}$ *be the product of* n *octonions in a fixed associative order* $\otimes_n$. *Then* $\sum_{\pi(l_1,\cdots,l_n)}\left(x_{l_1}x_{l_2}\cdots x_{l_n}\right)_{\otimes_n}$ *is independent of the associative order* $\otimes_n$, *where the sum runs over all* *distinguishable permutations of* $(l_1,\cdots,l_n)$

Proof. Let $x=\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k$, then $\sum_{\pi(l_1,\cdots,l_n)}\left(x_{l_1}x_{l_2}\cdots x_{l_n}\right)_{\otimes_n}$ is just the coefficient of $\lambda_{l_1}\lambda_{l_2}\cdots\lambda_{l_n}$ in the product of $x^n=\underbrace{(xx\cdots x)}_{n\ xs}{}_{\otimes_n}$. By induction and (2.2), one can easily prove that $x^n=\underbrace{(xx\cdots x)}_{n\ xs}{}_{\otimes_n}$ is independent of the associative order $\otimes_n$ for any $x\in\mathbf{O}$. Hence $\sum_{\pi(l_1,\cdots,l_n)}\left(x_{l_1}x_{l_2}\cdots x_{l_n}\right)_{\otimes_n}$ is also independent of the associative order $\otimes_n$. □

$\mu=(\mu_0,\mu_1,\cdots,\mu_n)$ is called a Stein-Weiss conjugate harmonic system if they satisfy the following equations (see [11]):

$$\sum_{i=0}^{n}\frac{\partial\mu_i}{\partial x_i}=0,\quad \frac{\partial\mu_i}{\partial x_j}=\frac{\partial\mu_j}{\partial x_i}\quad (0\le i<j\le n).$$

It is easy to see that if $F(x_0,x_1,\cdots,x_7)=(f_0,f_1,\cdots,f_7)$ is a Stein-Weiss conjugate harmonic system in an open set Ω of $\mathbf{R}^8$, then there exists a real-valued harmonic function Φ in Ω such that F is the gradient of Φ. Thus $\bar{F}=f_0e_0-f_1e_1-\cdots-f_7e_7=\bar{D}\Phi$ is an $\mathbf{O}$-analytic function. But inversely, this is not true [12].

Example. Observe the $\mathbf{O}$-analytic function $g(x)=\left(x_6^2-x_7^2\right)e_2-2x_6x_7e_3$. Since

$$\frac{\partial g_2}{\partial x_6}=2x_6\neq 0=\frac{\partial g_6}{\partial x_2},$$

$\bar{g}$ is not a Stein-Weiss conjugate harmonic system.

In [13] Li and Peng proved the octonionic analogue of the classical Taylor theorem. Taking account of Proposition 2.3, we obtain an improving of Taylor type theorem for $\mathbf{O}$-analytic functions (see [14] [15]).

Theorem A (Taylor). *If* $f(x)$ *is a left* $\mathbf{O}$*-analytic function in* Ω *which containing the origin, then it can be developed into Taylor series*

$$f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}V_{l_1\cdots l_k}(x)\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}f(0),$$

and if $f(x)$ is a right $\mathbf{O}$-analytic function, then the Taylor series of f at the origin is given by

$$f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}f(0)V_{l_1\cdots l_k}(x),$$

where $(l_1,\cdots,l_k)$ runs over all possible combinations of k elements out of $\{1,\cdots,7\}$ repetitions being allowed.

The polynomials $V_{l_1\cdots l_k}$ of order k in Theorem A is defined by

$$V_{l_1\cdots l_k}(x)=\frac{1}{k!}\sum_{\pi(l_1,\cdots,l_k)}\left(\cdots\left(\left(z_{l_1}z_{l_2}\right)z_{l_3}\right)\cdots\right)z_{l_k},$$

where the sum runs over all distinguishable permutations of $(l_1,\cdots,l_k)$ and

$$z_{l_j} = x_{l_j} e_0 - x_0 e_{l_j}, j = 1, \cdots, k.$$

We have the following uniqueness theorem for $\mathbf{O}$-analytic functions [7].

Proposition 2.4. *If* f *is left* (*right*) $\mathbf{O}$*-analytic in an open connect set* $\Omega \subset \mathbf{R}^8$ *and vanishes in the open set* $\mathcal{E} \subset \Omega \cap \{x_0 = a_0\} \neq \varnothing$, *then* f *is identically zero in* Ω.

Proof. Without loss of generality, we let $\mathcal{E}$ which containing the origin and let $x_0 = 0$. Then f can be developed into Taylor series

$$f(x) = \sum_{k=0}^{\infty} \sum_{(l_1, \cdots, l_k)} V_{l_1 \cdots l_k}(x) \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0).$$

Thus we have

$$f(\underline{x}) = \sum_{k=0}^{\infty} \sum_{(l_1, \cdots, l_k)} x_{l_1} x_{l_2} \cdots x_{l_k} \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0) \equiv 0.$$

By the uniqueness of the Taylor series for the real analytic function, we have $\partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0) = 0$ for any $(l_1, \cdots, l_k) \in \{1, 2, \cdots, 7\}^7$ and $k \in \mathbf{N}$. This shows that f is identically zero in $\mathcal{E}$ and also in Ω. □

For more references about octonions and octonionic analysis, we refer the reader to [7] [13]-[20].

3. Sufficient Conditions

In what follows we consider the complexification of $\mathbf{O}$, it is denoted by $\mathbf{O}^c$. Thus, $\mathbb{z} \in \mathbf{O}^c$ is of the form $\mathbb{z} = \sum_0^7 \mathbb{z}_i e_i, \mathbb{z}_i \in \mathbf{C}$. $\mathbb{z}_0$ and $\underline{\mathbb{z}} = \sum_0^7 \mathbb{z}_i e_i$ are still called the scalar part and vector part, respectively. The norm of $\mathbb{z} \in \mathbf{O}^c$ is $|\mathbb{z}| = \left(\sum_0^7 |\mathbb{z}_i|^2\right)^{\frac{1}{2}}$ and its conjugate is defined by $\overline{\mathbb{z}} = \sum_0^7 \overline{\mathbb{z}}_i \overline{e}_i$, where $\overline{\mathbb{z}}_i$ is of the conjugate in the complex numbers. We can easily show that for any $\mathbb{z}, \mathbb{z}' \in \mathbf{O}^c$, $|\mathbb{z}\mathbb{z}'| \leq \sqrt{2} |\mathbb{z}||\mathbb{z}'|$. For any $\mathbb{z} \in \mathbf{O}^c$, we may rewrite $\mathbb{z}$ as $\mathbb{z} = x + iy$, where $x, y \in \mathbf{O}$. The multiplication rules in $\mathbf{O}^c$ is the same as in (2.1). Note that $\mathbf{O}^c$ is no longer a division algebra. Finally, the properties of associator in (2.2) except that $[\mathbb{z}, \overline{\mathbb{z}}, \mathbb{u}] = 0$ are also true for any $\mathbb{z}, \mathbb{u}, \mathbb{v} \in \mathbf{O}^c$:

$$[\mathbb{z}, \mathbb{u}, \mathbb{v}] = [\mathbb{u}, \mathbb{v}, \mathbb{z}], \quad [\mathbb{z}, \mathbb{v}, \mathbb{u}] = -[\mathbb{z}, \mathbb{u}, \mathbb{v}], \quad [\mathbb{z}, \mathbb{z}, \mathbb{u}] = 0. \tag{3.1}$$

Example. Let $\mathbb{z} = e_1 + ie_2, \mathbb{u} = e_4$, then

$$[\mathbb{z}, \overline{\mathbb{z}}, \mathbb{u}] = [e_1 + ie_2, -e_1 + ie_2, e_4] = i[e_1, e_2, e_4] - i[e_2, e_1, e_4] = 4ie_7 \neq 0.$$

By (3.1) we can get the following lemma, which is useful to deduce our results.

Lemma 3.1. *Let* $\mathbb{z}, \mathbb{u}, \mathbb{v} \in \mathbf{O}^c$ *and there exists complex numbers* λ *and* $\mu(|\lambda| + |\mu| \neq 0)$ *such that* $\lambda \underline{\mathbb{z}} + \mu \underline{\mathbb{u}} = 0$ *or* $\lambda \underline{\mathbb{u}} + \mu \underline{\mathbb{v}} = 0$ *or* $\lambda \underline{\mathbb{v}} + \mu \underline{\mathbb{z}} = 0$, *then* $[\mathbb{z}, \mathbb{u}, \mathbb{v}] = 0$.

For functions, *f*, under study will be defined in an open set Ω of $\mathbf{R}^8$ and take values in $\mathbf{O}^c$, with the form $f(x) = \sum_0^7 f_i(x) e_i$, where $f_i(x)(i = 0, 1, \cdots, 7)$ are the complex-valued functions.

Hence, we say that, a function $f(x) = g(x) + ih(x)$ is left $\mathbf{O}^c$-analytic in an open set Ω of $\mathbf{R}^8$, if $g(x)$ and $h(x)$ are the left $\mathbf{O}$-analytic functions, since

$$Df=0 \Leftrightarrow Dg=Dh=0,$$

where $D=\sum_{i=0}^{7}\frac{\partial}{\partial x_i}e_i$ is the Dirac operator as in Section 1.

In the case of $\mathbf{O}^c$, we call $f(x)=g(x)+ih(x)$ a complex Stein-Weiss conjugate harmonic system, if $g(x),h(x)$ are the Stein-Weiss conjugate harmonic systems. A left (right) $\mathbf{O}^c$-analytic functions $g(x)$ also have the Taylor expansion as in Theorem A.

Now we consider the product $f(x)g(x)$ of two left $\mathbf{O}^c$-analytic functions $f(x),g(x)$ in Ω. In general, $f(x)g(x)$ is no longer left $\mathbf{O}^c$-analytic in Ω. But, in some particular cases, the product $f(x)g(x)$ can maintain the analyticity for two left $\mathbf{O}^c$-analytic functions $f(x)$ and $g(x)$.

Theorem 3.2. *Let $f(x),g(x)$ be two left $\mathbf{O}^c$-analytic functions in Ω. Then $f(x)g(x)$ is also left $\mathbf{O}$-analytic in Ω if $f(x),g(x)$ satisfy one of the following conditions:*

1) $\underline{f(x)}$ *or* $g(x)$ *is a complex constant function.*

2) $f(x)$ *is a complex Stein-Weiss conjugate harmonic system in* Ω *and* $g(x)$ *is an* $\mathbf{O}^c$*-constant function.*

3) $f(x)$ *is of the form* $f(x)=f_0e_0+f_ie_i\ (i\in\{1,2,\cdots,7\})$ *and* $f(x),g(x)$ *depend only on* x_0 *and* x_i, *where* f_0,f_i *are the complex-valued functions.*

4) $f(x)$ *and* $g(x)$ *belong to the following class*

$$\mathfrak{S}=\left\{h(x)\,|\,Dh(x)=0,\underline{h(x)}=\sum_{i=1}^{7}h_1(x)e_i,h_1(x)\in C^1(\Omega,\mathbf{C})\right\}. \tag{3.2}$$

5) $f(x)$ *is of the form* $f(x)=f_0e_0+f_\alpha e_\alpha+f_\beta e_\beta+f_\gamma e_\gamma$, $g=c_0e_0+c_\alpha e_\alpha+c_\beta e_\beta+c_\gamma e_\gamma$ *is a constant function, where* $(\alpha,\beta,\gamma)\in\mathscr{W}$, $c_0,c_\alpha,c_\beta,c_\gamma\in\mathbf{C}$ *and* $f(x)$ *depends only on* $x_0,x_\alpha,x_\beta,x_\gamma$.

Proof. 1) The proof is trivial.

2) In view of Proposition 2.1 we have $[e_i,e_j,\lambda]=0$ when $i=0$ or $j=0$ or $i=j$ for any $\lambda\in\mathbf{O}^c$. Then we have

$$\begin{aligned}D(f\lambda)&=\sum_{i,j=0}^{7}\frac{\partial f_j}{\partial x_i}e_i(e_j\lambda)\\&=\sum_{i,j=0}^{7}\frac{\partial f_j}{\partial x_i}(e_ie_j)\lambda-\sum_{i,j=0}^{7}\frac{\partial f_j}{\partial x_i}[e_i,e_j,\lambda]\\&=(Df)\lambda-\sum_{i,j=0}^{7}\frac{\partial f_j}{\partial x_i}[e_i,e_j,\lambda]\\&=(Df)\lambda-\sum_{1\le i\ne j\le 7}\frac{\partial f_j}{\partial x_i}[e_i,e_j,\lambda].\end{aligned}$$

Since $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system, thus $Df=0$ and $\frac{\partial f_j}{\partial x_i}=\frac{\partial f_i}{\partial x_j}$ for $i,j\ge 1,i\ne j$. But $[e_j,e_i,\lambda]=-[e_i,e_j,\lambda]$, therefore

$$D(f\lambda)=-\sum_{1\le i\ne j\le 7}\frac{\partial f_j}{\partial x_i}\left[e_i,e_j,\lambda\right]=\sum_{1\le i<j\le 7}\left(\frac{\partial f_i}{\partial x_j}-\frac{\partial f_j}{\partial x_i}\right)\left[e_i,e_j,\lambda\right]=0.$$

3) Since $f(x),g(x)$ are only related to variables x_0 and x_i, we have

$$D(fg)=\left(\frac{\partial}{\partial x_0}+\frac{\partial}{\partial x_i}e_i\right)\left(\left(f_0+f_ie_i\right)g\right)$$
$$=\frac{\partial f}{\partial x_0}g+e_i\left(\left(\frac{\partial f_0}{\partial x_i}+\frac{\partial f_i}{\partial x_i}e_i\right)g\right)+f\frac{\partial g}{\partial x_0}+e_i\left(\left(f_0+f_ie_i\right)\frac{\partial g}{\partial x_i}\right).$$

By Lemma 3.1 it follows that

$$e_i\left(\left(\frac{\partial f_0}{\partial x_i}+\frac{\partial f_i}{\partial x_i}e_i\right)g\right)=\left(e_i\left(\frac{\partial f_0}{\partial x_i}+\frac{\partial f_i}{\partial x_i}e_i\right)\right)g=\left(e_i\frac{\partial f}{\partial x_i}\right)g$$

and

$$e_i\left(\left(f_0+f_ie_i\right)\frac{\partial g}{\partial x_i}\right)=\left(e_i\left(f_0+f_ie_i\right)\right)\frac{\partial g}{\partial x_i}=\left(\left(f_0+f_ie_i\right)e_i\right)\frac{\partial g}{\partial x_i}=\left(f_0+f_ie_i\right)\left(e_i\frac{\partial g}{\partial x_i}\right).$$

Thus we get

$$D(fg)=\frac{\partial f}{\partial x_0}g+\left(e_i\frac{\partial f}{\partial x_i}\right)g+f\frac{\partial g}{\partial x_0}+f\left(e_i\frac{\partial g}{\partial x_i}\right)=(Df)g+f(Dg)=0.$$

4) Let $f(x)=f_0e_0+\sum_{i=1}^{7}f_1e_i$ and $g(x)=g_0e_0+\sum_{i=1}^{7}g_1e_i$, then we have

$$D\left(f(x)g(x)\right)=\sum_{j=0}^{7}e_j\frac{\partial}{\partial x_j}\left(\left(f_0e_0+f_1\sum_{i=1}^{7}e_i\right)\left(g_0e_0+g_1\sum_{i=1}^{7}e_i\right)\right)$$
$$=\sum_{j=0}^{7}e_j\left(\left(\frac{\partial f_0}{\partial x_j}e_0+\frac{\partial f_1}{\partial x_j}\sum_{i=1}^{7}e_i\right)\left(g_0e_0+g_1\sum_{i=1}^{7}e_i\right)\right)$$
$$+\sum_{j=0}^{7}e_j\left(\left(f_0e_0+f_1\sum_{i=1}^{7}e_i\right)\left(\frac{\partial g_0}{\partial x_j}e_0+\frac{\partial g_1}{\partial x_j}\sum_{i=1}^{7}e_i\right)\right).$$

By Lemma 3.1 we get

$$e_j\left(\left(\frac{\partial f_0}{\partial x_j}e_0+\frac{\partial f_1}{\partial x_j}\sum_{i=1}^{7}e_i\right)\left(g_0e_0+g_1\sum_{i=1}^{7}e_i\right)\right)=\left(e_j\frac{\partial f}{\partial x_j}\right)g$$

and

$$e_j\left(\left(f_0e_0+f_1\sum_{i=1}^{7}e_i\right)\left(\frac{\partial g_0}{\partial x_j}e_0+\frac{\partial g_1}{\partial x_j}\sum_{i=1}^{7}e_i\right)\right)$$
$$=e_j\left(\left(\frac{\partial g_0}{\partial x_j}e_0+\frac{\partial g_1}{\partial x_j}\sum_{i=1}^{7}e_i\right)\left(f_0e_0+f_1\sum_{i=1}^{7}e_i\right)\right)$$
$$=\left(e_j\frac{\partial g}{\partial x_j}\right)f.$$

Hence we obtain

$$D\big(f(x)g(x)\big)=\sum_{j=0}^{7}\left(\left(e_j\frac{\partial f}{\partial x_j}\right)g+\left(e_j\frac{\partial g}{\partial x_j}\right)f\right)=(Df)g+(Dg)f=0.$$

5) This case is equivalent to a left quaternionic analytic function right-multiplying by a quaternionic constant, the analyticity is obvious since the multiplication of the quaternion is associative.

The proof of Theorem 3.2 is complete. □

From Theorem 3.2(d), if $f(x),g(x)\in\mathfrak{S}$, then $f(x)g(x)\in\mathfrak{S}$; that is, the multiply operation in $\mathfrak{S}$ is closed. Also, the division operation is closed in $\mathfrak{S}$. Actually, let $f(x)=f_0(x)+\sum_{i=1}^{7}f_1(x)e_i\in\mathfrak{S}$, assume $f_0^2+7f_1^2\neq 0$, then

$$\big(f(x)\big)^{-1}=\frac{f_0-f_1(e_1+e_2+\cdots+e_7)}{f_0^2+7f_1^2}.$$

Thus we have

$$\begin{aligned}
D\big(f(x)\big)^{-1}&=\sum_{i=0}^{7}e_i\frac{\partial\big(f(x)\big)^{-1}}{\partial x_i}\\
&=\sum_{i=0}^{7}e_i\Bigg(\big(f_0^2+7f_1^2\big)^{-1}\left(\frac{\partial f_0}{\partial x_i}-\frac{\partial f_1}{\partial x_i}(e_1+e_2+\cdots+e_7)\right)\\
&\quad-\big(f_0-f_1(e_1+e_2+\cdots+e_7)\big)\left(2f_0\frac{\partial f_0}{\partial x_i}+14f_1\frac{\partial f_1}{\partial x_i}\right)\big(f_0^2+7f_1^2\big)^{-2}\Bigg)\\
&=\sum_{i=0}^{7}e_i\left(\left(\frac{\partial f_0}{\partial x_i}+\frac{\partial f_1}{\partial x_i}(e_1+\cdots+e_7)\right)\big(7f_1^2-f_0^2+2f_0f_1(e_1+\cdots+e_7)\big)\big(f_0^2+7f_1^2\big)^{-2}\right)\\
&=\sum_{i=0}^{7}\left(e_i\left(\frac{\partial f_0}{\partial x_i}+\frac{\partial f_1}{\partial x_i}(e_1+\cdots+e_7)\right)\right)\big(7f_1^2-f_0^2+2f_0f_1(e_1+\cdots+e_7)\big)\big(f_0^2+7f_1^2\big)^{-2}\\
&=\big(Df(x)\big)\big(7f_1^2-f_0^2+2f_0f_1(e_1+\cdots+e_7)\big)\big(f_0^2+7f_1^2\big)^{-2}\\
&=0.
\end{aligned}$$

An element belongs to $\mathfrak{S}$ is the exponential function:

$$\exp(x)=\mathrm{e}^{x_1+\cdots+x_7}\left(\cos\left(x_0\sqrt{7}\right)e_0+\left(-\frac{1}{\sqrt{7}}(e_1+\cdots+e_7)\right)\sin\left(x_0\sqrt{7}\right)\right).\tag{3.3}$$

The results in Theorem 3.2 also hold on octonions(no complexification), since $\mathbf{O}^c$ contains $\mathbf{O}$. If one switch the locations of $f(x),g(x)$, and the "left" change into "right" in Theorem 3.2, then this theorem is also true, since left and right is symmetric. These principles also hold in the rest of this paper.

4. Necessary and Sufficient Conditions

If we consider the product of a left $\mathbf{O}^c$-analytic function and an $\mathbf{O}^c$-constant, we can get the necessary and sufficient conditions for the analyticity(these results obtained in this section for $\mathbf{O}$-analytic functions are also described in [19]).

Applying Theorem 3.2(a) and (b), if $f(x)$ is a left $\mathbf{O}^c$-analytic function and λ is a complex constant, or $\overline{f(x)}$ is a complex Stein-Weiss conjugate harmonic system and λ is an $\mathbf{O}^c$-constant, then $f(x)\lambda$ is a left $\mathbf{O}^c$-analytic function. In what follows we will see that these conditions are also necessary in some sense.

Theorem 4.1. *Let* $\lambda \in \mathbf{O}^c$*, then* $f\lambda$ *is a left* $\mathbf{O}^c$*-analytic function for any left* $\mathbf{O}^c$*-analytic functions* f *if and only if* $\lambda \in \mathbf{C}$.

Proof. We only prove the necessity. Taking a left $\mathbf{O}^c$-analytic function $f = x_1e_2 - x_0e_3$, then

$$\begin{aligned} D(f\lambda) &= -\sum_{i,j,k=0}^{7} \frac{\partial f_j}{\partial x_i}\lambda_k [e_i, e_j, e_k] = \sum_{k=1}^{7} \frac{\partial f_2}{\partial x_1}\lambda_k [e_2, e_1, e_k] = \sum_{k=4}^{7} \lambda_k [e_2, e_1, e_k] \\ &= \lambda_4 [e_2, e_1, e_4] + \lambda_5 [e_2, e_1, e_5] + \lambda_6 [e_2, e_1, e_6] + \lambda_7 [e_2, e_1, e_7] \\ &= -2\lambda_4 e_7 + 2\lambda_5 e_6 - 2\lambda_6 e_5 + 2\lambda_7 e_4. \end{aligned}$$

Thus $\lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$. A similar technique yields $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence $\lambda \in \mathbf{C}$. □

Theorem 4.2. *Let* $f \in C^1(\Omega, \mathbf{O}^c)$*. Then* $D(f\lambda) = 0$ *for any* $\lambda \in \mathbf{O}^c$ *if and only if* f *is a complex Stein-Weiss conjugate harmonic system in* Ω.

Now we postpone the proof of Theorem 4.2 and consider a problem under certain conditions weaker than Theorem 4.2. In [2] the authors proposed an open problem as follows:

Find the necessary and sufficient conditions for an $\mathbf{O}^c$*-valued function* f*, such that the equality* $[\lambda, f(x), D] = 0$ *holds for any constant* $\lambda \in \mathbf{O}^c$.

Note that this problem is of no meaning for an associative system, but octonions is a non-associative algebra, therefore we usually encounter some difficulties while disposing some problems in octonionic analysis. In [2] the authors added the condition $[\lambda, f(x), D] = 0$ for $f(x)$ to study the Cauchy integrals on Lipschitz surfaces in octonions and then prove the analogue of Calderón's conjecture in octonionic space.

Next we give the answer to the Open Problem as follows.

Theorem 4.3. *Let* $f \in C^1(\Omega, \mathbf{O}^c)$*. Then* $[D, f, \lambda] = 0$ $([\lambda, f, D] = 0)$ *for any* $\lambda \in \mathbf{O}^c$ *if and only if*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad i, j = 1, 2, \cdots, 7. \tag{4.1}$$

Proof. By Proposition 2.1, we have

$$[D, f, \lambda] = \sum_{i,j=0}^{7} \frac{\partial f_j}{\partial x_i}[e_i, e_j, \lambda] = \sum_{1 \le i < j \le 7} \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right)[e_i, e_j, \lambda].$$

If f satisfies (4.1), then $[D, f, \lambda] = 0$.

Inversely, let $(\alpha, \beta, \gamma) \in \mathscr{W}$, $\{1, 2, \cdots, 7\} \setminus \{\alpha, \beta, \gamma\} = \{t_1, t_2, t_3, t_4\}$ and

$$e_{t_1}e_{t_2} = e_\gamma = -e_{t_2}e_{t_1},\ e_{t_3}e_{t_4} = e_\gamma = -e_{t_4}e_{t_3}.$$

From Propositions 2.1 and 2.2 we have $[e_\alpha, e_\beta, e_t] = 0$ and $[e_\alpha, e_\beta, e_t] = 2(e_\alpha e_\beta)e_t = 2e_\gamma e_t$ when $t = \alpha, \beta, \gamma$ and $t = t_1, t_2, t_3, t_4$, respectively.

Hence, taking $\lambda = e_{t_1}$ it follows that

$$\begin{aligned}
&\left[D, f, e_{t_1}\right] \\
&= \sum_{1\le i<j\le 7}\left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}\right)\left[e_i, e_j, e_{t_1}\right] \\
&= \left(\frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta}\right)\left[e_\alpha, e_\beta, e_{t_1}\right] + \left(\frac{\partial f_{t_4}}{\partial x_{t_3}} - \frac{\partial f_{t_3}}{\partial x_{t_4}}\right)\left[e_{t_3}, e_{t_4}, e_{t_1}\right] + \sum_{s\ne t_2} g_s e_s \\
&= 2\left(\frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_{t_4}}{\partial x_{t_3}} - \frac{\partial f_{t_3}}{\partial x_{t_4}}\right) e_{t_2} + \sum_{s\ne t_2} g_s e_s.
\end{aligned} \tag{4.2}$$

Similarly, we take $\lambda = e_{t_3}$, then

$$\left[D, f, e_{t_3}\right] = 2\left(\frac{\partial f_\beta}{\partial x_\alpha} - \frac{\partial f_\alpha}{\partial x_\beta} + \frac{\partial f_{t_2}}{\partial x_{t_1}} - \frac{\partial f_{t_1}}{\partial x_{t_2}}\right) e_{t_4} + \sum_{s\ne t_4} h_s e_s, \tag{4.3}$$

Also we can get

$$\left[D, f, e_\alpha\right] = 2\left(\frac{\partial f_{t_2}}{\partial x_{t_1}} - \frac{\partial f_{t_1}}{\partial x_{t_2}} + \frac{\partial f_{t_4}}{\partial x_{t_3}} - \frac{\partial f_{t_3}}{\partial x_{t_4}}\right) e_\beta + \sum_{s\ne \beta} y_s e_s. \tag{4.4}$$

If we require $[D, f, \lambda] = 0$ for any constants $\lambda \in \mathbf{O}^c$, from (4.2), (4.3) and (4.4) we obtain

$$\begin{cases}
\dfrac{\partial f_\beta}{\partial x_\alpha} - \dfrac{\partial f_\alpha}{\partial x_\beta} + \dfrac{\partial f_{t_4}}{\partial x_{t_3}} - \dfrac{\partial f_{t_3}}{\partial x_{t_4}} = 0, \\
\dfrac{\partial f_\beta}{\partial x_\alpha} - \dfrac{\partial f_\alpha}{\partial x_\beta} + \dfrac{\partial f_{t_2}}{\partial x_{t_1}} - \dfrac{\partial f_{t_1}}{\partial x_{t_2}} = 0, \\
\dfrac{\partial f_{t_2}}{\partial x_{t_1}} - \dfrac{\partial f_{t_1}}{\partial x_{t_2}} + \dfrac{\partial f_{t_4}}{\partial x_{t_3}} - \dfrac{\partial f_{t_3}}{\partial x_{t_4}} = 0.
\end{cases}$$

Combining above three equations with the randomicity of (α, β, γ) we have (4.1) holds. □

Proof of Theorem 4.2. The sufficient from Theorem 3.2(b). Inversely, if we take $\lambda = 1$ in $D(f\lambda) = 0$ it follows that f is a left $\mathbf{O}^c$-analytic function. Thus for any $\lambda \in \mathbf{O}^c$, we have

$$D(f\lambda) = (Df)\lambda - [D, f, \lambda] = -[D, f, \lambda] = 0.$$

By Theorem 4.3 we get that f satisfies (4.1). On the other hand,

$$Df = \left(\frac{\partial}{\partial x_0} + \nabla\right)\left(f_0 + \underline{f}\right) = \frac{\partial f_0}{\partial x_0} - \nabla\cdot\underline{f} + \frac{\partial \underline{f}}{\partial x_0} + \nabla f_0 + \nabla\times\underline{f} = 0. \tag{4.5}$$

From (4.1) it easily to get $\nabla\times\underline{f} = 0$, again by (4.5) it follows that

$$\frac{\partial f_0}{\partial x_0} - \nabla\cdot\underline{f} + \frac{\partial \underline{f}}{\partial x_0} + \nabla f_0 = 0,$$

namely

$$\frac{\partial f_0}{\partial x_0} - \nabla\cdot\underline{f} = 0, \quad \frac{\partial \underline{f}}{\partial x_0} + \nabla f_0 = 0.$$

Combining this with (4.1) it shows that $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system in Ω. □

5. Some Applications and Relations with the C-K Products

From Theorem A we can see that $V_{l_1\cdots l_k}(x)$ are the basic components for (left) $\mathbf{O}$-analytic functions. It is proved in [13] that the polynomials $V_{l_1\cdots l_k}(x)$ are all $\mathbf{O}$-analytic functions, therefore they are the suitable substitutions of the polynomial z^k in $\mathbf{C}$.

Again from Theorem A, since $V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k}$ is an item in the Taylor expansion of a left $\mathbf{O}$-analytic function, $V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k}$ should be also a left $\mathbf{O}$ analytic function. Applying Theorem 4.2, the conjugate of $V_{l_1\cdots l_k}(x)$ is probably a Stein-Weiss conjugate harmonic system. The following theorem prove this is true.

Theorem 5.1. *For any combination* $(l_1,\cdots,l_k)$ *of* k *elements out of* $\{1,\cdots,7\}$ *repetitions being allowed,* $\overline{V}_{l_1\cdots l_k}(x)$ *is a Stein-Weiss conjugate harmonic system in* $\mathbf{R}^8$.

Proof. Let $s_i\,(i=1,\cdots,7)$ be the appearing times of i in $(l_1,\cdots,l_k)$. Hence the following equality

$$V_{l_1\cdots l_k}(x)=\bar{D}\Phi_{s_1\cdots s_7}(x) \tag{5.1}$$

shows that $\overline{V}_{l_1\cdots l_k}(x)$ is a Stein-Weiss conjugate harmonic system in $\mathbf{R}^8$, where

$$\Phi_{s_1\cdots s_7}(x)=\sum_{\substack{\kappa_i=0\\ i=1,\cdots,7}}^{\left[\frac{s_i}{2}\right]}\left\{\frac{(-1)^{\kappa}\kappa!\,x_0^{2\kappa+1}}{(2\kappa+1)!}\prod_{j=1}^{7}\frac{x_j^{s_j-2\kappa_j}}{\kappa_j!\,(s_j-2\kappa_j)!}\right\}$$

is a real-valued harmonic function of order $(s_1+s_2+\cdots+s_7+1)$ with $\kappa=\sum_{i=1}^{7}\kappa_i$.

Actually, put $x_0=0$, the both sides of (5.1) equal to $\dfrac{1}{s_1!s_2!\cdots s_7!}x_1^{s_1}\cdots x_7^{s_7}$. On the other hand, $V_{l_1\cdots l_k}(x)$ is left $\mathbf{O}$-analytic in $\mathbf{R}^8$. Thus by Proposition 2.4 we have (5.1) holds. □

Combining Theorem 3.2(b) and Theorem 5.1 it really shows that all the $V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k}$ are left $\mathbf{O}^c$-analytic functions for any $\lambda_{l_1\cdots l_k}\in\mathbf{O}^c$. Hence the following series

$$\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k} \tag{5.2}$$

is a left $\mathbf{O}^c$-analytic function in some open neighborhood Λ of the origin if $\{\lambda_{l_1\cdots l_k}\}$ satisfies certain bounded conditions.

Theorem 5.2. For any combination $(l_1,\cdots,l_k)$ of k elements out of $\{1,\cdots,7\}$ repetitions being allowed, let $\lambda_{l_1\cdots l_k}\in\mathbf{O}^c, k\in\mathbf{N}$. If $\overline{\lim_{k\to\infty}}\dfrac{7^k}{k!}\sup_{(l_1\cdots l_k)}\left|\lambda_{l_1\cdots l_k}\right|=\gamma<\infty$, then the series (5.2) converges to a left $\mathbf{O}^c$-analytic function $f(x)$ in the following region

$$\Lambda_\gamma = \left\{ x \in \mathbf{R}^8 : \sqrt{x_0^2 + x_i^2} < \frac{1}{\gamma}, i = 1, 2, \cdots, 7 \right\}.$$

More over, $\lambda_{l_1 \cdots l_k} = \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0)$ Particularly, if $\sup_{\substack{(l_1 \cdots l_k) \\ k \in \mathbf{N}}} |\lambda_{l_1 \cdots l_k}| \le C < \infty$, then f will be a left $\mathbf{O}^c$-entire function.

Proof. Let

$$S_N(x) = \sum_{k=0}^{N} \sum_{(l_1, \cdots, l_k)} V_{l_1 \cdots l_k}(x) \lambda_{l_1 \cdots l_k}, N \in \mathbf{N}.$$

For any $x = \sum_0^7 x_i e_i \in \Lambda_\gamma$, there exists $\gamma' > \gamma$ such that $\sqrt{x_0^2 + x_i^2} < \frac{1}{\gamma'}, i = 1, 2, \cdots, 7$. Thus

$$\begin{aligned}
&\sup_{x \in \Lambda_{\gamma'}} |S_N(x) - S_M(x)| \\
&\le \sup_{x \in \Lambda_{\gamma'}} \sum_{k=M}^{N} \sum_{(l_1, \cdots, l_k)} |V_{l_1 \cdots l_k}(x)| |\lambda_{l_1 \cdots l_k}| \\
&\le \sup_{x \in \Lambda_{\gamma'}} \sum_{k=M}^{N} \frac{1}{k!} \sum_{l_1, \cdots, l_k = 1}^{7} |z_{l_1}| \cdots |z_{l_k}| |\lambda_{l_1 \cdots l_k}| \\
&\le \sum_{k=M}^{N} \frac{7^k}{k!} \frac{1}{\gamma'^k} |\lambda_{l_1 \cdots l_k}| \to 0 \quad (\inf(M, N) \to \infty).
\end{aligned}$$

From Weierstrass Theorem on octonions [13] and the analyticity of $V_{l_1 \cdots l_k}(x) \lambda_{l_1 \cdots l_k}$, then there exists a left $\mathbf{O}^c$-analytic function f in Λ_γ such that

$$f(x) = \lim_{N \to \infty} S_N(x) = \sum_{k=0}^{\infty} \sum_{(l_1, \cdots, l_k)} V_{l_1 \cdots l_k}(x) \lambda_{l_1 \cdots l_k},$$

and the series uniformly converges to $f(x)$ in each compact subset $K \subset \Lambda_\gamma$. Again from the expansion of $f(x)$ we easily get that $\lambda_{l_1 \cdots l_k} = \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0)$.

If $\sup_{\substack{(l_1 \cdots l_k) \\ k \in \mathbf{N}}} |\lambda_{l_1 \cdots l_k}| \le C < \infty$, then $\Lambda_\gamma = \mathbf{R}^8$, since $\lim_{k \to \infty} \frac{7^k}{k!} = 0$. Therefore f is a left $\mathbf{O}^c$-entire function. □

Example. Taking $\lambda_{l_1 \cdots l_k} \equiv 1$ for all $k \in \mathbf{N}$ in (5.2), then

$$\sum_{k=0}^{\infty} \sum_{(l_1, \cdots, l_k)} V_{l_1 \cdots l_k}(x) \tag{5.3}$$

is an $\mathbf{O}$-entire function. In fact, (5.3) is the Taylor expansion of the exponential function $\exp(x)$ as in (3.3). From (3.3) we can find $\exp(x)$ satisfies

$$\exp(0) = 1, \quad \exp(x + y) = \exp(x) \cdot \exp(y) = \exp(y) \cdot \exp(x).$$

Corollary 5.3. *For any left* $\mathbf{O}^c$*-analytic function* f, *if the coefficients in its Taylor series about the origin satisfy*

$$\begin{cases} \partial_{x_i^k} f(0) \in \mathbf{C} + e_i \mathbf{C}, & k \in \mathbf{N}, \ i = 1, 2, \cdots, 7 \\ \partial_{x_{l_1}} \cdots \partial_{x_{l_k}} f(0) \in \mathbf{C}, & \text{otherwise.} \end{cases} \tag{5.4}$$

Then $\overline{f}$ is a complex Stein-Weiss conjugate harmonic system.

Proof. From (5.4), we easily obtain that all the conjugates of $V_{l_1\cdots l_k}(x)\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}f(0)$ are complex Stein-Weiss conjugate harmonic systems. Hence by Weierstrass Theorem, $\overline{f}$ also is a complex Stein-Weiss conjugate harmonic system in its convergent area. □

Combining Theorem 3.2(b), Theorems 5.1 and 5.2, by an analogous method in [6] we can define the Cauchy-Kowalewski product for any two left $\mathbf{O}^c$ analytic functions *f* and *g* in Ω which containing origin. We let their Taylor expansions be

$$f(x)=\sum_{k=0}^{\infty}\sum_{(l_1,\cdots,l_k)}V_{l_1\cdots l_k}(x)\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}f(0)$$

and

$$g(x)=\sum_{t=0}^{\infty}\sum_{(s_1,\cdots,s_t)}V_{s_1\cdots s_t}(x)\partial_{x_{s_1}}\cdots\partial_{x_{s_t}}g(0).$$

Then the (left) Cauchy-Kowalewski product of *f* and *g* is defined by

$$f\odot_L g(x)$$
$$=\sum_{k,t=0}^{\infty}\sum_{\substack{(l_1,\cdots,l_k)\\(s_1,\cdots,s_t)}}\left(\prod_{i=1}^{7}\frac{(n_i+n_i')!}{n_i!n_i'!}\right)V_{l_1\cdots l_k s_1\cdots s_t}(x)\left(\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}f(0)\cdot\partial_{x_{s_1}}\cdots\partial_{x_{s_t}}g(0)\right),$$

where n_i and n_i' are the appearing times of *i* in $(l_1,\cdots,l_k)$ and $(s_1,\cdots,s_t)$, respectively.

We have the following relation for the product and the left Cauchy-Kowalewski product between two left $\mathbf{O}^c$-analytic functions.

Theorem 5.4. *Let* $f(x),g(x)$ *be two left* $\mathbf{O}^c$*-analytic functions in* Ω *which containing origin. If* $D(f(x)g(x))=0$ *then*

$$f(x)g(x)=f\odot_L g(x).$$

Proof. It is easy to see that $f(\underline{x})g(\underline{x})=f\odot_L g(\underline{x})$, then by Proposition 2.4 and the analyticity of $f(x)g(x)$ and $f\odot_L g(x)$ we get $f(x)g(x)=f\odot_L g(x)$. □

Remark. In this paper we study the analyticity of the product of two left $\mathbf{O}^c$-analytic functions. Theorem 3.2 give some sufficient conditions for the product of two left $\mathbf{O}^c$-analytic functions is also a left $\mathbf{O}^c$-analytic function. From Theorem 5.4 we can see that $D(f(x)g(x))=0$ for two left $\mathbf{O}^c$-analytic functions $f(x),g(x)$ if and only if this product is just equal to their left Cauchy-Kowalewski product. Since $\mathbf{H}\subseteq\mathbf{O}$, our result is also true for quaternionic cases.

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Squeezed Coherent States in Non-Unitary Approach and Relation to Sub- and Super-Poissonian Statistics

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Abstract

After developing the concept of displaced squeezed vacuum states in the non-unitary approach and establishing the connection to the unitary approach we calculate their quasiprobabilities and expectation values $a^{\dagger k}a$ in general form. Then we consider the displacement of the squeezed vacuum states and calculate their photon statistics and their quasiprobabilities. The expectation values of the displaced states are related to the expectation values of the undisplaced states and are calculated for some simplest cases which are sufficient to discuss their categorization as sub-Poissonian and super-Poissonian statistics. A large set of these states do not belong to sub- or to super-Poissonian states but are also not Poissonian states. We illustrate in examples their photon distributions. This shows that the notions of sub- and of super-Poissonian statistics and their use for the definition of nonclassicality of states are problematic. In Appendix A we present the most important relations for $SU(1,1)$ treatment of squeezing and the disentanglement of their operators. Some initial members of sequences of expectation values for squeezed vacuum states are collected in Appendix E.

Keywords

$SU(1,1)$ Group of Squeezing and Rotation, Wigner Quasiprobability, Unitary Approach to Squeezing, Nonclassical States, Uncertainty Matrix, Distance of States, Jacobi, Ultraspherical, Legendre and Hermite Polynomials, Poisson Statistics

1. Introduction

Besides the number states $|n\rangle$ and the coherent states $|\alpha\rangle$ the squeezed coherent states or, what is the same, the displaced squeezed vacuum states belong to the most interesting states in quantum optics for which, practically, all interesting parameters and quasiprobabilities may be calculated in closed exact way. The coherent states are the vacuum states $|n=0\rangle$ displaced by a complex parameter β in the phase space (for one mode). The higher number states $|n\rangle$ with $n\neq 0$ are the discrete excitations of the ground state $|n=0\rangle$ of a harmonic oscillator and they also can be displaced and squeezed but this we do not consider in present article. All minimum uncertainty states belong to the squeezed coherent states and therefore some aspects of these states were already considered in the early years of the development of quantum mechanics although not under this name, for example, by Schrödinger [1], Pauli [2] and Louisell [3]. The name "squeezed states" appeared in the eighties by Walls [4] and others and numerous articles and reviews are published since this time, e.g., [5]-[14] and, e.g., [15] [16] [17] [18] [19].

In the narrow sense the squeezing operations form together with rotations in a plane (the two-dimensional phase plane) the Lie group $SU(1,1)$ with 3 real parameters. This Lie group possesses different realizations in quantum optics of a single mode and also a basic nontrivial realization in a two-mode system. We will deal with in this article a single mode where the basic operators of the Lie group $SU(1,1)$ are realized by quadratic combinations of the annihilation and creation operators $(a,a^{\dagger})$ of this mode but in **Appendix A** we represent in detail the basic relations for $SU(1,1)$. Besides this, the Lie group $SU(1,1)$ may find application within a single mode also for the treatment of phase states and as mentioned possesses a basic realization in a two-mode system (e.g., [15]). Dynamical squeezing appears if the Hamiltonian or Liouvillean of a process is described by quadratic combinations of annihilation and creation operators.

The main purpose of this paper is the representation of the formalism of $SU(1,1)$ squeezing in two approaches concerning the complex squeezing parameter which we call non-unitary and unitary approach and the calculation of expectation values and of the basic quasiprobabilities for squeezed vacuum and squeezed coherent states. The squeezed coherent states are well appropriate to demonstrate some problems of the distinction of sub- and super-Poissonian photon statistics because the whole set of these states can be not assigned to only one of these two kinds of statistics and it requires substantial efforts to find out to which of these statistics it belongs in a special case. The cases when they are neither sub- nor super-Poissonian statistics may be very far from a Poisson statistics that can be seen by the distance parameter. This shows in an example of nonclassical states the problems of classification of statistics in quantum optics in this way and is discussed in Section 11.

2. Squeezed Vacuum States in Non-Unitary Approach and Their Photon Statistics

In this section we begin with the discussion of squeezed vacuum states in the non-unitary approach. For their definition we apply the non-unitary operator

$\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)$ with ζ as a complex parameter $(|\zeta|<1)$ onto the vacuum state $|0\rangle$. As usual, $(a,a^{\dagger})$ denote a pair of annihilation and creation operators of a single boson mode with the commutation relations $\left[a,a^{\dagger}\right]\equiv aa^{\dagger}-a^{\dagger}a=I$, ($I$ unity operator) and they act onto the number states $|n\rangle$ which are orthonormalized and complete

$$
\begin{aligned}
&a|n\rangle=\sqrt{n}|n-1\rangle, \quad a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle, \quad N|n\rangle\equiv a^{\dagger}a|n\rangle=n|n\rangle,\\
&\langle m|n\rangle=\delta_{m,n}, \quad \sum_{n=0}^{\infty}|n\rangle\langle n|=I, \quad |n\rangle=\frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle, \quad (n=0,1,2,\cdots). \qquad (2.1)
\end{aligned}
$$

Now, we define the squeezed vacuum states $|0,\zeta\rangle$ in the non-unitary approach by

$$
|0,\zeta\rangle\propto\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle=\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}\zeta^m a^{\dagger 2m}|0\rangle=\sum_{m=0}^{\infty}\frac{(-1)^m\sqrt{(2m)!}}{2^m m!}\zeta^m|2m\rangle, \qquad (2.2)
$$

where the zero in the notation $|0,\zeta\rangle$ is arranged for the substitution by a complex displacement parameter β in the later generalization to displaced vacuum states $|\beta,\zeta\rangle$ (see **Figure 1** and from Section 7 on).

Since $\mathrm{e}^{-\frac{\zeta}{2}a^{\dagger 2}}$ is not a unitary operator the right-hand side of (2) is not normalized and using the Taylor series $\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}q^m=\frac{1}{\sqrt{1-q}}$ we find the normalization factor

$$
\begin{aligned}
|0,\zeta\rangle&=\left(1-\zeta\zeta^*\right)^{\frac{1}{4}}\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle\\
&=\left(1-\zeta\zeta^*\right)^{\frac{1}{4}}\sum_{m=0}^{\infty}\frac{(-1)^m\sqrt{(2m)!}}{2^m m!}\zeta^m|2m\rangle, \langle 0,\zeta|0,\zeta\rangle=1, |\zeta|\equiv\sqrt{\zeta\zeta^*}<1.
\end{aligned} \qquad (2.3)
$$

The complex parameter ζ is restricted in the non-unitary approach for normalizable states to $|\zeta|<1$ but can be continued to non-normalizable states for $|\zeta|\geq 1$. In the unitary approach (2) we apply a unitary operator $\exp\left(\frac{\zeta'^*}{2}a^2-\frac{\zeta'}{2}a^{\dagger 2}\right)$ to the vacuum state $|0\rangle$ according to

$$
|0,\zeta\rangle\equiv\exp\left\{\frac{\operatorname{Arth}|\zeta|}{|\zeta|}\left(\frac{\zeta^*}{2}a^2-\frac{\zeta}{2}a^{\dagger 2}\right)\right\}|0\rangle\equiv\exp\left(\frac{\zeta'^*}{2}a^2-\frac{\zeta'}{2}a^{\dagger 2}\right)|0\rangle. \qquad (2.4)
$$

The connection between the two parameters ζ and ζ' is given by[1]

$$
\zeta'=\frac{\operatorname{Arth}(|\zeta|)}{|\zeta|}\zeta, \quad \zeta'^*=\frac{\operatorname{Arth}(|\zeta|)}{|\zeta|}\zeta^*, \quad |\zeta'|=\operatorname{Arth}|\zeta|, \quad \frac{\zeta'}{|\zeta'|}=\frac{\zeta}{|\zeta|},
$$

[1]Note that the operators $S\left(\zeta'^*,0,\zeta'\right)\equiv\exp\left(\frac{\zeta'^*}{2}a^2-\frac{\zeta'}{2}a^{\dagger 2}\right)$ with complex ζ' do not form a group that means the product of two such operators with different parameters is, in general, not an operator of this type but by a small extension one comes to the $SU(1,1)$ group of squeezing and rotation operators; see **Appendix A**.

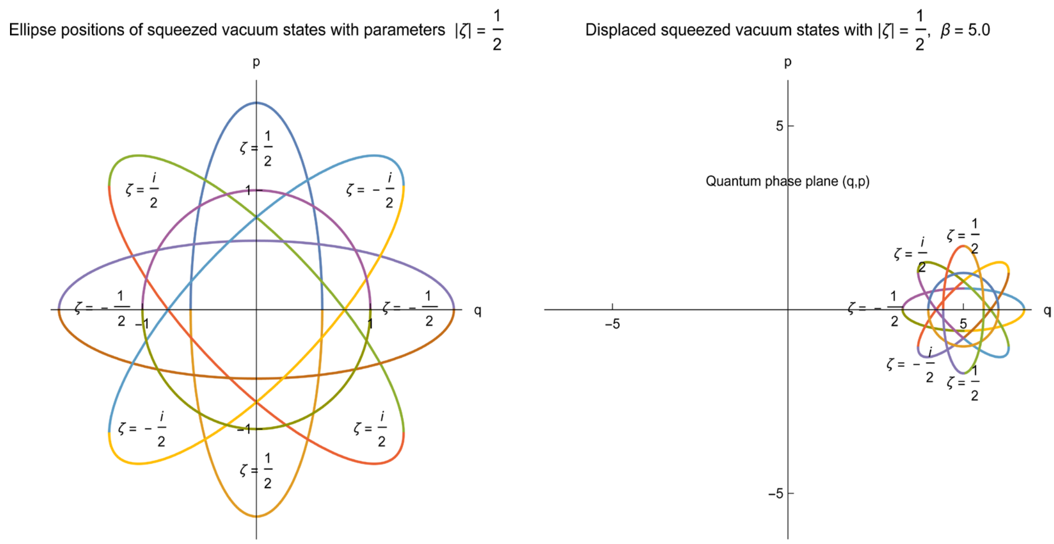

Figure 1. Position of ellipses of squeezed vacuum states in dependence on the phase of squeezing parameter ζ. The drawn squeezing ellipses for $|\zeta| = \frac{1}{2}$ are a contour of equal height of the Wigner quasiprobability $W(q,p)$. The mean value $\bar{N}$ of the number operator N depends for the squeezed vacuum states only on the modulus $|\zeta|$ of the squeezing parameter ζ and is $\bar{N} = \frac{|\zeta|^2}{1-|\zeta|^2}$ and the variance is $\overline{(\Delta N)^2} = \frac{2|\zeta|^2}{\left(1-|\zeta|^2\right)^2}$ that means $\bar{N} = \frac{1}{3}$ and $\overline{(\Delta N)^2} = \frac{8}{9}$ in our case. This shows that all squeezed vacuum states possess a super-Poissonian photon statistics. In the second picture we have shown a displacement of squeezed vacuum states in the quantum phase plane (see from Section 7 on). The squeezing parameter remains the same under displacements. The circle on the left figure corresponds to the vacuum state and on the right figure to a coherent state.

$$\zeta = \frac{\mathrm{th}\left(|\zeta'|\right)}{|\zeta'|}\zeta', \quad \zeta^* = \frac{\mathrm{th}\left(|\zeta'|\right)}{|\zeta'|}\zeta'^*, \quad |\zeta| = \mathrm{th}\left(|\zeta'|\right). \tag{2.5}$$

The parameter ζ' is stretched in comparison to ζ and takes on the whole complex plane for normalizable states but ζ and ζ' possess the same directions in the complex plane. It is easy to rewrite the formulae derived in the following from parameters (ζ,ζ^*) to parameters (ζ',ζ'^*) using (5), for example

$$\frac{1}{1-|\zeta|^2} = \mathrm{ch}^2\left(|\zeta'|\right), \quad \frac{|\zeta|^2}{1-|\zeta|^2} = \mathrm{sh}^2\left(|\zeta'|\right), \quad \sqrt{\frac{1\pm|\zeta|}{1\mp|\zeta|}} = \exp\left(\pm|\zeta'|\right). \tag{2.6}$$

The complex parameter ζ has often some advantages in comparison to the complex parameter ζ' concerning compactness of formulae but sometimes, e.g., in the dynamics to quadratic Hamiltonians in $(a,a^\dagger)$, the representation by ζ' is to prefer and ζ in literature notations $|\alpha,\zeta\rangle$ corresponds mostly to our ζ'. In Appendix A we consider the relations in detail.

The equivalence of the two approaches is given by the following general disentanglement of the unitary squeezing operator in (almost) normal ordering

$$\exp\left\{\frac{\operatorname{Arth}|\zeta|}{|\zeta|}\left(\frac{\zeta^*}{2}a^2-\frac{\zeta}{2}a^{\dagger 2}\right)\right\}$$
$$=\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)\left(1-\zeta\zeta^*\right)^{\frac{1}{4}\left(aa^\dagger+a^\dagger a\right)}\exp\left(\frac{\zeta^*}{2}a^2\right). \tag{2.7}$$

The basic relations for squeezing operators between unitary and non-unitary approach were developed already earlier (e.g., [18]) for the $SU(1,1)$ group and we collect the most important relations in **Appendix A**.

From (2.3) follow the probabilities p_n to the photon statistics of squeezed vacuum states

$$p_n \equiv \langle n|0,\zeta\rangle\langle 0,\zeta|n\rangle$$
$$=\begin{cases} p_{2m}=\sqrt{1-|\zeta|^2}\,\frac{(2m)!}{2^{2m}m!^2}|\zeta|^{2m}=\sqrt{1-|\zeta|^2}\,\frac{(2m-1)!!}{2^m m!}|\zeta|^{2m} \\ p_{2m+1}=0. \end{cases} \tag{2.8}$$

The sum over the p_n are normalized according to

$$\sum_{n=0}^{\infty}p_n=\sqrt{1-|\zeta|^2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}|\zeta|^{2m}=1, \tag{2.9}$$

as one affirms from the Taylor series of the function $\left(1-|\zeta|^2\right)^{-\frac{1}{2}}$. Only the probabilities for even $n=2m$ are non-vanishing and the probabilities p_{2m} are strictly decreasing ($p_{2(m+1)}\le p_{2m}$) for increasing m.

From the commutation relations

$$\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)a\exp\left(\frac{\zeta}{2}a^{\dagger 2}\right)$$
$$=a-\frac{\zeta}{2}\left[a^{\dagger 2},a\right]+\frac{1}{2!}\left(\frac{\zeta}{2}\right)^2\left[a^{\dagger 2},\left[a^{\dagger 2},a\right]\right]-\cdots=a+\zeta a^\dagger, \tag{2.10}$$

in connection with $a|0\rangle=0$ follows that the states $|0,\zeta\rangle$ are eigenstates of the operator $a+\zeta a^\dagger$ to the eigenvalue zero

$$0=\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)a|0\rangle=\left(a+\zeta a^\dagger\right)\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle, \tag{2.11}$$

that means

$$\left(a+\zeta a^\dagger\right)|0,\zeta\rangle=0,\quad\Leftrightarrow\quad a|0,\zeta\rangle=-\zeta a^\dagger|0,\zeta\rangle. \tag{2.12}$$

In representation by canonical operators (Q,P) this is equivalent to

$$\left((1+\zeta)Q+\mathrm{i}(1-\zeta)P\right)|0,\zeta\rangle=0,\quad\Leftrightarrow\quad P|0,\zeta\rangle=\mathrm{i}\frac{1+\zeta}{1-\zeta}Q|0,\zeta\rangle. \tag{2.13}$$

Thus both the states $a|0,\zeta\rangle$ and $a^\dagger|0,\zeta\rangle$ as well as the states $Q|0,\zeta\rangle$ and $P|0,\zeta\rangle$ are linearly dependent. Furthermore, from (2.12) follows

$$a^{\dagger k}a|0,\zeta\rangle=-\zeta a^{\dagger k+1}|0,\zeta\rangle,\quad a^l|0,\zeta\rangle=-\zeta a^{l-1}a^\dagger|0,\zeta\rangle. \tag{2.14}$$

If one forms the scalar products of these relations by multiplication with $\langle 0,\zeta|$ one obtains the expectation values of $\overline{a^{\dagger k}a}$ and $-\zeta\overline{a^{\dagger k+1}}$ and the expectation values $\overline{a^{l}}$ and $-\zeta\overline{a^{l-1}a^{\dagger}}$, respectively, with equality relations between them. By differentiations of (2.3) with respect to ζ and to ζ^{*} in connection with (2.14) for $k=1$ we find

$$N|0,\zeta\rangle=-\zeta a^{\dagger 2}|0,\zeta\rangle=2\left(\zeta\frac{\partial}{\partial\zeta}-\zeta^{*}\frac{\partial}{\partial\zeta^{*}}\right)|0,\zeta\rangle, \tag{2.15}$$

as a further interesting relation which can also be written as eigenvalue equation for $|0,\zeta\rangle$ to eigenvalue zero.

Another interesting characteristics of a state is its (Hilbert-Schmidt) distance to the nearest coherent state which in case of squeezed vacuum states $|0,\zeta\rangle$ is the vacuum state $|0\rangle$. It may be considered as a measure of nonclassicality of a state [20] [21] [22]. For this distance one finds (see (5) for ζ and ζ')

$$\begin{aligned}d\left(|0,\zeta\rangle\langle 0,\zeta|,|0\rangle\langle 0|\right)&=\sqrt{2\left(1-\langle 0|0,\zeta\rangle\langle 0,\zeta|0\rangle\right)}\\&=\sqrt{2\left(1-\sqrt{1-|\zeta|^{2}}\right)}=\sqrt{2\left(1-\frac{1}{\mathrm{ch}\left(|\zeta'|\right)}\right)}\\&=|\zeta|\left(1+\frac{1}{8}|\zeta|^{2}+\cdots\right)=|\zeta'|\left(1-\frac{5}{24}|\zeta'|^{2}+\cdots\right).\end{aligned} \tag{2.16}$$

It depends only on the modulus of ζ. For strong squeezing $|\zeta|\to 1$ this distance goes to $\sqrt{2}$ that is the largest distance for normalized states in Hilbert space and means orthogonality of the two states (**Figure 2**).

3. Wave Functions of Squeezed Vacuum States and Uncertainty Matrix

The wave functions of squeezed vacuum states are the scalar products $\langle q|0,\zeta\rangle$ and $\langle p|0,\zeta\rangle$ with the eigenstates $|q\rangle$ and $|p\rangle$ of the operators Q and P. Their number representations are

$$\begin{aligned}|q\rangle&=\frac{1}{(\hbar\pi)^{\frac{1}{4}}}\exp\left(-\frac{q^{2}}{2\hbar}\right)\sum_{n=0}^{\infty}\frac{1}{\sqrt{2^{n}n!}}\mathrm{H}_{n}\left(\frac{q}{\sqrt{\hbar}}\right)|n\rangle,\\|p\rangle&=\frac{1}{(\hbar\pi)^{\frac{1}{4}}}\exp\left(-\frac{p^{2}}{2\hbar}\right)\sum_{n=0}^{\infty}\frac{\mathrm{i}^{n}}{\sqrt{2^{n}n!}}\mathrm{H}_{n}\left(\frac{p}{\sqrt{\hbar}}\right)|n\rangle,\end{aligned} \tag{3.1}$$

where $\mathrm{H}_{n}(z)$ denotes the Hermite polynomials. They are not normalizable as it is well known and are only normalized by means of the delta function

$$\begin{aligned}&\langle q|q'\rangle=\delta(q-q'),\quad\langle p|p'\rangle=\delta(p-p'),\\&\langle q|p\rangle=\frac{1}{\sqrt{2\hbar\pi}}\exp\left(\mathrm{i}\frac{qp}{\hbar}\right)=\langle p|q\rangle^{*}.\end{aligned} \tag{3.2}$$

Using the number-state representation (3.3) of squeezed vacuum states we find by applying the first of the generating functions (3.3) for even Hermite polynomials

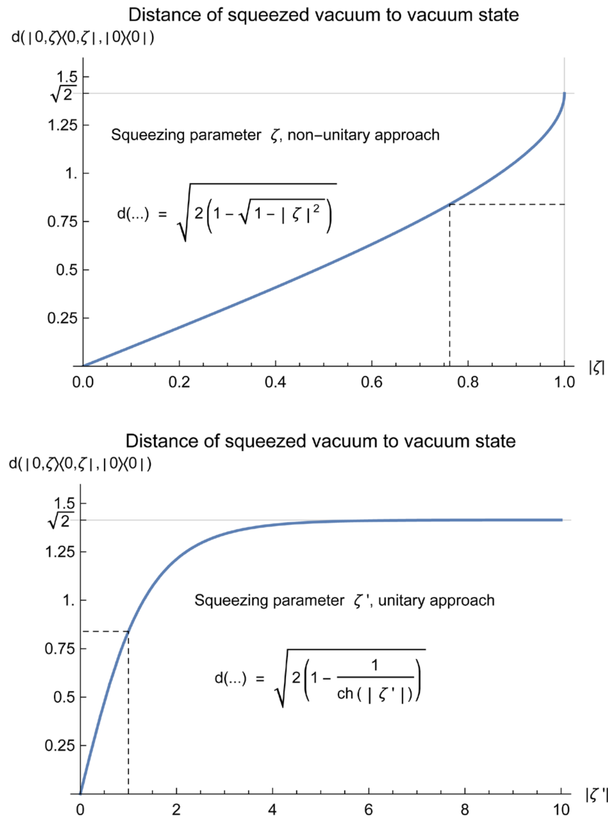

Figure 2. Distance of squeezed vacuum states to vacuum state in dependence on squeezing parameters ζ and ζ' in non-unitary and in unitary approach. The figures show that for $|\zeta|\to 1$ the parameter $|\zeta'|$ is stretched up to ∞ in comparison to $|\zeta|$. The maximal distance of two pure normalized states in Hilbert space is $\sqrt{2}$.

$$\begin{aligned}
\langle q|0,\zeta\rangle &= \left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{q^2}{2\hbar}\right)\sum_{m=0}^{\infty}\frac{(-1)^m\zeta^m}{2^{2m}m!}\mathrm{H}_{2m}\left(\frac{q}{\sqrt{2\hbar}}\right)\\
&= \left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{1-\zeta}}\exp\left(-\frac{(1+\zeta)q^2}{(1-\zeta)2\hbar}\right),\\
\langle p|0,\zeta\rangle &= \left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{p^2}{2\hbar}\right)\sum_{m=0}^{\infty}\frac{\zeta^m}{2^{2m}m!}\mathrm{H}_{2m}\left(\frac{p}{\sqrt{2\hbar}}\right)\\
&= \left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{1+\zeta}}\exp\left(-\frac{(1-\zeta)p^2}{(1+\zeta)2\hbar}\right).
\end{aligned} \tag{3.3}$$

From this follows

$$W(q) \equiv \langle q|0,\zeta\rangle\langle 0,\zeta|q\rangle = \sqrt{\frac{1-\zeta\zeta^*}{(1-\zeta)(1-\zeta^*)\hbar\pi}}\exp\left(-\frac{(1-\zeta\zeta^*)q^2}{(1-\zeta)(1-\zeta^*)\hbar}\right),$$

$$W(p) \equiv \langle p|0,\zeta\rangle\langle 0,\zeta|p\rangle = \sqrt{\frac{1-\zeta\zeta^*}{(1+\zeta)(1+\zeta^*)\hbar\pi}}\exp\left(-\frac{(1-\zeta\zeta^*)p^2}{(1+\zeta)(1+\zeta^*)\hbar}\right), \tag{3.4}$$

with the normalization

$$\int_{-\infty}^{+\infty} \mathrm{d}q W(q) = \int_{-\infty}^{+\infty} \mathrm{d}p W(p) = 1. \tag{3.5}$$

The functions $W(q)$ and $W(p)$ are the Wigner quasiprobability $W(q,p)$ integrated over one of the canonical variable p or q. The functions (3.4) remain invariant by interchanging the squeezing parameter ζ with its complex conjugate $\zeta \leftrightarrow \zeta^*$. This shows in an example that, in general, a state (here $|0,\zeta\rangle$) cannot uniquely be reconstructed from $W(q)$ and $W(p)$ alone.

The functions $W(q)$ and $W(p)$ are two normalized Gaussian distribution with the expectation values

$$\bar{Q} = 0, \quad \bar{P} = 0, \quad \overline{(\Delta Q)^2} = \frac{(1-\zeta)(1-\zeta^*)}{1-\zeta\zeta^*}\frac{\hbar}{2}, \quad \overline{(\Delta P)^2} = \frac{(1+\zeta)(1+\zeta^*)}{1-\zeta\zeta^*}\frac{\hbar}{2}. \tag{3.6}$$

The product of their uncertainties $\overline{(\Delta Q)^2}$ and $\overline{(\Delta P)^2}$ (note inequality $z^2 + z^{*2} \le 2zz^*$ for arbitrary complex $z = x + \mathrm{i}y$ in contrast to $x^2 + y^2 \ge 2xy$ for arbitrary real x and y)

$$\overline{(\Delta Q)^2}\,\overline{(\Delta P)^2} = \frac{(1-\zeta^2)(1-\zeta^{*2})}{(1-\zeta\zeta^*)^2}\frac{\hbar^2}{4} \ge \frac{\hbar^2}{4}. \tag{3.7}$$

It depends on the phase χ of the complex squeezing parameter $\zeta = |\zeta|\mathrm{e}^{\mathrm{i}\chi}$ that means on the position of the principal axes of squeezing in comparison to the axes of the canonical coordinates (q,p) (see **Figure 1**). In contrast, the sum of the uncertainties

$$\overline{(\Delta Q)^2} + \overline{(\Delta P)^2} = \frac{1+\zeta\zeta^*}{1-\zeta\zeta^*}\hbar, \tag{3.8}$$

does not depend on the phase of the squeezing parameter ζ. For real squeezing parameter $\zeta = \zeta^*$ we find for the uncertainties

$$\overline{(\Delta Q)^2} = \frac{1-\zeta}{1+\zeta}\frac{\hbar}{2}, \quad \overline{(\Delta P)^2} = \frac{1+\zeta}{1-\zeta}\frac{\hbar}{2}, \quad (\zeta = \zeta^*), \tag{3.9}$$

and their product is equal to $\frac{\hbar^2}{4}$ the minimal possible one. The principal axes of the squeezing ellipses are then in direction of the coordinate axes (q,p) (see **Figure 1**). Clearly, it is not satisfying to consider squeezed vacuum states $|0,\zeta\rangle$ with real ζ as minimum uncertainty states and such with complex ζ which are only rotated in the phase plane (**Figure 1**), in general, not as minimum uncertainty states. The satisfying solution of this problem is to consider in addition the uncertainty correlation $\frac{1}{2}\overline{\Delta Q\Delta P + \Delta P\Delta Q}$ [23] [24].

The uncertainty correlation $\frac{1}{2}\overline{QP + PQ}$ arises in a natural way as non-diagonal elements of the (symmetrical) uncertainty matrix S if we consider the cumulant expansion of the Wigner quasiprobability $W(q,p)$ or the corresponding expansion of its Fourier transform $\tilde{W}(u,v)$ [24]

$$\mathrm{S} \equiv \begin{pmatrix} \overline{(\Delta Q)^2} & \frac{1}{2}\overline{\Delta Q\Delta P + \Delta P\Delta Q} \\ \frac{1}{2}\overline{\Delta Q\Delta P + \Delta P\Delta Q} & \overline{(\Delta P)^2} \end{pmatrix}$$
$$= \begin{pmatrix} \overline{\Delta a\Delta a^\dagger + \Delta a^\dagger \Delta a} + \left(\overline{(\Delta a)^2} + \overline{(\Delta a^\dagger)^2}\right) & -\mathrm{i}\left(\overline{(\Delta a)^2} - \overline{(\Delta a^\dagger)^2}\right) \\ -\mathrm{i}\left(\overline{(\Delta a)^2} - \overline{(\Delta a^\dagger)^2}\right) & \overline{\Delta a\Delta a^\dagger + \Delta a^\dagger \Delta a} - \left(\overline{(\Delta a)^2} + \overline{(\Delta a^\dagger)^2}\right) \end{pmatrix} \frac{\hbar}{2}. \tag{3.10}$$

It is also called variance matrix [23] and is related to the covariance matrix [10] (3.p. 61). The trace of the uncertainty matrix S denoted by $\langle \mathrm{S} \rangle$

$$\langle \mathrm{S} \rangle = \overline{(\Delta Q)^2} + \overline{(\Delta P)^2} = \left(\overline{\Delta a\Delta a^\dagger + \Delta a^\dagger \Delta a}\right)\hbar, \tag{3.11}$$

is the uncertainty sum and the determinant of the matrix S denoted by $[\mathrm{S}]$ is essentially the uncertainty product but modified by the uncertainty correlations

$$\begin{aligned} [\mathrm{S}] &= \overline{(\Delta Q)^2}\,\overline{(\Delta P)^2} - \frac{1}{4}\left(\overline{\Delta Q\Delta P + \Delta P\Delta Q}\right)^2 \\ &= \left\{\left(\overline{\Delta a\Delta a^\dagger + \Delta a^\dagger \Delta a}\right)^2 - 4\overline{(\Delta a)^2}\,\overline{(\Delta a^\dagger)^2}\right\}\frac{\hbar^2}{4} \geq \frac{\hbar^2}{4}. \end{aligned} \tag{3.12}$$

The chain of inequalities which generalizes the basic uncertainty relation of quantum mechanics is ([18] [24])

$$\frac{\hbar}{2} \leq \sqrt{[\mathrm{S}]} \leq \sqrt{\overline{(\Delta Q)^2}\,\overline{(\Delta P)^2}} \leq \langle \mathrm{S} \rangle. \tag{3.13}$$

Both quantities $\langle \mathrm{S} \rangle$ and $[\mathrm{S}]$ are invariant with respect to rotations and displacement of the states in the quantum phase plane and $\langle \mathrm{S} \rangle$ is additionally invariant with respect to squeezing [24].

For squeezed vacuum states $|0,\zeta\rangle$ we find using their number representation (2.2)

$$\frac{1}{2}\left(\overline{\Delta Q\Delta P + \Delta P\Delta Q}\right) = -\mathrm{i}\left(\overline{(\Delta a)^2} - \overline{(\Delta a^\dagger)^2}\right)\frac{\hbar}{2} = \mathrm{i}\frac{\zeta - \zeta^*}{1 - \zeta\zeta^*}\frac{\hbar}{2}, \tag{3.14}$$

and therefore for the modified uncertainty product $[\mathrm{S}]$ in (3.12) using the explicit expressions (3.7) and (3.14)

$$[\mathrm{S}] = \frac{\hbar^2}{4}. \tag{3.15}$$

This modified uncertainty product for squeezed vacuum states does no more depend on the position of the principal axes of squeezing ellipses in phase plane of canonical coordinates (q,p) shown in **Figure 1**. It is the minimum possible one characterizing the states as minimum uncertainty states in this generalized sense.

By a rotation of the canonical coordinates (q,p) to new canonical coordinates (q',p') one may bring them in the position of the principal axes of the squeezing ellipses and since this is fully obvious we do not give the explicit transformation relations. However, this suggests that it is better not to exclude squeezed vacuum states with arbitrary positions of the squeezing ellipses from the minimum uncertainty states.

4. Bargmann Representation and Quasiprobabilities for Squeezed Vacuum States

The Bargmann representation of a state is a representation by an analytic function which, in particular, leads immediately to the Husimi-Kano quasiprobability $Q\left(\alpha,\alpha^{*}\right)$ [25] (chap. 7) and [26]. For this purpose we calculate the scalar product of a state with the analytic (but non-normalized) coherent states $\|\alpha\rangle \equiv \exp\left(\frac{\alpha\alpha^{*}}{2}\right)|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle$ with arbitrary complex α. For the squeezed vacuum state $|0,\zeta\rangle$ this provides its Bargmann representations

$$\begin{aligned}\exp\left(\frac{\alpha\alpha^{*}}{2}\right)\langle 0,\zeta|\alpha\rangle &= \left(1-\zeta\zeta^{*}\right)^{\frac{1}{4}}\sum_{m=0}^{\infty}\frac{(-1)^{m}}{2^{m}m!}\zeta^{*m}\alpha^{2m} \\ &= \left(1-\zeta\zeta^{*}\right)^{\frac{1}{4}}\exp\left(-\frac{\zeta^{*}}{2}\alpha^{2}\right) = \left(\exp\left(\frac{\alpha\alpha^{*}}{2}\right)\langle\alpha|0,\zeta\rangle\right)^{*}.\end{aligned} \tag{4.1}$$

From this one obtains the Husimi-Kano quasiprobability $Q\left(\alpha,\alpha^{*}\right)$ for density operator $\rho = |0,\zeta\rangle\langle 0,\zeta|$

$$\begin{aligned}&Q\left(\alpha,\alpha^{*}\right) \equiv \frac{\langle\alpha|0,\zeta\rangle\langle 0,\zeta|\alpha\rangle}{\pi} = \frac{\sqrt{1-\zeta\zeta^{*}}}{\pi}\exp\left\{-\left(\alpha\alpha^{*}+\frac{\zeta^{*}}{2}\alpha^{2}+\frac{\zeta}{2}\alpha^{*2}\right)\right\}, \\ &\int\frac{\mathrm{i}}{2}\mathrm{d}\alpha\wedge\mathrm{d}\alpha^{*}Q\left(\alpha,\alpha^{*}\right) = 1.\end{aligned} \tag{4.2}$$

In representation by real canonical variables (q,p) this is

$$\begin{aligned}&Q(q,p) = \frac{\sqrt{1-\zeta\zeta^{*}}}{2\hbar\pi}\exp\left\{-\frac{2}{\hbar}\left(\left(1+\frac{\zeta+\zeta^{*}}{2}\right)q^{2}+\left(1-\frac{\zeta+\zeta^{*}}{2}\right)p^{2}-\mathrm{i}\left(\zeta-\zeta^{*}\right)qp\right)\right\}, \\ &\int\mathrm{d}q\wedge\mathrm{d}pQ(q,p) = 1.\end{aligned} \tag{4.3}$$

There are different possibilities to calculate the Wigner quasprobability [27] [28]. In most convenient way the Wigner quasiprobability $W\left(\alpha,\alpha^{*}\right)$ of an arbitrary squeezed state can be calculated from the Wigner quasiprobability $W_{0}\left(\alpha,\alpha^{*}\right)$ of the corresponding non-squeezed state using the relation (4.9) of **Appendix B**. The parameters for our non-unitary approach (4.4) are derived in (4.18) where we have to set $\xi = \zeta^{*}$ and have

$$W\left(\alpha,\alpha^{*}\right) = W_{0}\left(\frac{\alpha+\zeta\alpha^{*}}{\sqrt{1-\zeta\zeta^{*}}},\frac{\alpha^{*}+\zeta^{*}\alpha}{\sqrt{1-\zeta\zeta^{*}}}\right). \tag{4.4}$$

Taking into account the well-known Wigner quasiprobability for the vacuum state

$$W_0\left(\alpha,\alpha^*\right)=\frac{2}{\pi}\exp\left(-2\alpha\alpha^*\right),\quad \left(\rho=|0\rangle\langle 0|\right), \tag{4.5}$$

we obtain

$$\begin{aligned}W\left(\alpha,\alpha^*\right)&=\frac{2}{\pi}\exp\left\{-2\frac{\left(\alpha+\zeta\alpha^*\right)\left(\alpha^*+\zeta^*\alpha\right)}{1-\zeta\zeta^*}\right\}\\&=\frac{2}{\pi}\exp\left\{-2\frac{\left(1+\zeta\zeta^*\right)\alpha\alpha^*+\zeta^*\alpha^2+\zeta\alpha^{*2}}{1-\zeta\zeta^*}\right\},\quad \int\frac{\mathrm{i}}{2}\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*W\left(\alpha,\alpha^*\right)=1.\end{aligned} \tag{4.6}$$

In representation by the real canonical variables (q,p)

$$\begin{aligned}&W(q,p)\\&=\frac{1}{\hbar\pi}\exp\left\{-\frac{(1+\zeta)\left(1+\zeta^*\right)q^2+(1-\zeta)\left(1-\zeta^*\right)p^2-\mathrm{i}\left(\zeta-\zeta^*\right)2qp}{\left(1-\zeta\zeta^*\right)\hbar}\right\}\\&=\frac{1}{\hbar\pi}\exp\left\{-\frac{\left((1+\zeta)q+\mathrm{i}(1-\zeta)p\right)\left(\left(1+\zeta^*\right)q-\mathrm{i}\left(1-\zeta^*\right)p\right)}{\left(1-\zeta\zeta^*\right)\hbar}\right\},\\&\int\mathrm{d}q\wedge\mathrm{d}pW(q,p)=1.\end{aligned} \tag{4.7}$$

Another often easy way to calculate the Wigner quasiprobability $W\left(\alpha,\alpha^*\right)$ is to make first the normal ordering of the operator involved in the representation (4.1) that leads to [22]

$$\begin{aligned}W\left(\alpha,\alpha^*\right)&=\left\langle\rho\exp\left(-a\frac{\partial}{\partial\alpha}-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\right\rangle\delta\left(\alpha,\alpha^*\right)\\&=\left\langle\rho\exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle\exp\left(\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\delta\left(\alpha,\alpha^*\right)\\&=\left\langle\rho\exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle\frac{2}{\pi}\exp\left(-2\alpha\alpha^*\right).\end{aligned} \tag{4.8}$$

For example, one obtains then immediately from it the already used Wigner quasiprobability for the vacuum state (4.5) since the application of the operator $\exp\left(-a\frac{\partial}{\partial\alpha}\right)$ to the state $|0\rangle$ and of $\exp\left(-a^\dagger\frac{\partial}{\partial\alpha^*}\right)$ from the right to $\langle 0|$ reproduces them. In the real representation one may use the following equivalent definitions where the first corresponds to the definition given by Wigner [27] [28] for pure states (written here with density operator ρ and with Dirac's notations for states)

$$\begin{aligned}W(q,p)&=\frac{1}{\hbar\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\langle q-x|\rho|q+x\rangle\exp\left(\mathrm{i}\frac{2px}{\hbar}\right)\\&=\frac{1}{\hbar\pi}\int_{-\infty}^{+\infty}\mathrm{d}y\langle p+y|\rho|p-y\rangle\exp\left(\mathrm{i}\frac{2qy}{\hbar}\right).\end{aligned} \tag{4.9}$$

We checked (4.7) by these formulae using the wave functions $\langle q|0,\zeta\rangle$ and $\langle p|0,\zeta\rangle$ derived in (3.3).

Without presenting its detailed calculation let us give the more general quasiprobabilities $F_r\left(\alpha,\alpha^*\right)$ with the parameter r and defined by [29] ('*' means convolution)

$$
\begin{aligned}
F_r\left(\alpha,\alpha^*\right) &= \exp\left(\frac{r}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)W\left(\alpha,\alpha^*\right) \\
&= \frac{2}{\pi\sqrt{r^2}}\exp\left(-\frac{2\alpha\alpha^*}{r}\right)*W\left(\alpha,\alpha^*\right),
\end{aligned}
\tag{4.10}
$$

Our result for the squeezed vacuum states $|0,\zeta\rangle$ is

$$
\begin{aligned}
F_r\left(\alpha,\alpha^*\right) &= \frac{1}{\pi}\sqrt{\frac{1-\zeta\zeta^*}{\left(\frac{1+r}{2}\right)^2-\left(\frac{1-r}{2}\right)^2\zeta\zeta^*}} \\
&\times\exp\left\{-\frac{\left(\frac{1+r}{2}+\frac{1-r}{2}\zeta\zeta^*\right)\alpha\alpha^*+\frac{\zeta^*}{2}\alpha^2+\frac{\zeta}{2}\alpha^{*2}}{\left(\frac{1+r}{2}\right)^2-\left(\frac{1-r}{2}\right)^2\zeta\zeta^*}\right\}.
\end{aligned}
\tag{4.11}
$$

For $r=0$ one obtains the Wigner quasiprobability $W\left(\alpha,\alpha^*\right)$, for $r=+1$ the Husimi-Kano quasiprobability $Q\left(\alpha,\alpha^*\right)$ and for $r=-1$ the Glauber-Sudarshan quasiprobability $P\left(\alpha,\alpha^*\right)$. The Glauber-Sudarshan quasiprobability $P\left(\alpha,\alpha^*\right)$ for squeezed vacuum states is a singular generalized function and makes for $\zeta=0$ the transition to the delta function $\delta\left(\alpha,\alpha^*\right)$. The representation of (4.10) by real canonical variables (q,p) is easily to make. Furthermore, with some calculation one may bring all quasiprobabilities in representation by real variable to principal axes form that, however, we do not demonstrate here.

5. Expectation Values of Powers of Number Operator and Related Ordered Operators for Squeezed Vacuum States

We now calculate expectation values for the squeezed vacuum states. In particular interesting are the expectation values of ordered powers of the annihilation and creation operators $\left(a,a^\dagger\right)$. We begin with the special expectation values $\overline{a^{\dagger k}a^k}$ since from these expectation values one may calculate the expectation values $\overline{N^l}$ of the number operator *N*.

Using the number representation (2.3) of the squeezed vacuum states $|0,\zeta\rangle$ we proceed quickly for the expectation values $\overline{a^{\dagger k}a^k}$ which depend only on $|\zeta|\equiv\sqrt{\zeta\zeta^*}$ to the following intermediate result

$$
\begin{aligned}
\overline{a^{\dagger k}a^k} &\equiv \langle 0,\zeta|a^{\dagger k}a^k|0,\zeta\rangle = \sqrt{1-|\zeta|^2}\sum_{m=0}^{\infty}\frac{(2m)!^2}{2^{2m}m!^2(2m-k)!}|\zeta|^{2m} \\
&= \sqrt{1-|\zeta|^2}\,|\zeta|^k\frac{\partial^k}{\partial|\zeta|^k}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2}|\zeta|^{2m} \\
&= \sqrt{1-|\zeta|^2}\,|\zeta|^k\frac{\partial^k}{\partial|\zeta|^k}\frac{1}{\sqrt{1-|\zeta|^2}},
\end{aligned}
\tag{5.1}
$$

where we used that the last sum on the right-hand side is the Taylor series of $\left(1-|\zeta|^2\right)^{-\frac{1}{2}}$. This result can be expressed by the Ultraspherical polynomials $\mathrm{P}_n^{(\alpha,\alpha)}(u)$ as special case $\alpha=\beta$ of the Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(u)$ (e.g., [30] [31]) in two essentially different ways. We represent this in a slightly more general form as needed here in **Appendix C**. The result specialized for (5.1) is

$$\begin{aligned}\overline{a^{\dagger k}a^k} &= k!\left(-\frac{2|\zeta|}{1-|\zeta|^2}\right)^k \mathrm{P}_k^{\left(-\frac{1}{2}-k,\frac{1}{2}-k\right)}\left(|\zeta|\right)\\ &= k!\left(-\frac{\mathrm{i}|\zeta|}{\sqrt{1-|\zeta|^2}}\right)^k \mathrm{P}_k\left(\frac{\mathrm{i}|\zeta|}{\sqrt{1-|\zeta|^2}}\right),\end{aligned} \tag{5.2}$$

where $\mathrm{P}_n(u)\equiv\mathrm{P}_n^{(0,0)}(u)$ are the Legendre polynomials as special case of the Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(u)$. More explicitly this provides

$$\overline{a^{\dagger k}a^k} = k!\left(\frac{|\zeta|}{2\left(1-|\zeta|^2\right)}\right)^k \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{k!}{j!^2(k-2j)!}\left(2|\zeta|\right)^{k-2j} \geq 0. \tag{5.3}$$

Remarkable in these transformations is that we could split an essential factor $\left(\frac{|\zeta|}{1-|\zeta|^2}\right)^k$ multiplied by a polynomial in comparison to the infinite sums in (5.1).

The expectation values of symmetrically ordered power operators $\overline{\mathcal{S}\left\{a^{\dagger k}a^k\right\}}$ are connected with $\overline{a^{\dagger k}a^k}$ by (see, e.g., Equation (7.6) in [22])

$$\overline{\mathcal{S}\left\{a^{\dagger k}a^k\right\}} = \sum_{l=0}^{k}\frac{k!^2}{l!(k-l)!^2}\left(\frac{1}{2}\right)^l \overline{a^{\dagger k-l}a^{k-l}}. \tag{5.4}$$

Inserting for $\overline{a^{\dagger k-l}a^{k-l}}$ the result (5.3) one may transform the arising double sum by reordering to

$$\begin{aligned}\overline{\mathcal{S}\left\{a^{\dagger k}a^k\right\}} &= \frac{k!}{\left(2\left(1-|\zeta|^2\right)\right)^k}\sum_{j=0}^{\left[\frac{k}{2}\right]}\frac{k!}{j!^2(k-2j)!}|\zeta|^{2j}\left(1+|\zeta|^2\right)^{k-2j}\\ &= \frac{k!}{\left(2\left(1-|\zeta|^2\right)\right)^k}\sum_{l=0}^{k}\left(\frac{k!}{l!(k-l)!}\right)^2|\zeta|^{2l}.\end{aligned} \tag{5.5}$$

The transition from the first line to the second line is possible after Taylor series expansion of $\left(1+|\zeta|^2\right)^{k-2j}$ and applying then Vandermonde's convolution identity which provides a particularly interesting representation with the squared binomial coefficients involved. One may also directly make the transition to representations of the result by the Jacobi polynomials and by its special case of Legendre polynomials using their explicit representations and transformation formulae as follows

$$\overline{\mathcal{S}\left\{a^{\dagger k}a^k\right\}} = \frac{k!}{2^k}\mathrm{P}_k\left(\frac{1+|\zeta|^2}{1-|\zeta|^2}\right) = \frac{k!}{\left(2\left(1-|\zeta|^2\right)\right)^k}\mathrm{P}_k^{(0,-2k-1)}\left(1-2|\zeta|^2\right)$$
$$= k!\left(\frac{|\zeta|^2}{2\left(1-|\zeta|^2\right)}\right)^k \mathrm{P}_k^{(0,-2k-1)}\left(1-\frac{2}{|\zeta|^2}\right). \tag{5.6}$$

In **Appendix E** we give some first members of the explicit representation of the sequence $\overline{\mathcal{S}\left\{a^{\dagger k}a^k\right\}}$.

The expectation values $\overline{N^k}$ of powers of the number operator can be calculated from the expectation values $\overline{a^{\dagger k}a^k}$ by the relation

$$\overline{N^k} = \sum_{l=0}^{k}\mathrm{S}(k,l)\overline{a^{\dagger l}a^l}, \tag{5.7}$$

where $\mathrm{S}(k,l)$ denotes the Stirling numbers of second kind. We could not find up to now a closed representation of the coefficients in front of powers of $|\zeta|^2$ in the polynomial in the numerator over the denominator $\left(1-|\zeta|^2\right)^k$ in $\overline{N^k}$ and we give explicit results in **Appendix E** up to $k=5$.

From the calculated expectation values we find in first order

$$\overline{a^\dagger a} = \overline{N} = \frac{|\zeta|^2}{1-|\zeta|^2}, \quad \overline{\mathcal{S}\left\{a^\dagger a\right\}} = \frac{1+|\zeta|^2}{2\left(1-|\zeta|^2\right)} = \overline{N}+\frac{1}{2}, \tag{5.8}$$

and in second order (see also **Appendix E** for more and higher-order expectation values)

$$\overline{a^{\dagger 2}a^2} = \frac{|\zeta|^2\left(1+2|\zeta|^2\right)}{\left(1-|\zeta|^2\right)^2} = \overline{N}\left(3\overline{N}+1\right), \quad \overline{N^2} = \frac{|\zeta|^2\left(2+|\zeta|^2\right)}{\left(1-|\zeta|^2\right)^2} = \overline{N}\left(3\overline{N}+2\right),$$
$$\overline{\mathcal{S}\left\{a^{\dagger 2}a^2\right\}} = \frac{1+4|\zeta|^2+|\zeta|^4}{2\left(1-|\zeta|^2\right)^2} = 3\overline{N}\left(\overline{N}+1\right)+\frac{1}{2}. \tag{5.9}$$

From this follows for squeezed vacuum states using $|\zeta|^2 = \frac{\overline{N}}{\overline{N}+1} < 1$

$$\overline{(\Delta N)^2} \equiv \overline{N^2}-\left(\overline{N}\right)^2 = \frac{2|\zeta|^2}{\left(1-|\zeta|^2\right)^2} = 2\overline{N}\left(\overline{N}+1\right), \tag{5.10}$$

that means the number uncertainty for squeezed vacuum states is larger than, for example, for coherent states ($\overline{(\Delta N)^2} = \overline{N}$) and furthermore

$$\overline{a^{\dagger 2}a^2}-\left(\overline{a^\dagger a}\right)^2 = \frac{|\zeta|^2\left(1+|\zeta|^2\right)}{\left(1-|\zeta|^2\right)^2} = \overline{(\Delta N)^2}-\overline{N} = \overline{N}\left(2\overline{N}+1\right),$$
$$\overline{\mathcal{S}\left\{a^{\dagger 2}a^2\right\}}-\left(\overline{\mathcal{S}\left\{a^\dagger a\right\}}\right)^2 = \frac{1+6|\zeta|^2+|\zeta|^4}{4\left(1-|\zeta|^2\right)^2} = \frac{1}{4}+2\overline{N}\left(\overline{N}+1\right) \geq \frac{1}{4}. \tag{5.11}$$

This shows that sub-Poissonian statistics $\overline{a^{\dagger 2}a^2}-\left(\overline{a^{\dagger}a}\right)^2<0$ does not exist for squeezed vacuum states and that $\overline{\mathcal{S}\left\{a^{\dagger 2}a^2\right\}}-\left(\overline{\mathcal{S}\left\{a^{\dagger}a\right\}}\right)^2\geq\frac{1}{4}$ satisfies the general inequality for this quantity [22].

6. Further Expectation Values for Squeezed Vacuum States

In this Section we calculate more general expectation values for the squeezed vacuum states of the form $\overline{a^{\dagger k}a^l}=\overline{a^{\dagger l}a^k}^*$ than in the preceding Section. They are only non-vanishing if the difference $k-l$ is an even number $2m$. This follows from symmetry considerations of the squeezed vacuum states or from their number-state representation which contains only even number states $|2m\rangle$ Therefore, we now calculate separately the expectation values $\overline{a^{\dagger 2k}a^{2l}}$ and $\overline{a^{\dagger 2k+1}a^{2l+1}}$. They depend on ζ and ζ^* separately and, therefore, we use here the pair of variables (ζ,ζ^*) for the representation of the results.

For $\overline{a^{\dagger 2k}a^{2l}}$ using (2.2) and doubling relations for the Gamma function applied to $(2x)!$ we arrive at the intermediate result

$$\begin{aligned}&\overline{a^{\dagger 2k}a^{2l}}\equiv\langle 0,\zeta|a^{\dagger 2k}a^{2l}|0,\zeta\rangle\\&=\sqrt{\frac{1-\zeta\zeta^*}{\pi}}(-2)^{k+l}\zeta^{*k}\zeta^{l}\sum_{j=0}^{\infty}\frac{\left(j+k-\frac{1}{2}\right)!\left(j+l-\frac{1}{2}\right)!}{j!\left(j-\frac{1}{2}\right)!}\left(\zeta\zeta^*\right)^j,\\&\left(\left(-\frac{1}{2}\right)!\equiv\sqrt{\pi}\right).\end{aligned}\tag{6.1}$$

By comparison of this expression with explicit expressions for the Hypergeometric function ${}_2\mathrm{F}_1(\alpha,\beta;\gamma;z)$ we see that this is a polynomial case with the Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(u)$ involved. The main polynomial case of the Hypergeometric function is

$$\begin{aligned}{}_2\mathrm{F}_1(-n,\beta;\gamma;z)&=\frac{k!(\gamma-1)!}{(n+\gamma-1)!}\mathrm{P}_n^{(\gamma-1,\beta-\gamma-n)}(1-2z)\\&=\frac{n!(\gamma-1)!}{(n+\gamma-1)!}z^n\mathrm{P}_n^{(-\beta-n,\beta-\gamma-n)}\left(1-\frac{2}{z}\right)\\&={}_2\mathrm{F}_1(\beta,-n;\gamma;z),\end{aligned}\tag{6.2}$$

and the other possible polynomial case is (see [32] Section 5 there)

$$z^n\,{}_2\mathrm{F}_1\left(-\frac{n}{2},-\frac{n-1}{2};\alpha+1;-\frac{1}{z^2}\right)=\frac{n!\alpha!}{(n+\alpha)!}\left(\sqrt{1+z^2}\right)^n\mathrm{P}_n^{(\alpha,\alpha)}\left(\frac{z}{\sqrt{1+z^2}}\right).\tag{6.3}$$

By transformation relations of the Hypergeometric function, in general, and in possible special cases we find the relation between the Jacobi polynomials

$$(1-z)^n\mathrm{P}_n^{(\alpha,\beta)}\left(\frac{1+z}{1-z}\right)=\mathrm{P}_n^{(\alpha,-2n-\alpha-\beta-1)}(1-2z),\tag{6.4}$$

and in special case if one of the upper indices is an integer

$$\frac{n!}{(n+\beta)!}\left(\frac{2}{u-1}\right)^{n}\mathrm{P}_{n}^{(m-n,\beta)}(u)=\frac{m!}{(m+\beta)!}\left(\frac{2}{u-1}\right)^{m}\mathrm{P}_{m}^{(n-m,\beta)}(u). \quad (6.5)$$

Together with the more trivial relation

$$\mathrm{P}_{n}^{(\alpha,\beta)}(u)=(-1)^{n}\,\mathrm{P}_{n}^{(\beta,\alpha)}(-u), \quad (6.6)$$

this provides many possibilities to represent our final results.

Our final result admits the following two equivalent representations showing some symmetry with respect to interchanging $k\leftrightarrow l$

$$\begin{aligned}\overline{a^{\dagger 2k}a^{2l}}&=(-2)^{k+l}\frac{k!\left(l-\frac{1}{2}\right)!}{\left(-\frac{1}{2}\right)!}\frac{\zeta^{*k}\zeta^{l}}{(1-\zeta\zeta^{*})^{l}}\mathrm{P}_{k}^{\left(-\frac{1}{2},l-k\right)}\left(\frac{1+\zeta\zeta^{*}}{1-\zeta\zeta^{*}}\right)\\&=(-2)^{k+l}\frac{l!\left(k-\frac{1}{2}\right)!}{\left(-\frac{1}{2}\right)!}\frac{\zeta^{*k}\zeta^{l}}{(1-\zeta\zeta^{*})^{k}}\mathrm{P}_{l}^{\left(-\frac{1}{2},k-l\right)}\left(\frac{1+\zeta\zeta^{*}}{1-\zeta\zeta^{*}}\right),\end{aligned} \quad (6.7)$$

or alternatively

$$\begin{aligned}\overline{a^{\dagger 2k}a^{2l}}&=(-2)^{k+l}\frac{k!\left(l-\frac{1}{2}\right)!}{\left(-\frac{1}{2}\right)!}\frac{\zeta^{*k}\zeta^{l}}{(1-\zeta\zeta^{*})^{k+l}}\mathrm{P}_{k}^{\left(-\frac{1}{2},-k-l-\frac{1}{2}\right)}\left(1-2\zeta\zeta^{*}\right)\\&=(-2)^{k+l}\frac{l!\left(k-\frac{1}{2}\right)!}{\left(-\frac{1}{2}\right)!}\frac{\zeta^{*k}\zeta^{l}}{(1-\zeta\zeta^{*})^{k+l}}\mathrm{P}_{l}^{\left(-\frac{1}{2},-k-l-\frac{1}{2}\right)}\left(1-2\zeta\zeta^{*}\right).\end{aligned} \quad (6.8)$$

In case of the expectation values $\overline{a^{\dagger 2k+1}a^{2l+1}}$ the analogous intermediate result to (6.1) is

$$\overline{a^{\dagger 2k+1}a^{2l+1}}=\sqrt{\frac{1-\zeta\zeta^{*}}{\pi}}(-1)^{k+l}\,2^{k+l+1}\zeta^{*k}\zeta^{l}\sum_{j=1}^{\infty}\frac{\left(j+k-\frac{1}{2}\right)!\left(j+l-\frac{1}{2}\right)!}{\left(j-\frac{1}{2}\right)!(j-1)!}\left(\zeta\zeta^{*}\right)^{j}. \quad (6.9)$$

Using the relation to the Hypergeometric function and the Jacobi polynomials from this follows in analogy to (6.7)

$$\begin{aligned}\overline{a^{\dagger 2k+1}a^{2l+1}}&=(-2)^{k+l}\frac{k!\left(l+\frac{1}{2}\right)!}{\left(\frac{1}{2}\right)!}\frac{\zeta^{*k+1}\zeta^{l+1}}{(1-\zeta\zeta^{*})^{l+1}}\mathrm{P}_{k}^{\left(\frac{1}{2},l-k\right)}\left(\frac{1+\zeta\zeta^{*}}{1-\zeta\zeta^{*}}\right)\\&=(-2)^{k+l}\frac{l!\left(k+\frac{1}{2}\right)!}{\left(\frac{1}{2}\right)!}\frac{\zeta^{*k+1}\zeta^{l+1}}{(1-\zeta\zeta^{*})^{k+1}}\mathrm{P}_{l}^{\left(\frac{1}{2},k-l\right)}\left(\frac{1+\zeta\zeta^{*}}{1-\zeta\zeta^{*}}\right),\end{aligned} \quad (6.10)$$

or, alternatively, in analogy to (6.8)

$$\overline{a^{\dagger 2k+1}a^{2l+1}} = (-2)^{k+l}\frac{k!\left(l+\frac{1}{2}\right)!}{\left(\frac{1}{2}\right)!}\frac{\zeta^{*k+1}\zeta^{l+1}}{(1-\zeta\zeta^*)^{k+l+1}}\mathrm{P}_k^{\left(\frac{1}{2},-k-l-\frac{3}{2}\right)}(1-2\zeta\zeta^*)$$
$$= (-2)^{k+l}\frac{l!\left(k+\frac{1}{2}\right)!}{\left(\frac{1}{2}\right)!}\frac{\zeta^{*k+1}\zeta^{l+1}}{(1-\zeta\zeta^*)^{k+l+1}}\mathrm{P}_l^{\left(\frac{1}{2},-k-l-\frac{3}{2}\right)}(1-2\zeta\zeta^*). \quad (6.11)$$

We checked the special cases $k=l$ in comparison with the representation (5.2) by the Legendre polynomials. We checked too that the right-hand sides of this formula gives for all the four different representations the same result.

There is yet an interesting mathematical aspect. In (5.2) Section 5 we calculated the expectation values $\overline{a^{\dagger k}a^l}$ in the special case $k=l$ by formulae which involve Jacobi polynomials (or their special case Legendre polynomials) without distinction of even and odd *k*, whereas in present Section we calculated the more general cases $k \neq l$ and had to distinguish the cases of even *k* and *l* and of odd *k* and *l* and in the specialization $k=l$ there are involved Jacobi polynomials which are different from that for $k=l$ and it is not possible (or simple) to join these polynomials for $k=l$ to one common formula. The manifold of different transformation relations for Jacobi polynomials is very astonishing (see **Appendix C** and [27]).

As alternative to the calculation of expectation values by the number representation of squeezed vacuum states one may calculate them from the quasiprobabilities that, however, is also not very simple. With the Wigner quasiprobability $W(\alpha,\alpha^*)$ one may calculate basically the expectation values of symmetrically (Weyl) ordered operators, for example, by integration over the function $\alpha\alpha^*$ for the expectation value of the operator $\frac{1}{2}(aa^\dagger + a^\dagger a) = N + \frac{1}{2}I$. With the Husimi-Kano quasiprobability $Q(\alpha,\alpha^*)$ one may calculate basically the expectation values of anti-normally operators which one has then to transform to the more interesting expectation values of normally ordered operators. The expectation values of $a^{\dagger k}$ and of a^l can be calculated with an arbitrary quasiprobability with the parameter *r* considered in Section 4.

7. Displaced Squeezed Vacuum States or Squeezed Coherent States in Non-Unitary Approach

As a generalization of squeezed vacuum states we derive here shortly their representation in the basis of number state and discuss a very interesting aspect. It is difficult to deal with squeezing in full generality and one may find in literature many approaches which are special ones (squeezing only in directions of coordinate axes (q,p)) or with absent calculation of basic functions connected with them.

We define displaced squeezed vacuum states in the non-unitary approach by applying the displacement operator $D(\beta,\beta^*)$ to squeezed vacuum states

$|0,\zeta\rangle$ according to

$$\begin{aligned}|\beta,\zeta\rangle &\equiv D(\beta,\beta^*)|0,\zeta\rangle = D(\beta,\beta^*)S(\zeta'^*,0,\zeta')|0\rangle,\\ (\zeta',\zeta'^*) &= \frac{\mathrm{Arth}(|\zeta|)}{|\zeta|}(\zeta,\zeta^*),\end{aligned} \tag{7.1}$$

where $S(\zeta'^*,0,\zeta')$ is the squeezing operator in unitary approach. We may change the order of operations of displacement and squeezing where the squeezing operator remains the stable part and the displacement operator has to be change. The basic relations for this provides the fundamental representation (A.7) setting there $\eta=0$ and substituting $\zeta\to-\zeta'$

$$\begin{aligned}&\left(S(\zeta'^*,0,\zeta')\right)^\dagger(a,a^\dagger)S(\zeta'^*,0,\zeta')\\ &= S(-\zeta'^*,0,-\zeta')(a,a^\dagger)\left(S(-\zeta'^*,0,-\zeta')\right)^\dagger\\ &=(a,a^\dagger)\begin{pmatrix}\frac{1}{\sqrt{1-|\zeta|^2}}, & -\frac{\zeta^*}{\sqrt{1-|\zeta|^2}}\\ -\frac{\zeta}{\sqrt{1-|\zeta|^2}}, & \frac{1}{\sqrt{1-|\zeta|^2}}\end{pmatrix}.\end{aligned} \tag{7.2}$$

These transformations of $(a,a^\dagger)$ after transition to the Hermitean basis of operators (Q,P) are very similar to Special Lorentz transformations of (x,ct) with x one space coordinate and t the time and this is not incidental since it is for real parameter ζ the same one-parameter Lie group. From (7.2) follows

$$\begin{aligned}&\left(S(\zeta'^*,0,\zeta')\right)^\dagger D(\beta,\beta^*)S(\zeta'^*,0,\zeta')\\ &=\left(S(\zeta'^*,0,\zeta')\right)^\dagger \exp(\beta a^\dagger-\beta^*a)S(\zeta'^*,0,\zeta')\\ &=\exp\left(\beta\frac{a^\dagger-\zeta^*a}{\sqrt{1-|\zeta|^2}}-\beta^*\frac{a-\zeta a^\dagger}{\sqrt{1-|\zeta|^2}}\right)\\ &=\exp\left(\frac{\beta+\zeta\beta^*}{\sqrt{1-|\zeta|^2}}a^\dagger-\frac{\beta^*+\zeta^*\beta}{\sqrt{1-|\zeta|^2}}a\right)\\ &=D\left(\frac{\beta+\zeta\beta^*}{\sqrt{1-|\zeta|^2}},\frac{\beta^*+\zeta^*\beta}{\sqrt{1-|\zeta|^2}}\right).\end{aligned} \tag{7.3}$$

Applied to (7.1) we find (see also Schleich [13], p. 125)

$$\begin{aligned}|\beta,\zeta\rangle &= D(\beta,\beta^*)|0,\zeta\rangle = D(\beta,\beta^*)S(\zeta'^*,0,\zeta')|0\rangle\\ &= S(\zeta'^*,0,\zeta')D\left(\frac{\beta+\zeta\beta^*}{\sqrt{1-|\zeta|^2}},\frac{\beta^*+\zeta^*\beta}{\sqrt{1-|\zeta|^2}}\right)|0\rangle\\ &= S(\zeta'^*,0,\zeta')|\beta',0\rangle,\quad (\beta',\beta'^*)\equiv\left(\frac{\beta+\zeta\beta^*}{\sqrt{1-|\zeta|^2}},\frac{\beta^*+\zeta^*\beta}{\sqrt{1-|\zeta|^2}}\right),\end{aligned} \tag{7.4}$$

as alternative representation of squeezed vacuum states. This means that we make first a displacement of the vacuum state $|0\rangle$ to a coherent state $|\beta',0\rangle$ with the changed displacement parameter β' and after this the squeezing of the coherent state with the same squeezing operator $S(\zeta'^{*},0,\zeta')$ as in the first variant. Therefore, displaced squeezed vacuum states are fully equivalent to squeezed coherent states with the changed displacement parameters as seen from (7.1) and (7.4). For $\zeta=0$ one obtains from $|\beta,\zeta\rangle$ the coherent states and for $\beta=0$ the squeezed vacuum states

$$\begin{aligned} &|\beta,0\rangle = D(\beta,\beta^{*})|0,0\rangle \equiv |\beta\rangle, \\ &|0,\zeta\rangle = S(\zeta'^{*},0,\zeta')|0,0\rangle, \quad |0,0\rangle \equiv |0\rangle. \end{aligned} \tag{7.5}$$

The squeezing operator is the stable part in these two alternative representations. Figure 3 shows schematically the displacement of squeezed vacuum states under fixed complex squeezing parameters ζ in different directions of the complex phase plane described by the complex parameter β ($|\beta|=\text{const}$ in the four particular pictures).

In generalization of the well-known eigenvalue equation of coherent states $|\beta,0\rangle \equiv |\beta\rangle$ the displaced squeezed vacuum states $|\beta,\zeta\rangle$ are right-hand eigenstates of the operator $a+\zeta a^{\dagger}$ to eigenvalues $\beta+\zeta\beta^{*}$ according to

$$\left(a+\zeta a^{\dagger}\right)|\beta,\zeta\rangle = \left(\beta+\zeta\beta^{*}\right)|\beta,\zeta\rangle. \tag{7.6}$$

This follows from the relation

$$\begin{aligned} &\left(a+\zeta a^{\dagger}\right)\exp\left(\beta a^{\dagger}-\beta^{*}a\right)|0,\zeta\rangle \\ &= \exp\left(\beta a^{\dagger}-\beta^{*}a\right)\left(a+\beta I+\zeta\left(a^{\dagger}+\beta^{*}I\right)\right)|0,\zeta\rangle \\ &= \left(\beta+\zeta\beta^{*}\right)\exp\left(\beta a^{\dagger}-\beta^{*}a\right)|0,\zeta\rangle, \end{aligned} \tag{7.7}$$

using that $|0,\zeta\rangle$ are right-hand eigenstates of $a+\zeta a^{\dagger}$ to eigenvalue zero (see (2.12)).

We now derive the number representation of displaced squeezed vacuum states. Using the normally order form (2.7) of the squeezing operator and the following normally ordered form of the displacement operator

$$D(\beta,\beta^{*}) = \exp\left(-\frac{\beta\beta^{*}}{2}\right)\exp\left(\beta a^{\dagger}\right)\exp\left(-\beta^{*}a\right), \tag{7.8}$$

from the definition (7.1) follows

$$\begin{aligned} &|\beta,\zeta\rangle \\ &= \exp\left(-\frac{\beta\beta^{*}}{2}\right)\left(1-\zeta\zeta^{*}\right)^{\frac{1}{4}}\exp\left(\beta a^{\dagger}\right)\exp\left(-\beta^{*}a\right)\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle \\ &= \left(1-\zeta\zeta^{*}\right)^{\frac{1}{4}}\exp\left(-\frac{\beta\beta^{*}}{2}\right)\exp\left(\beta a^{\dagger}\right)\exp\left(-\frac{\zeta}{2}\left(a^{\dagger}-\beta^{*}I\right)^{2}\right)\exp\left(-\beta^{*}a\right)|0\rangle \\ &= \left(1-\zeta\zeta^{*}\right)^{\frac{1}{4}}\exp\left(-\frac{\beta\beta^{*}+\zeta\beta^{*2}}{2}\right)\exp\left(\left(\beta+\zeta\beta^{*}\right)a^{\dagger}-\frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle. \end{aligned} \tag{7.9}$$

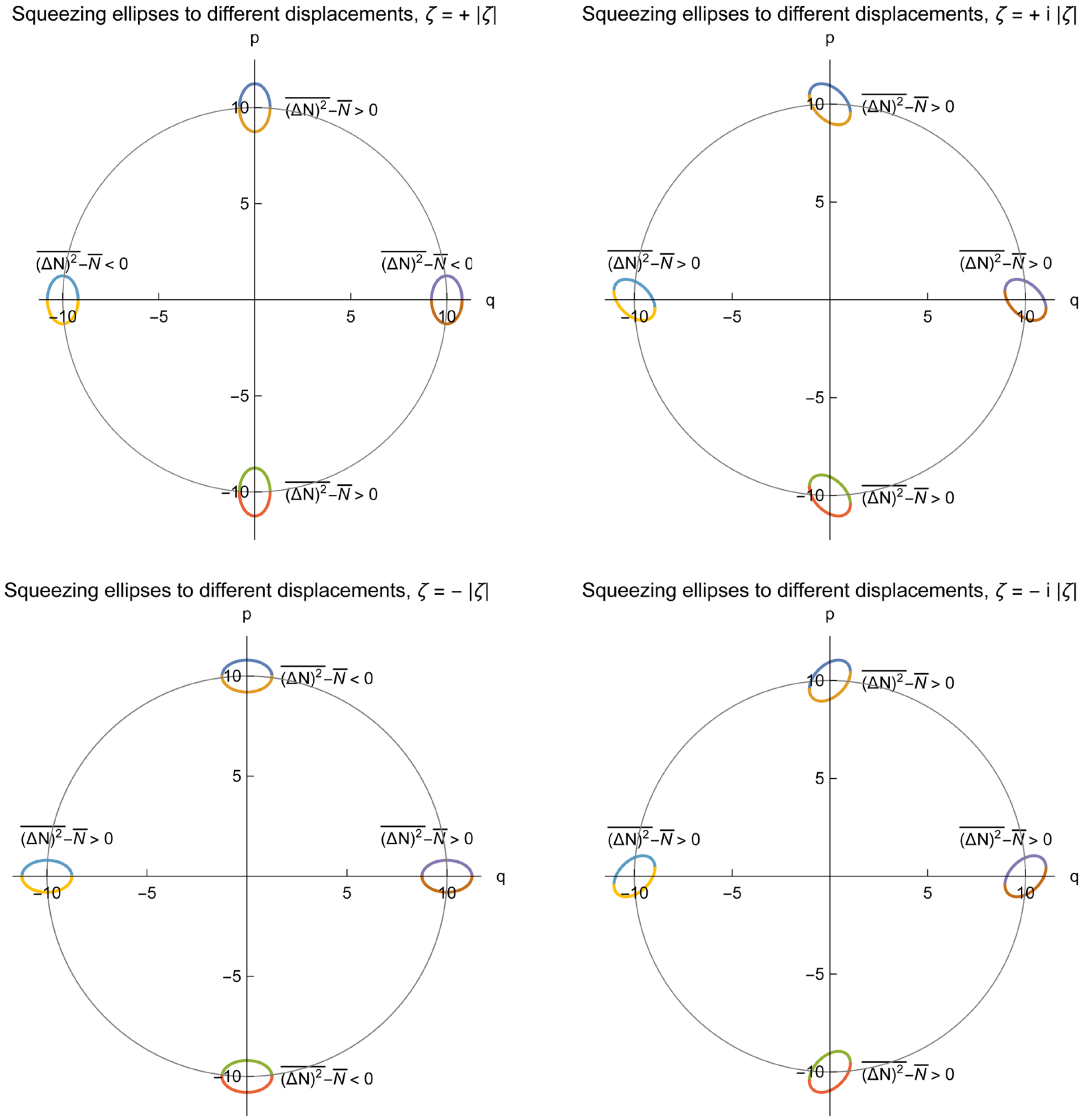

Figure 3. Squeezing ellipses in relation to the displacement parameter of squeezed coherent states (schematically). The mean value $|\overline{N}|$ of the number operator N depends for squeezed coherent states only on the modulus $|\zeta|$ of the squeezing parameter ζ left constant here and on the modulus $|\beta|$ of the complex displacement parameter β and, therefore, is the same on the shown circles in all four partial figures. The case of squeezing in direction of the displacement parameter β is also called "amplitude squeezing" (in q-direction in first figure) and in opposite direction to the displacement parameter β "phase squeezing" (in p-direction in first figure).

If we now apply the generating function (D.1) for Hermite polynomials to the factor in front of $|0\rangle$ we obtain the following form of the representation of displaced squeezed vacuum states

$$|\beta,\zeta\rangle = \left(1-\zeta\zeta^*\right)^{\frac{1}{4}} \exp\left(-\frac{\beta\beta^* + \zeta\beta^{*2}}{2}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\sqrt{2\zeta}}{2} a^\dagger\right)^n \mathrm{H}_n\left(\frac{\beta+\zeta\beta^*}{\sqrt{2\zeta}}\right) |0\rangle, \quad (7.10)$$

and using the generation of number states $|n\rangle$ from the vacuum state $|0\rangle$ (see (2.1)) one finds the final form of the number-state representation[2]

$$|\beta,\zeta\rangle=\left(1-\zeta\zeta^{*}\right)^{\frac{1}{4}}\exp\left(-\frac{\left(\beta+\zeta\beta^{*}\right)\beta^{*}}{2}\right)\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(\frac{\sqrt{2\zeta}}{2}\right)^{n}\mathrm{H}_{n}\left(\frac{\beta+\zeta\beta^{*}}{\sqrt{2\zeta}}\right)|n\rangle. \quad (7.11)$$

It is easy to see that for $\beta=0$ using $\mathrm{H}_{2m}(0)=\frac{(-1)^{m}(2m)!}{m!}, \mathrm{H}_{2m+1}(0)=0$ one obtains the number representation (2.3) of squeezed vacuum states $|0,\zeta\rangle$ and for $\zeta=0$ using $\lim_{z\to\infty}\frac{1}{(2z)^{n}}\mathrm{H}_{n}(z)=1$ the number representation of coherent states $|\beta\rangle$.

From (7.10) one finds the probabilities p_n of the photon statistics

$$\begin{aligned}
p_{n} &\equiv \langle n|\beta,\zeta\rangle\langle\beta,\zeta|n\rangle \\
&=\sqrt{1-\zeta\zeta^{*}}\exp\left(-\frac{2\beta\beta^{*}+\zeta^{*}\beta^{2}+\zeta\beta^{*2}}{2}\right)\frac{1}{n!}\left(\frac{\sqrt{\zeta\zeta^{*}}}{2}\right)^{n} \\
&\quad\times\mathrm{H}_{n}\left(\frac{\beta+\zeta\beta^{*}}{\sqrt{2\zeta}}\right)\mathrm{H}_{n}\left(\frac{\beta^{*}+\zeta^{*}\beta}{\sqrt{2\zeta^{*}}}\right).
\end{aligned} \quad (7.12)$$

By means of the generating function (D.2) for products of two Hermite polynomials it can be affirmed that the states $|\beta,\zeta\rangle$ are normalized and that the probabilities p_n satisfy the necessary relation

$$\sum_{n=0}^{\infty}p_{n}=1. \quad (7.13)$$

In contrast to the photon distributions of coherent states $|\beta\rangle\equiv|\beta,0\rangle$ which depends only on $|\beta|^{2}\equiv\beta\beta^{*}$ and of squeezed vacuum states $|0,\zeta\rangle$ which depends only on $|\zeta|^{2}\equiv\zeta\zeta^{*}<1$ the photon distribution (7.12) depends in addition to the moduli also on the phases of ζ and β in the complex plane.

The nearest coherent state to the state $|\beta,\zeta\rangle$ is the state $|\beta\rangle$ and for this distance $d\left(|\beta,\zeta\rangle\langle\beta,\zeta|,|\beta\rangle\langle\beta|\right)$ one obtains the same value as on the right-hand side of (2.16). This means that it does not depend on β.

8. Wave Functions of Displaced Squeezed Vacuum States or Squeezed Coherent States

As important characteristics of the displaced squeezed vacuum states $|\beta,\zeta\rangle$ we now calculate their wave functions $\langle q|\beta,\zeta\rangle$ and $\langle p|\beta,\zeta\rangle$ in the eigenstates $|q\rangle$ and $|p\rangle$ of the canonical operators Q and P (in the usual standardizations $\langle q|q'\rangle=\delta(q-q')$, $\int_{-\infty}^{+\infty}\mathrm{d}q|q\rangle\langle q|=I$ and similar for $|p\rangle$). From the number representation (7.11) of the states and the well-known number representation of $|q\rangle$ and $|p\rangle$ follows as the first step in the calculation

[2]In both parts with $\sqrt{2\zeta}$ one has to choose the same sign of the root but which sign does not matter.

$$\begin{aligned}
\langle q|\beta,\zeta\rangle &= \left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{q^2}{2\hbar}-\frac{\beta^*\left(\beta+\zeta\beta^*\right)}{2}\right)\\
&\quad\times\sum_{n=0}^{\infty}\frac{\left(\sqrt{\zeta}\right)^n}{2^n n!}\mathrm{H}_n\left(\frac{q}{\sqrt{\hbar}}\right)\mathrm{H}_n\left(\frac{\beta+\zeta\beta^*}{\sqrt{2\zeta}}\right),\\
\langle p|\beta,\zeta\rangle &= \left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{p^2}{2\hbar}-\frac{\beta^*\left(\beta+\zeta\beta^*\right)}{2}\right)\\
&\quad\times\sum_{n=0}^{\infty}\frac{\left(-\mathrm{i}\sqrt{\zeta}\right)^n}{2^n n!}\mathrm{H}_n\left(\frac{p}{\sqrt{\hbar}}\right)\mathrm{H}_n\left(\frac{\beta+\zeta\beta^*}{\sqrt{2\zeta}}\right).
\end{aligned}\tag{8.1}$$

The infinite sums can be calculated in closed form using the generating function (D.2) for the product of two Hermite polynomials. We represent the result by the mean values $\bar{Q}\equiv\langle\beta,\zeta|Q|\beta,\zeta\rangle$ and $\bar{P}\equiv\langle\beta,\zeta|P|\beta,\zeta\rangle$ of the canonical operators Q and P. For squeezed vacuum states $|0,\zeta\rangle$ these mean values vanish already due to the symmetry and for the displaced squeezed vacuum states they are simply connected with the complex displacement parameter (β,β^*) according to

$$\bar{Q}=\sqrt{\frac{\hbar}{2}}\left(\beta+\beta^*\right),\quad \bar{P}=-\mathrm{i}\sqrt{\frac{\hbar}{2}}\left(\beta-\beta^*\right).\tag{8.2}$$

With these parameters the result of the evaluation of the sums in (8.1) can be represented in the form

$$\begin{aligned}
&\langle q|\beta,\zeta\rangle\\
&=\left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{1-\zeta}}\exp\left(\mathrm{i}\frac{\bar{P}\left(2q-\bar{Q}\right)}{2\hbar}\right)\exp\left\{-\frac{(1+\zeta)\left(q-\bar{Q}\right)^2}{2(1-\zeta)\hbar}\right\},\\
&\langle p|\beta,\zeta\rangle\\
&=\left(\frac{1-\zeta\zeta^*}{\hbar\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{1+\zeta}}\exp\left(-\mathrm{i}\frac{\bar{Q}\left(2p-\bar{P}\right)}{2\hbar}\right)\exp\left\{-\frac{(1-\zeta)\left(p-\bar{P}\right)^2}{2(1+\zeta)\hbar}\right\}.
\end{aligned}\tag{8.3}$$

One may make cross checks of these relations using the pair of Fourier transformations

$$\begin{aligned}
\langle p|\beta,\zeta\rangle &= \frac{1}{\sqrt{2\hbar\pi}}\int_{-\infty}^{+\infty}\mathrm{d}q\exp\left(-\mathrm{i}\frac{pq}{\hbar}\right)\langle q|\beta,\zeta\rangle,\\
\langle q|\beta,\zeta\rangle &= \frac{1}{\sqrt{2\hbar\pi}}\int_{-\infty}^{+\infty}\mathrm{d}p\exp\left(\mathrm{i}\frac{qp}{\hbar}\right)\langle p|\beta,\zeta\rangle.
\end{aligned}\tag{8.4}$$

From (8.3) one finds the Gaussian distributions

$$\begin{aligned}
&W(q)\equiv\langle q|\beta,\zeta\rangle\langle\beta,\zeta|q\rangle\\
&\qquad=\sqrt{\frac{1-\zeta\zeta^*}{(1-\zeta)\left(1-\zeta^*\right)\hbar\pi}}\exp\left\{-\frac{1-\zeta\zeta^*}{(1-\zeta)\left(1-\zeta^*\right)}\frac{\left(q-\bar{Q}\right)^2}{\hbar}\right\},
\end{aligned}$$

$$\begin{aligned} W(p) &\equiv \langle p|\beta,\zeta\rangle\langle\beta,\zeta|p\rangle \\ &= \sqrt{\frac{1-\zeta\zeta^*}{(1+\zeta)(1+\zeta^*)\hbar\pi}}\exp\left\{-\frac{1-\zeta\zeta^*}{(1+\zeta)(1+\zeta^*)}\frac{(p-\bar{P})^2}{\hbar}\right\}, \end{aligned} \tag{8.5}$$

with the normalization

$$\int_{-\infty}^{+\infty}\mathrm{d}qW(q) = \int_{-\infty}^{+\infty}\mathrm{d}pW(p) = 1. \tag{8.6}$$

The functions $W(q)$ and $W(p)$ are equal to the Wigner quasiprobability $W(q,p)$ integrated over one of the canonical variables p or q and nothing speaks as known against an interpretation of genuine one-dimensional probability densities. They remain invariant with interchanging $\zeta \leftrightarrow \zeta^*$ and the states are not uniquely reconstructible from $W(q)$ and $W(p)$ alone. In our case of (8.5) they are one-dimensional normalized Gaussian distributions around $\bar{Q}$ and $\bar{P}$, respectively, with the variances of $\bar{Q}$ and $\bar{P}$

$$\begin{aligned} \overline{(\Delta Q)^2} &= \frac{(1-\zeta)(1-\zeta^*)}{1-\zeta\zeta^*}\frac{\hbar}{2}, \\ \overline{(\Delta P)^2} &= \frac{(1+\zeta)(1+\zeta^*)}{1-\zeta\zeta^*}\frac{\hbar}{2}, \end{aligned} \tag{8.7}$$

The uncertainty product (see also (3.7))

$$\overline{(\Delta Q)^2}\,\overline{(\Delta P)^2} = \frac{(1-\zeta^2)(1-\zeta^{*2})}{(1-\zeta\zeta^*)^2}\frac{\hbar^2}{4} \ge \frac{\hbar^2}{4}, \tag{8.8}$$

is only for real squeezing parameter ζ but not for complex ζ the minimal possible one.

In case of real squeezing parameter $\zeta = \zeta^*$ one obtains from (8.3)

$$\begin{aligned} \langle q|\beta,\zeta\rangle &= \frac{1}{\left(2\sqrt{\overline{(\Delta Q)^2}}\pi\right)^{\frac{1}{4}}}\exp\left(\mathrm{i}\frac{\bar{P}(2q-\bar{Q})}{2\hbar}\right)\exp\left\{-\frac{(q-\bar{Q})^2}{4\overline{(\Delta Q)^2}}\right\}, \\ \langle p|\beta,\zeta\rangle &= \frac{1}{\left(2\sqrt{\overline{(\Delta P)^2}}\pi\right)^{\frac{1}{4}}}\exp\left(-\mathrm{i}\frac{\bar{Q}(2p-\bar{P})}{2\hbar}\right)\exp\left\{-\frac{(p-\bar{P})^2}{4\overline{(\Delta P)^2}}\right\}, \end{aligned} \tag{8.9}$$

with the variances of Q and P

$$\overline{(\Delta Q)^2} = \frac{1-\zeta}{1+\zeta}\frac{\hbar}{2}, \quad \overline{(\Delta P)^2} = \frac{1+\zeta}{1-\zeta}\frac{\hbar}{2}, \quad \overline{(\Delta Q)^2}\,\overline{(\Delta P)^2} = \frac{\hbar^2}{4}, \quad (\zeta = \zeta^*), \tag{8.10}$$

As already explained in Section 3 only in case of real $\zeta = \zeta^*$ the squeezing axes coincide with the axes of (q,p) and the uncertainty product becomes the minimal one but taking into account the uncertainty correlation the Gaussian states with other positions of the squeezing axes can be included into the minimum uncertainty states (see Section 3).

By differentiation of relations (8.9) with respect to variables q and p follows

$$
\begin{aligned}
-\mathrm{i}\hbar\left(\frac{\partial}{\partial q}+\frac{q}{2(\Delta Q)^2}\right)\langle q|\beta,\zeta\rangle &\equiv \langle q|\left(P-\frac{\mathrm{i}\hbar}{2(\Delta Q)^2}Q\right)|\beta,\zeta\rangle \\
&=\left(\bar{P}-\frac{\mathrm{i}\hbar\bar{Q}}{2(\Delta Q)^2}\right)\langle q|\beta,\zeta\rangle, \\
\mathrm{i}\hbar\left(\frac{\partial}{\partial p}+\frac{p}{2(\Delta P)^2}\right)\langle p|\beta,\zeta\rangle &\equiv \langle p|\left(Q+\frac{\mathrm{i}\hbar}{2(\Delta Q)^2}P\right)|\beta,\zeta\rangle \\
&=\left(\bar{Q}+\frac{\mathrm{i}\hbar\bar{P}}{2(\Delta P)^2}\right)\langle p|\beta,\zeta\rangle,
\end{aligned}
\tag{8.11}
$$

where we used $\langle q|P=-\mathrm{i}\hbar\frac{\partial}{\partial q}\langle q|$ and $\langle p|Q=\mathrm{i}\hbar\frac{\partial}{\partial p}\langle p|$. The normalization of the states in (8.10) and the pure phase factors $\exp\left(\mp\mathrm{i}\frac{\bar{P}\bar{Q}}{2\hbar}\right)$ do not follow from the differential equations (8.11) and must be determined where the mentioned phase factors must be present to get full agreement with the usual definitions of the phases of the states $|q\rangle$ and $|p\rangle$.

Pauli in his hand-book article [2] (pp. 20, 21) derives the first of the differential equations (8.11), however, with the right-hand side equal to zero that means he does not take into account the possible displacements $\bar{Q}$ and $\bar{P}$ that corresponds to squeezed vacuum states (displaced squeezed vacuum states were not known or discussed at that time but Schrödinger considered already coherent states although not under this name in [1]). More general and complete are the derivations in the monographs of Louisell [3] (p. 50, Equation (1.12.23)) and of Leonhardt [14] (p. 32, Equation (2.85)). The derivations of Pauli are the usual ones in the theory of elements (states) and operators in a Hilbert space known already at that time. Starting from the axiom of positive definiteness of the norm (square root of scalar product of an element with itself) of non-zero elements they consider the superpositions of two arbitrary states and from their norms they derive first the Cauchy-Bunyakovski-Schwarz inequality and second from the vanishing of the superpositions of the two states that they have to be linearly dependent [33]. In the derivations of uncertainty relations this is applied to two states which are generated from one state $|\psi\rangle$ by applying two different operators to this one state (here of $Q-\bar{Q}I$ and $P-\bar{P}I$). For the minimum of the norm (equal to zero) this superposition has to be linearly dependent as seen in (8.11). This leads then to the superposition of the two states $\left(Q-\bar{Q}I\right)|\beta,\zeta\rangle$ and $\left(P-\bar{P}I\right)|\beta,\zeta\rangle$ (see also (2.13)) with real ζ and β in which Pauli in foresight of the result introduced $(\Delta Q)^2$ instead of a more indifferent real parameter λ. Thus the result is connected with the basic assumption that all states in quantum optics may be considered as states with positive norm in a Hilbert space.

The displaced squeezed vacuum states or squeezed coherent states $|\beta,\zeta\rangle$ are the most general pure states with Gaussian distributions.

9. Bargmann Representation and Wigner Quasiprobability for Displaced Squeezed Vacuum States

The Bargmann representation of states is the representation by an analytic function of α obtain by forming the scalar product of the state with the analytic but non-normalized coherent state $\|\alpha\rangle \equiv \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|0\rangle$ (see Section 4). From the number representation of displaced squeezed vacuum states (7.11) using the generating function for Hermite polynomials (9.1) we find

$$\begin{aligned} \exp\left(\frac{\alpha\alpha^*}{2}\right)\langle 0,\zeta|\alpha\rangle &= \left(1-\zeta\zeta^*\right)^{\frac{1}{4}} \exp\left(-\frac{\beta\beta^*}{2}+\alpha\beta^*-\frac{\zeta^*}{2}(\alpha-\beta)^2\right) \\ &= \left(\exp\left(\frac{\alpha\alpha^*}{2}\right)\langle\alpha|0,\zeta\rangle\right)^*. \end{aligned} \tag{9.1}$$

The Bargmann representation of a state contains the full information about the state. From (9.1) one finds the Husimi-Kano quasiprobability

$$\begin{aligned} Q\left(\alpha,\alpha^*\right) &\equiv \frac{\langle\alpha|\beta,\zeta\rangle\langle\beta,\zeta|\alpha\rangle}{\pi} \\ &= \frac{\sqrt{1-\zeta\zeta^*}}{\pi}\exp\left\{-\left(\left(\alpha-\beta\right)\left(\alpha^*-\beta^*\right)+\frac{\zeta^*}{2}\left(\alpha-\beta\right)^2+\frac{\zeta}{2}\left(\alpha^*-\beta^*\right)^2\right)\right\}, \\ \int \frac{\mathrm{i}}{2}\mathrm{d}\alpha\wedge\mathrm{d}\alpha^* Q\left(\alpha,\alpha^*\right) &= 1. \end{aligned} \tag{9.2}$$

This is the corresponding Husimi-Kano quasiprobability (4.2) for squeezed vacuum states with argument displacement.

In **Appendix B** it is shown by a very simple transformation that the Wigner quasiprobability $W\left(\alpha,\alpha^*\right)$ of a displaced state can be obtained from Wigner quasiprobability $W_0\left(\alpha,\alpha^*\right)$ of the corresponding undisplaced state by a simple argument displacement $\left(\alpha,\alpha^*\right)\to\left(\alpha-\beta,\alpha^*-\beta^*\right)$. This is also true for other quasiprobabilities such as for example the Husimi-Kano quasiprobability $Q\left(\alpha,\alpha^*\right)$ as we saw and for the Glauber-Sudarshan quasiprobability $P\left(\alpha,\alpha^*\right)$. In this way one obtains from (4.6) for the Wigner quasiprobability of displaced squeezed vacuum states

$$\begin{aligned} W\left(\alpha,\alpha^*\right) &= \frac{2}{\pi}\exp\left\{-2\frac{\left(\alpha-\beta+\zeta\left(\alpha^*-\beta^*\right)\right)\left(\alpha^*-\beta^*+\zeta\left(\alpha-\beta\right)\right)}{1-\zeta\zeta^*}\right\}, \\ \int \frac{\mathrm{i}}{2}\mathrm{d}\alpha\wedge\mathrm{d}\alpha^* W\left(\alpha,\alpha^*\right) &= 1. \end{aligned} \tag{9.3}$$

As discussed in Section 7 as alternative we may first make a displacement of the vacuum state to a coherent state with the displacement parameter $\left(\frac{\beta+\zeta\beta^*}{\sqrt{1-\zeta\zeta^*}},\frac{\beta^*+\zeta^*\beta}{\sqrt{1-\zeta\zeta^*}}\right)$ that provides the Wigner quasiprobability

$$W_0\left(\alpha,\alpha^*\right)=\frac{2}{\pi}\exp\left\{-2\left(\alpha-\frac{\beta+\zeta\beta^*}{\sqrt{1-\zeta\zeta^*}}\right)\left(\alpha^*-\frac{\beta^*+\zeta^*\beta}{\sqrt{1-\zeta\zeta^*}}\right)\right\}. \tag{9.4}$$

If we now make the squeezing of coherent state with the same squeezing operator as in the other variant (see (7.4)) we have to transform the arguments in (9.4) according to $\left(\alpha,\alpha^*\right)\to\left(\frac{\alpha+\zeta\alpha^*}{\sqrt{1-\zeta\zeta^*}},\frac{\alpha^*+\zeta^*\alpha}{\sqrt{1-\zeta\zeta^*}}\right)$ (see also (4.4)) and we obtain again the Wigner quasiprobability (9.3) that affirms the interchangeability of squeezing and displacement according to (7.4).

The Wigner quasiprobability for displaced squeezed vacuum states in the representation by canonical variables (q,p) can be obtained from (4.7) by the following substitution of the canonical variables (q,p)

$$(q,p)\to\left(q-\bar{Q},p-\bar{P}\right),\left(\bar{a},\bar{a}^*\right)\equiv\left(\frac{\bar{Q}+\mathrm{i}\bar{P}}{\sqrt{2\hbar}},\frac{\bar{Q}-\mathrm{i}\bar{P}}{\sqrt{2\hbar}}\right)=\left(\beta,\beta^*\right),\left(\overline{a^\dagger}=\bar{a}^*\right). \tag{9.5}$$

We do not write down this. In analogous way we may find the other quasiprobabilities in the representation by canonical variables.

The given quasiprobabilities can be used for the calculation of expectation values for displaced squeezed vacuum states or squeezed coherent states but in next Section we present an alternative for such calculations.

10. Calculation of Expectation Values for Displaced States from the Expectation Values of the Undisplaced States

An alternative for the calculation of expectation values of ordered powers of the annihilation and creation operator for displaced states from that for the undisplaced states is the following possibility presented here for normal ordering. Using the unitary displacement operator $D\left(\beta,\beta^*\right)$ one finds applied to displaced squeezed vacuum states

$$\begin{aligned}\left(\overline{a^{\dagger k}a^l}\right)_\beta &\equiv\langle\beta,\zeta|a^{\dagger k}a^l|\beta,\zeta\rangle\\ &=\langle 0,\zeta|\left(D(\beta,\beta^*)\right)^\dagger a^{\dagger k}a^l D\left(\beta,\beta^*\right)|0,\zeta\rangle\\ &=\langle 0,\zeta|\left(a^\dagger+\beta^* I\right)^k\underbrace{\left(D\left(\beta,\beta^*\right)\right)^\dagger D\left(\beta,\beta^*\right)}_{=I}\left(a+\beta I\right)^l|0,\zeta\rangle\end{aligned} \tag{10.1}$$

that after Taylor series expansion of the binomials can be written

$$\left(\overline{a^{\dagger k}a^l}\right)_\beta=\sum_{i=0}^{k}\sum_{j=0}^{l}\frac{k!l!}{i!(k-i)!j!(l-j)!}\beta^{*i}\beta^j\left(\overline{a^{\dagger k-i}a^{l-j}}\right)_0, \tag{10.2}$$

where index 0 at expectation values means the expectation values before the displacement and with index β after the displacement with the complex parameter β.

For squeezed vacuum states the expectation values $\left(\overline{a^{\dagger k}a^l}\right)_0$ are only non-vanishing if the difference $|k-l|$ is an even number. Taking this into account we find for displaced squeezed vacuum states the expectation values of the operators a and $a^\dagger$

$$\left(\overline{a}\right)_\beta = \beta\left(\overline{I}\right)_0 = \beta, \quad \left(\overline{a^\dagger}\right)_\beta = \beta^*\left(\overline{I}\right)_0 = \beta^*, \tag{10.3}$$

and for the expectation values of the operators a^2 and $a^{\dagger 2}$

$$\begin{aligned} \left(\overline{a^2}\right)_\beta &= \left(\overline{a^2}\right)_0 + \beta^2\left(\overline{I}\right)_0 = -\frac{\zeta}{1-\zeta\zeta^*} + \beta^2, \\ \left(\overline{a^{\dagger 2}}\right)_\beta &= \left(\overline{a^{\dagger 2}}\right)_0 + \beta^{*2}\left(\overline{I}\right)_0 = -\frac{\zeta^*}{1-\zeta\zeta^*} + \beta^{*2}. \end{aligned} \tag{10.4}$$

Next we calculate expectation values of operators which are important for the photon statistics. The expectation value of the number operator $N = a^\dagger a$ depends only on the squared moduli $|\zeta|^2 \equiv \zeta\zeta^*$ and $|\beta|^2 \equiv \beta\beta^*$ of the complex ζ and β and we find

$$\left(\overline{N}\right)_\beta = \left(\overline{a^\dagger a}\right)_\beta = \left(\overline{a^\dagger a}\right)_0 + \beta\beta^*\left(\overline{I}\right)_0 = \frac{\zeta\zeta^*}{1-\zeta\zeta^*} + \beta\beta^* = \frac{|\zeta|^2}{1-|\zeta|^2} + |\beta|^2. \tag{10.5}$$

For the discussion of sub- and super-Poissonian statistics of displaced squeezed vacuum states we need in addition the expectation value of the operator $a^{\dagger 2}a^2$. It does not only depend on the moduli of ζ and β but also on their phases and from (10.2) we find

$$\begin{aligned} \left(\overline{a^{\dagger 2}a^2}\right)_\beta &= \left(\overline{a^{\dagger 2}a^2}\right)_0 + 4\beta\beta^*\left(\overline{a^\dagger a}\right)_0 + \left(\beta\beta^*\right)^2\left(\overline{I}\right)_0 + \beta^2\left(\overline{a^{\dagger 2}}\right)_0 + \beta^{*2}\left(\overline{a^2}\right)_0 \\ &= \frac{\zeta\zeta^*\left(1+2\zeta\zeta^*\right)}{\left(1-\zeta\zeta^*\right)^2} + 4\beta\beta^*\frac{\zeta\zeta^*}{1-\zeta\zeta^*} + \left(\beta\beta^*\right)^2 \\ &\quad - \beta^2\frac{\zeta^*}{1-\zeta\zeta^*} - \beta^{*2}\frac{\zeta}{1-\zeta\zeta^*}. \end{aligned} \tag{10.6}$$

The expectation values for the squeezed vacuum states can be taken from Appendix E where they are collected. From this follows for the expectation value of the operator $N^2 = a^{\dagger 2}a^2 + a^\dagger a$

$$\left(\overline{N^2}\right)_\beta = \frac{\zeta\zeta^*\left(2+\zeta\zeta^*\right)}{\left(1-\zeta\zeta^*\right)^2} + \beta\beta^*\frac{1+3\zeta\zeta^*}{1-\zeta\zeta^*} + \left(\beta\beta^*\right)^2 - \beta^2\frac{\zeta^*}{1-\zeta\zeta^*} - \beta^{*2}\frac{\zeta}{1-\zeta\zeta^*}, \tag{10.7}$$

and for the variance of the number operator

$$\begin{aligned} \left(\overline{(\Delta N)^2}\right)_\beta &= \left(\overline{N^2}\right)_\beta - \left(\overline{N}\right)_\beta^2 \\ &= \frac{2\zeta\zeta^*}{\left(1-\zeta\zeta^*\right)^2} + \beta\beta^*\frac{1+\zeta\zeta^*}{1-\zeta\zeta^*} - \beta^2\frac{\zeta^*}{1-\zeta\zeta^*} - \beta^{*2}\frac{\zeta}{1-\zeta\zeta^*} \\ &= \frac{2\zeta\zeta^*}{\left(1-\zeta\zeta^*\right)^2} + \frac{\left(\beta-\zeta\beta^*\right)\left(\beta^*-\zeta^*\beta\right)}{1-\zeta\zeta^*} \ge 0. \end{aligned} \tag{10.8}$$

Finally we calculate the expectation value which plays a role for the definition of sub- and super-Poissonian statistics by its sign

$$\begin{aligned}
&\left(\overline{a^{\dagger 2}a^{2}}\right)_{\beta}-\left(\overline{a^{\dagger}a}\right)_{\beta}^{2}=\left(\overline{(\Delta N)^{2}}\right)_{\beta}-\left(\bar{N}\right)_{\beta}\\
&=\frac{\zeta\zeta^{*}\left(1+\zeta\zeta^{*}\right)}{\left(1-\zeta\zeta^{*}\right)^{2}}+2\beta\beta^{*}\frac{\zeta\zeta^{*}}{1-\zeta\zeta^{*}}-\beta^{2}\frac{\zeta^{*}}{1-\zeta\zeta^{*}}-\beta^{*2}\frac{\zeta}{1-\zeta\zeta^{*}}\\
&=\frac{\zeta\zeta^{*}\left(1+\zeta\zeta^{*}\right)}{\left(1-\zeta\zeta^{*}\right)^{2}}-\beta\beta^{*}+\frac{\left(\beta-\zeta\beta^{*}\right)\left(\beta^{*}-\zeta^{*}\beta\right)}{1-\zeta\zeta^{*}}.
\end{aligned}\tag{10.9}$$

The expectation values (10.6)-(10.9) depend not only on the moduli of β and ζ but also on their phases $\mathrm{e}^{\mathrm{i}\varphi}$ and $\mathrm{e}^{\mathrm{i}\chi}$. For comparison with (10.9) one finds for the corresponding symmetrically (Weyl) ordered quantity

$$\begin{aligned}
&\left(\overline{\mathcal{S}\left\{a^{\dagger 2}a^{2}\right\}}\right)_{\beta}-\left(\overline{\mathcal{S}\left\{a^{\dagger}a\right\}}\right)_{\beta}^{2}\\
&=\left(\overline{a^{\dagger 2}a^{2}}\right)_{\beta}-\left(\overline{a^{\dagger}a}\right)_{\beta}^{2}+\left(\overline{a^{\dagger}a}\right)_{\beta}+\frac{1}{4}\\
&=\frac{2\zeta\zeta^{*}}{\left(1-\zeta\zeta^{*}\right)^{2}}+\frac{\left(\beta-\zeta\beta^{*}\right)\left(\beta^{*}-\zeta^{*}\beta\right)}{1-\zeta\zeta^{*}}+\frac{1}{4}\geq\frac{1}{4}.
\end{aligned}\tag{10.10}$$

This quantity is greater than or equal to $1/4$ in every case but depends also on the phases of β and ζ.

11. Sub- and Super-Poissonian Photon Statistics for Displaced Squeezed Vacuum States or Squeezed Coherent States

The sign of the quantity (10.9) was taken by Mandel to define of sub- and super-Poissonian statistics as follows [11]

$$\begin{aligned}
&\overline{a^{\dagger 2}a^{2}}-\left(\overline{a^{\dagger}a}\right)^{2}=\overline{(\Delta N)^{2}}-\bar{N}<0,\quad(\text{sub-Poissonian}),\\
&\overline{a^{\dagger 2}a^{2}}-\left(\overline{a^{\dagger}a}\right)^{2}=\overline{(\Delta N)^{2}}-\bar{N}>0,\quad(\text{super-Poissonian}).
\end{aligned}\tag{11.1}$$

We now investigate the photon statistics of this quantity for displaced squeezed vacuum states starting from (10.9). First we denote the phases of β and ζ in the the complex plane as follows

$$\beta=|\beta|\mathrm{e}^{\mathrm{i}\varphi},\quad\zeta=|\zeta|\mathrm{e}^{\mathrm{i}\chi}.\tag{11.2}$$

The mean value $\bar{N}$ of the number operator N according to (10.5) does not depend on the angle between β and ζ and if we change only the angle φ of β leaving $|\beta|$ and ζ constant the mean value $\bar{N}$ remains constant and the squeezing ellipses change in their position in comparison to the displacement β. This is shown in **Figure 2**. In the following we show that the photon statistics may change from sub- to super-Poissonian statistics if the displacement β moves around the circles shown schematically in **Figure 2** and if $|\beta|$ is sufficiently large and somewhere between this has to be a position where it possesses the same values of $\bar{N}=\overline{a^{\dagger}a}$ and $\overline{a^{\dagger 2}a^{2}}-\left(\overline{a^{\dagger}a}\right)^{2}$ as the Poisson statistics of coherent states with the same $\bar{N}$.

From (10.9) and from definition (11.1) follows as condition for sub-Poissonian statistics of squeezed coherent states $|\beta,\zeta\rangle$

$$\frac{|\zeta|^{2}\left(1+|\zeta|^{2}\right)}{\left(1-|\zeta|^{2}\right)^{2}}-2|\beta|^{2}\frac{|\zeta|\left(\cos(\chi_{0})-|\zeta|\right)}{1-|\zeta|^{2}}<0,\quad\left(\chi_{0}\equiv\chi-2\varphi\right),\qquad(11.3)$$

or resolved to $|\beta|^{2}$ for $\cos(\chi_{0})-|\zeta|>0$ (pay attention that in case of $\cos(\chi_{0})-|\zeta|<0$ the inequality (11.3) cannot be satisfied or changes ">" into "<" if divided by $\cos(\chi_{0})-|\zeta|<0$!)

$$|\beta|^{2}>\frac{|\zeta|\left(1+|\zeta|^{2}\right)}{2\left(1-|\zeta|^{2}\right)\left(\cos(\chi_{0})-|\zeta|\right)}>0,\quad\left(\cos(\chi_{0})>|\zeta|\right).\qquad(11.4)$$

For possible sub-Poissonian statistics it is necessary that $\cos(\chi_{0})>|\zeta|$ whereas in case of $\cos(\chi_{0})<|\zeta|$ we have super-Poissonian statistics. For $\chi_{0}=0$ or $\cos(\chi_{0})-|\zeta|=1-|\zeta|>0$ that means for squeezing in direction of the displacement parameter β (amplitude squeezing, see **Figure 3**) the necessary condition for sub-Poissonian statistics is satisfied but according to (11.4) the modulus $|\beta|$ of the displacement parameter has to be greater than a minimal value defined by the equality sign in (11.4). For $\chi_{0}=\pi$ or $\cos(\chi_{0})-|\zeta|=-1-|\zeta|<0$ the left-hand side of (11.3) is positive and we have super-Poissonian statistics. Thus one of the necessary conditions for sub-Poissonian statistics is (see **Figure 4**, inner circle)

$$\cos(\chi_{0})-|\zeta|>0,\qquad(11.5)$$

with the limiting curve by substitution of ">" by the equality sign "=" in (11.5). In the other necessary condition (11.4) it is better to substitute the displacement parameter $|\beta|$ by the expectation value $\bar{N}$ of the number operator since it is then easier to compare the results with coherent states with the same $\bar{N}$.

We express now the displacement parameter β in (10.9) by the expectation value $\bar{N}$ of the number operator N using (10.5)

$$|\beta|^{2}=\bar{N}-\frac{|\zeta|^{2}}{1-|\zeta|^{2}}\geq 0,\qquad(11.6)$$

where we have omitted the index β in $\left(\bar{N}\right)_{\beta}$ since we use it in the following also for coherent states. If we do so then we obtain from (10.9) with abbreviation $\chi_{0}\equiv\chi-2\varphi$

$$\begin{aligned}\left(\overline{a^{\dagger 2}a^{2}}\right)_{\beta}-\left(\overline{a^{\dagger}a}\right)_{\beta}^{2}&=\frac{|\zeta|^{2}\left(1+|\zeta|^{2}\right)}{\left(1+|\zeta|^{2}\right)^{2}}-2\left(\bar{N}-\frac{|\zeta|^{2}}{1-|\zeta|^{2}}\right)\frac{|\zeta|\left(\cos(\chi_{0})-|\zeta|\right)}{1-|\zeta|^{2}}\\&=\frac{|\zeta|}{\left(1-|\zeta|^{2}\right)^{2}}\left\{|\zeta|\left(1+|\zeta|^{2}\right)-2\left(\bar{N}-\left(\bar{N}+1\right)|\zeta|^{2}\right)\left(\cos(\chi_{0})-|\zeta|\right)\right\}.\end{aligned}\qquad(11.7)$$

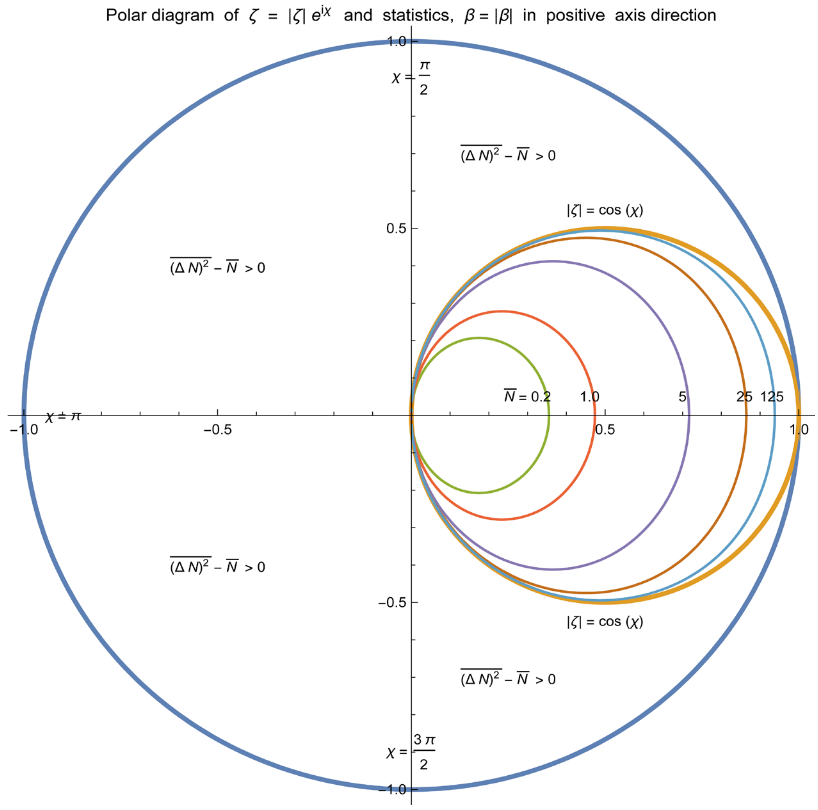

Figure 4. Polar diagram $|\zeta| = f(\chi)$ of sub- and super-Poissonian statistics in dependence on the complex squeezing parameter in non-unitary approach $\zeta = |\zeta| e^{i\chi}$ and with displacement parameter $\beta = |\beta| e^{i\varphi}$ with fixed $\varphi = 0$ (in positive axis direction) and therefore $\chi_0 \equiv \chi - 2\varphi = \chi$. Sub-Poissonian statistics is only possible within the inner circle $|\zeta| = \cos(\chi_0)$. Within this circle the ellipse-like curves with constant $\overline{N}$ separate regions of sub-Poissonian statistics from regions of super-Poissonian statistics where sub-Poissonian statistics is within the ellipse-like inner regions to the crescent-like outer regions with super-Poissonian statistics. The center $\zeta = 0$ (χ arbitrary) corresponds to coherent states of arbitrary $\overline{N}$. Constant $|\zeta|$ (circles around coordinate origin) correspond for fixed $\beta = |\beta|$ to constant $\overline{N}$ touching the separatrices with the same $\overline{N}$ at $\chi = 0$. Using the squeezing parameter ζ' in the unitary approach (see (5) and (6)) this scheme would take on the whole complex plane and, in particular, for parameters ζ with $|\zeta| \to 1$ one has a large stretching of corresponding parameters ζ' with $|\zeta'| \to \infty$ and the inner circle $|\zeta| = \cos(\chi)$ becomes $|\zeta'| = \mathrm{Arth}(\cos(\chi))$.

Setting this expression equal to zero one obtains an equation for states which belong neither to sub- nor to super-Poissonian statistics. This is a third-order equation for $|\zeta|$ in dependence on $\overline{N}$ and on the angle χ_0 as follows

$$|\zeta|^3 - \frac{2(\overline{N}+1)\cos(\chi_0)}{2\overline{N}+1}|\zeta|^2 - |\zeta| + \frac{2\overline{N}\cos(\chi_0)}{2\overline{N}+1} = 0. \tag{11.8}$$

As a third-order equation for $|\zeta|$ with real coefficients in dependence on $\overline{N}$ and χ_0 it may possess, in principal, three real or one real and two complex conjugate solutions but for our purpose the real solutions have to be positive ones and have to be restricted to $|\zeta|<1$. The results are presented in **Figure 4** as polar diagram of $\zeta=|\zeta|\mathrm{e}^{\mathrm{i}\chi}$. The ellipse-like regions of the inner smaller circle (11.5) in **Figure 4** belong to sub-Poissonian and the outer crescent-like regions to super-Poissonian statistics in dependence on $\bar{N}$ shown as separatrices.

Figure 5 shows that between sub- and super-Poissonian statistics may lie statistics which are very far from a Poisson statistics, in case of $|\zeta|\to 1$ even maximally far (distance d to nearest coherent states goes to $\sqrt{2}$). Alternatively to **Figure 5** one may demonstrate the existence of displaced squeezed vacuum states with $\overline{(\Delta N)^2}-\bar{N}=0$ also in the following way. One begins with an arbitrarily chosen displacement parameter ζ and looks according to condition (11.4) for displacement parameters β above the minimal possible one for

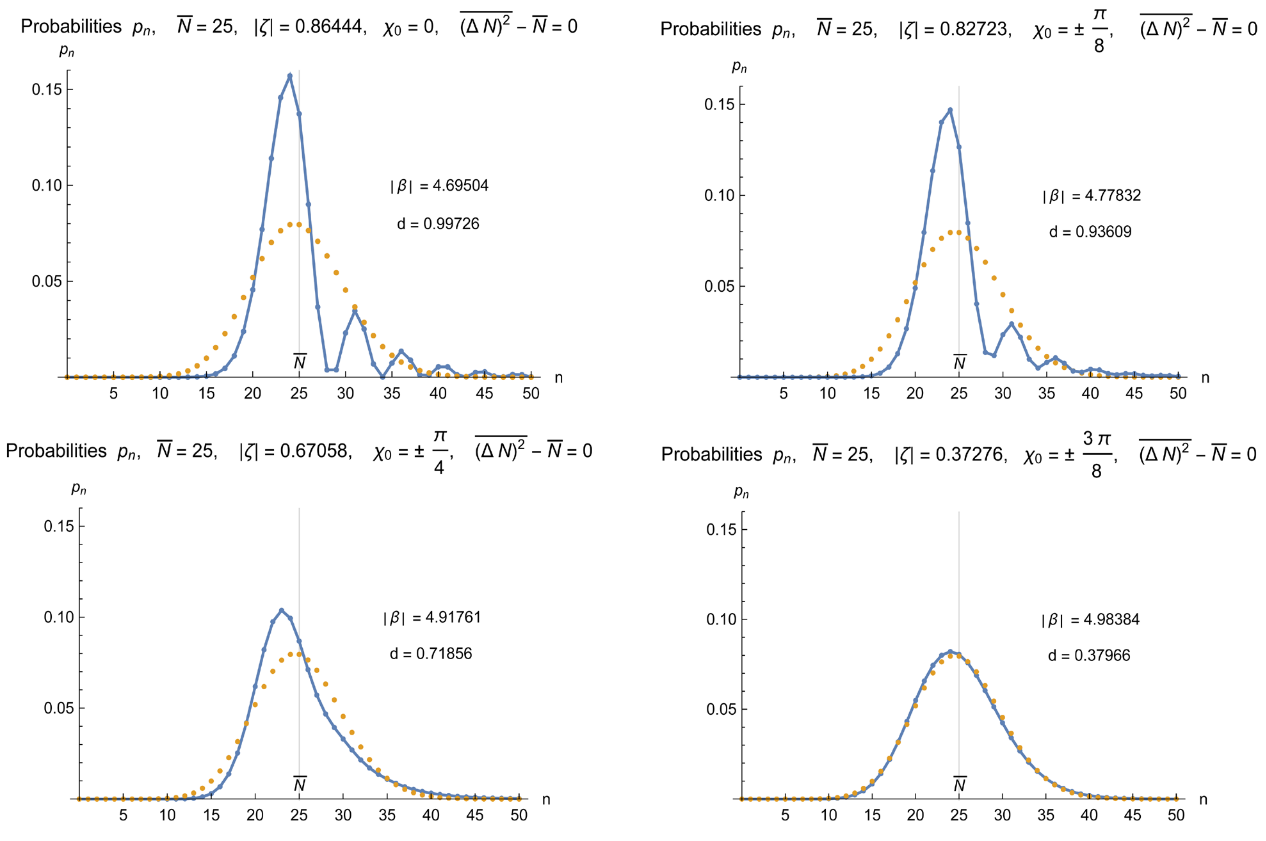

Figure 5. Probabilities p_n for squeezed coherent states (joined points) with $\bar{N}=25$ and $\overline{(\Delta N)^2}-\bar{N}=0$ such as for a coherent state (orange points) with $\bar{N}=25$ and nearest distance d to coherent states $|\beta\rangle$ with $\bar{N}=\beta\beta^*$ or $\bar{a}=\beta$. Additionally, the displacement parameter $|\beta|$ is given. The last partial figure with small squeezing parameter $|\zeta|$ is in its photon statistics already "visibly" near to a coherent state with $\bar{N}=25$ but the nearest distance to a coherent state (with $\bar{N}=\beta\beta^*=24.8387$) is not yet very small and shows that this measure is very sensible. All four partial figures belong to states with photon statistics which is neither sub- nor super-Poissonian but also not a Poisson statistics. For $|\zeta|=0$, $\zeta=0,\chi=\chi_0=\pm\frac{\pi}{2}$ (but, in principle, arbitrary) one has exactly a coherent state (see also **Figure 4**, center of diagram).

which one has sub-Poissonian statistics if β possesses the same or the opposite direction of ζ (amplitude squeezing). Then by rotating the phase $\mathrm{e}^{\mathrm{i}\chi}$ of ζ from $\chi = 0$ (amplitude squeezing) up to $\chi = \pi$ (phase squeezing) leaving constant the modulus $|\zeta|$ (see **Figure 1**, right partial figure) the photon statistics makes the transition from sub- to super-Poissonian statistics and one comes unavoidable to a value of the phase χ where $(\Delta N)^2 - \bar{N} = 0$ but which does not belong to Poisson statistics. The same effect one has if we rotate the displacement parameter β leaving constant the squeezing parameter ζ in modulus and phase (see **Figure 3**). If the time evolution of a squeezed state is determined by a Hamiltonian H which is a linear combination of $a^2, a^{\dagger 2}$ and of $\frac{1}{2}\left(aa^\dagger + a^\dagger a\right)$ then there may appear a complicated picture of changing with time from sub- to super-Poissonian statistics or from amplitude to phase squeezing since then also the modulus of the squeezing parameter changes with time.

Clearly, one may make the division of photon statistics in sub- and super-Poissonian ones but both categories are very inhomogeneous concerning the comprised states and the set of states which are neither sub- nor super-Poissonian ones is also very large and inhomogeneous and the prefixes "sub" and "super" are here problematic. There are hardly to expect clear differences and correlations in experiments with states of both statistics or, moreover, even qualitatively different behavior. In general, a photon statistics is determined by a countable infinite number of parameters (e.g., p_n or moments of the distribution) and for states which belong neither to sub- nor to super-Poissonian statistics only one from this countable infinite set is fixed ($(\Delta N)^2 - \bar{N} = 0$) and this can be considered in dependence on arbitrary $\bar{N}$. Therefore, also the classification of states with sub-Poissonian photon statistics as nonclassical states is highly problematic. In the same way it is also problematic to define the states with no regions of negativity of the Wigner quasiprobability as the classical states since then the set of these states is too large and inhomogeneous to be useful for comparative purpose (all squeezed coherent states belong then to them). Better seems to be for this purpose to use the nearest distance to a coherent state as quantitative measure but this measure cannot change its sign and is in every case positive or zero [20] [21]. In this definition it is evident that the category of states with large distance to the nearest coherent states is very inhomogeneous and large.

12. Conclusions

A main purpose of this article was to discuss the distinction of cases of sub- and super-Poissonian statistics within the displaced squeezed vacuum states where the non-unitary approach is preferable. For this case it was necessary to calculate the expectation values of powers of the number operator for these states. We have chosen for this purpose mainly its calculation from the number-state representation and posed this in a more general connection to the calculation of properties of these states. For squeezed coherent states, practically, all interesting parameters can be calculated in exact and not very difficult way and, therefore, this category of states is very suited to demonstrate in examples more principal definitions for all states.

The means developed in this article can be applied without substantial changes to squeezing of the number state $|1\rangle$ and then to its displacement since the operator $\exp\left(-\frac{\zeta^*}{2}a^2\right)$ in its disentanglement (2.7)) acts on $|1\rangle$ by reproducing it (case $k=\frac{3}{4}$ in (A.4)). This becomes more complicated if we apply it to the number states $|n\rangle$ with $n \geq 2$ and to extend the theory to these cases.

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Appendix A

Squeezing Operators and Their Disentanglement

Squeezing of states in the narrow sense is connected with the Lie group $SU(1,1) \cong Sp(2,\mathbb{R}) \cong SL(2,\mathbb{R})$ with 3 real parameters (or one complex and one real in our representation) of squeezing operators [7] [8] [9]. Its complex extension $Sp(2,\mathbb{C}) \cong SL(2,\mathbb{C})$ contains 3 complex parameters (or 6 real ones) and comprises also the $SU(2)$ group in the right specialization. In this Appendix we give a short but general and systematic representation of the action of $SU(1,1)$ squeezing operators in quantum optics. We cannot take care without making too clumsy notations in some cases to the notations in the main text but this concerns mostly the notations of the squeezing parameters with and without prime (or in non-unitary and unitary approaches) and one must pay attention to this when going from formulae of this Appendix to that of the main text.

We define operators $S(\xi,\eta,\zeta)$ with 3 complex parameters which are operators in $SL(2,\mathbb{C})$ as follows

$$S(\xi,\eta,\zeta) \equiv \exp\{\xi K_- + \mathrm{i}2\eta K_0 - \zeta K_+\} \tag{A.1}$$

where (K_-, K_0, K_+) are three abstract operators of the Lie algebra $sl(2,\mathbb{C})$ to the Lie group $SL(2,\mathbb{C})$ satisfying the commutation relation [7] [8] [9] [34]

$$[K_-, K_+] = 2K_0, \quad [K_0, K_-] = -K_-, \quad [K_0, K_+] = +K_+, \tag{A.2}$$

with Casimir operator C

$$C \equiv K_0^2 - K_1^2 - K_2^2 = K_0^2 - \frac{1}{2}(K_- K_+ + K_+ K_-), \quad [C, K_i] = 0. \tag{A.3}$$

One basic discrete realization of these operators in quantum optics of a single mode is

$$\begin{aligned}
K_- &\equiv \frac{1}{2}a^2 = \frac{1}{4\hbar}\left(Q^2 - P^2 + \mathrm{i}(QP + PQ)\right),\\
K_0 &\equiv \frac{1}{4}\left(aa^\dagger + a^\dagger a\right) = \frac{1}{4\hbar}\left(Q^2 + P^2\right),\\
K_+ &\equiv \frac{1}{2}a^{\dagger 2} = \frac{1}{4\hbar}\left(Q^2 - P^2 - \mathrm{i}(QP + PQ)\right),\\
C &\equiv k(k-1)I = -\frac{3}{16}, \quad k = \frac{1}{4}, \quad \left(\text{or } k = \frac{3}{4}\right),
\end{aligned} \tag{A.4}$$

where $(a, a^\dagger)$ is a pair of boson annihilation and creation operators (A.1) and $k = \frac{3}{4}$ belongs to squeezed number states $|1\rangle$ (and their displacement) not dealt with in this article. Each realization of the commutation relations (A.2) is appropriate for the following derivations but for the two-dimensional fundamental representation it is important to know two basic operators for which $(a, a^\dagger)$ are particularly convenient in combination with the realization (A.4). Squeezing with $SU(1,1)$ operators within two modes and genuinely different from single-mode squeezing is also possible [8] [15]. A further realization of $SU(1,1)$ for a single mode is connected with coherent phase states.

The transition from $SL(2,\mathbb{C})$ to $SL(2,\mathbb{R})\cong SU(1,1)$ can be made by specializing the parameters (ξ,η,ζ) in (A.1) according to

$$\begin{aligned} S\left(\zeta^*,\eta=\eta^*,\zeta\right) &= \exp\left\{\frac{\zeta^*}{2}a^2+\mathrm{i}\frac{\eta}{2}\left(aa^\dagger+a^\dagger a\right)-\frac{\zeta}{2}a^{\dagger 2}\right\} \\ &= \left(S\left(-\zeta^*,-\eta=-\eta^*,-\zeta\right)\right)^\dagger \end{aligned} \tag{A.5}$$

with complex parameter ζ and real parameter η. The operators $S\left(\zeta^*,\eta=\eta^*,\zeta\right)$ are then unitary operators $S^{-1}=S^\dagger$ in the infinite-dimensional unitary representation in Hilbert space and we have $\left(K_-^\dagger,K_0^\dagger,K_+^\dagger\right)=\left(K_+,K_0,K_-\right)$. Instead of (K_-,K_+) one may introduce Hermitean operators (K_1,K_2) by $K_\mp\equiv K_1\mp\mathrm{i}K_2$ but we do not write down all relations for these new operators that is easy to make and is diligent work.

We now consider the generation of the two-dimensional fundamental representation of the group $SL(2,\mathbb{C})$ by calculating the matrix with elements (κ,λ,μ,ν) in the following relation

$$S(\xi,\eta,\zeta)\left(a,a^\dagger\right)\left(S(\xi,\eta,\zeta)\right)^{-1}=\left(a,a^\dagger\right)\begin{pmatrix}\kappa & \lambda\\ \mu & \nu\end{pmatrix}=\left(\kappa a+\mu a^\dagger,\lambda a+\nu a^\dagger\right). \tag{A.6}$$

This step is essentially a Bogolyubov transformation. It can be represented by (e.g., [18])

$$\begin{aligned} \begin{pmatrix}\kappa & \lambda\\ \mu & \nu\end{pmatrix} &= \begin{pmatrix}\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon} & \xi\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\\ \zeta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon} & \mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\end{pmatrix}. \\ \varepsilon &\equiv \sqrt{\xi\zeta-\eta^2}. \end{aligned} \tag{A.7}$$

For $\varepsilon^2>0$, (ε real) we call this transformation squeezing-like and for $\varepsilon^2<0$, (ε imaginary) rotation-like. In last case it is better to write the hyperbolic functions by trigonometric functions. The special case $\varepsilon=0$

$$\begin{pmatrix}\kappa & \lambda\\ \mu & \nu\end{pmatrix}=\begin{pmatrix}1-\mathrm{i}\eta, & \xi\\ \zeta, & 1+\mathrm{i}\eta\end{pmatrix},\quad \xi\zeta=\eta^2, \tag{A.8}$$

could be called cone-like. The matrices (A.7) are unimodular

$$\left|\begin{pmatrix}\kappa & \lambda\\ \mu & \nu\end{pmatrix}\right|=\kappa\nu-\lambda\mu=1, \tag{A.9}$$

but in general, not unitary.

The inversion of (A.7) which is unique can be simply written

$$\xi=\lambda\frac{\mathrm{Arsh}(\vartheta)}{\vartheta},\quad \eta=\mathrm{i}\frac{\kappa-\nu}{2}\frac{\mathrm{Arsh}(\vartheta)}{\vartheta},\quad \zeta=\mu\frac{\mathrm{Arsh}(\vartheta)}{\vartheta}, \tag{A.10}$$

with the following relations between the abbreviations ε and ϑ

$$\varepsilon = \operatorname{Arch}\left(\frac{\kappa+\nu}{2}\right) = \operatorname{Arsh}\left(\sqrt{\left(\frac{\kappa+\nu}{2}\right)^2 - 1}\right) = \operatorname{Arsh}\left(\sqrt{\left(\frac{\kappa-\nu}{2}\right)^2 + \lambda\mu}\right) \equiv \operatorname{Arsh}(\vartheta),$$

$$\vartheta \equiv \operatorname{sh}(\varepsilon) = \sqrt{\left(\frac{\kappa+\nu}{2}\right)^2 - 1} = \sqrt{\left(\frac{\kappa-\nu}{2}\right)^2 + \lambda\mu}. \tag{11}$$

From $\operatorname{Arch}(z)$ the sign of $\operatorname{Arsh}\left(\sqrt{z^2-1}\right)$ does not follow uniquely but $\frac{\operatorname{Arsh}\left(\sqrt{z^2-1}\right)}{\sqrt{z^2-1}}$ is then uniquely determined if one chooses the same sign of $\sqrt{z^2-1}$ in numerator and denominator.

We now write down some special correspondences between the operators $S(\xi,\eta,\zeta)$ and its two-dimensional matrices in the fundamental representation

$$\begin{aligned}
S(\xi,0,0) &= \exp(\xi K_-) \rightarrow \exp\left(\frac{\xi}{2}a^2\right), \quad \Leftrightarrow \quad \begin{pmatrix} 1, & \xi \\ 0, & 1 \end{pmatrix}, \\
S(0,\eta,0) &= \exp(\mathrm{i}\eta 2K_0) \rightarrow \exp\left\{\mathrm{i}\frac{\eta}{2}\left(aa^\dagger + a^\dagger a\right)\right\} \quad \Leftrightarrow \quad \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\eta}, & 0 \\ 0, & \mathrm{e}^{\mathrm{i}\eta} \end{pmatrix}, \\
S(0,0,\zeta) &= \exp(-\zeta K_+) \rightarrow \exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right), \quad \Leftrightarrow \quad \begin{pmatrix} 1, & 0 \\ \zeta, & 1 \end{pmatrix}.
\end{aligned} \tag{A.12}$$

The triangular operators $S(\xi,0,0)$ and $S(0,0,\zeta)$ and their correspondent two-dimensional matrices form a group for themselves and in the same way the operators $S(0,\eta,0)$ together with its matrices. However, the operators $S(\xi,0,\zeta)$ do not form a group its extension to a group needs all operators $S(\xi,\eta,\zeta)$.

We may decompose the matrices in (A.7) into products of special matrices, in particular, in the following for us important ways

$$\begin{aligned}
\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \frac{\mu}{\kappa} & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda\kappa \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \kappa & 0 \\ 0 & \frac{1}{\kappa} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \frac{\mu}{\kappa} & 1 \end{pmatrix}\begin{pmatrix} \kappa & 0 \\ 0 & \frac{1}{\kappa} \end{pmatrix}\begin{pmatrix} 1 & \frac{\lambda}{\kappa} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \kappa & 0 \\ 0 & \frac{1}{\kappa} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mu\kappa & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{\lambda}{\kappa} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\nu} & 0 \\ 0 & \nu \end{pmatrix}\begin{pmatrix} 1 & \lambda\nu \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\mu}{\nu} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{\lambda}{\nu} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\nu} & 0 \\ 0 & \nu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\mu}{\nu} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{\lambda}{\nu} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mu\nu & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\nu} & 0 \\ 0 & \nu \end{pmatrix}, \quad \kappa\nu - \lambda\mu = 1.
\end{aligned} \tag{A.13}$$

As consequences we obtain the following disentanglements of the general operator $S(\xi,\eta,\zeta)$ into products of partial operators

$$\begin{aligned} S(\xi,\eta,\zeta) &\equiv \exp(\xi K_- + \mathrm{i}2\eta K_0 - \zeta K_+) \\ &= \exp\left(-\frac{\mu}{\kappa}K_+\right)\exp(\lambda\kappa K_-)\exp(-2\log(\kappa)K_0) \\ &= \exp\left(-\frac{\mu}{\kappa}K_+\right)\exp(-2\log(\kappa)K_0)\exp\left(\frac{\lambda}{\kappa}K_-\right) \\ &= \exp(-2\log(\kappa)K_0)\exp(-\mu\kappa K_+)\exp\left(\frac{\lambda}{\kappa}K_-\right) \\ &= \exp(2\log(\nu)K_0)\exp(\lambda\nu K_-)\exp\left(-\frac{\mu}{\nu}K_+\right) \\ &= \exp\left(\frac{\lambda}{\nu}K_-\right)\exp(2\log(\nu)K_0)\exp\left(-\frac{\mu}{\nu}K_+\right) \\ &= \exp\left(\frac{\lambda}{\nu}K_-\right)\exp(-\mu\nu K_+)\exp(2\log(\nu)K_0). \end{aligned} \tag{A.14}$$

For the operators (K_-,K_0,K_+) we may insert the realization (A.4) of $SU(1,1)$ but every other realization of $SL(2,\mathbb{C})$ is also appropriate.

In the special case $\eta = 0$ of operators $S(\xi,\eta,\zeta)$ the matrices (A.7) specialize

$$S(\xi,0,\zeta) \quad \Leftrightarrow \quad \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\left(\sqrt{\xi\zeta}\right) & \xi\dfrac{\mathrm{sh}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}} \\ \zeta\dfrac{\mathrm{sh}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}} & \mathrm{ch}\left(\sqrt{\xi\zeta}\right) \end{pmatrix}, \tag{A.15}$$

that leads to the disentanglement relations

$$\begin{aligned} S(\xi,0,\zeta) &= \exp(\xi K_- - \zeta K_+) \\ &= \exp\left(-\zeta\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_+\right)\exp\left(\xi\frac{\mathrm{sh}\left(2\sqrt{\xi\zeta}\right)}{2\sqrt{\xi\zeta}}K_-\right)\left(\mathrm{ch}\left(\sqrt{\xi\zeta}\right)\right)^{-2K_0} \\ &= \exp\left(-\zeta\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_+\right)\left(\mathrm{ch}\left(\sqrt{\xi\zeta}\right)\right)^{-2K_0}\exp\left(\xi\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_-\right) \\ &= \left(\mathrm{ch}\left(\sqrt{\xi\zeta}\right)\right)^{-2K_0}\exp\left(-\zeta\frac{\mathrm{sh}\left(2\sqrt{\xi\zeta}\right)}{2\sqrt{\xi\zeta}}K_+\right)\exp\left(\xi\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_-\right) \\ &= \left(\mathrm{ch}\left(\sqrt{\xi\zeta}\right)\right)^{2K_0}\exp\left(\xi\frac{\mathrm{sh}\left(2\sqrt{\xi\zeta}\right)}{2\sqrt{\xi\zeta}}K_-\right)\exp\left(-\zeta\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_+\right) \\ &= \exp\left(\xi\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_-\right)\left(\mathrm{ch}\left(\sqrt{\xi\zeta}\right)\right)^{2K_0}\exp\left(-\zeta\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_+\right) \\ &= \exp\left(\xi\frac{\mathrm{th}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}K_-\right)\exp\left(-\zeta\frac{\mathrm{sh}\left(2\sqrt{\xi\zeta}\right)}{2\sqrt{\xi\zeta}}K_+\right)\left(\mathrm{ch}\left(\sqrt{\xi\zeta}\right)\right)^{2K_0}, \end{aligned} \tag{A.16}$$

where again we may insert the realization (A.4). In addition to the operators K_- and K_+ we have involved then on the right-hand side also the operator K_0.

If we make in (A.15) the transition to the variables in the non-unitary approach

$$S(\xi,0,\zeta)\to S\left(\frac{\mathrm{Arth}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}\xi,0,\frac{\mathrm{Arth}\left(\sqrt{\xi\zeta}\right)}{\sqrt{\xi\zeta}}\zeta\right), \tag{A.17}$$

then the matrix in (A.15) makes the transition to

$$\begin{pmatrix}\kappa, & \lambda\\ \mu, & \nu\end{pmatrix}\to\begin{pmatrix}\frac{1}{\sqrt{1-\xi\zeta}} & \frac{\xi}{\sqrt{1-\xi\zeta}}\\ \frac{\zeta}{\sqrt{1-\xi\zeta}} & \frac{1}{\sqrt{1-\xi\zeta}}\end{pmatrix}, \tag{A.18}$$

as it is easily to see.

The next considerations are the effort to split the general operator $S(\xi,\eta,\zeta)$ into products of a proper squeezing operator $S(\xi',0,\zeta')$ and a proper rotation operator $S(0,\eta',0)$. The general two-dimensional matrix (A.7) can be split also in the following two ways

$$\begin{aligned}\begin{pmatrix}\kappa, & \lambda\\ \mu, & \nu\end{pmatrix}&=\begin{pmatrix}\sqrt{\kappa\nu} & \lambda\sqrt{\frac{\kappa}{\nu}}\\ \mu\sqrt{\frac{\nu}{\kappa}} & \sqrt{\kappa\nu}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\kappa}{\nu}} & 0\\ 0 & \sqrt{\frac{\nu}{\kappa}}\end{pmatrix}\\ &=\begin{pmatrix}\sqrt{\frac{\kappa}{\nu}} & 0\\ 0 & \sqrt{\frac{\nu}{\kappa}}\end{pmatrix}\begin{pmatrix}\sqrt{\kappa\nu} & \lambda\sqrt{\frac{\nu}{\kappa}}\\ \mu\sqrt{\frac{\kappa}{\nu}} & \sqrt{\kappa\nu}\end{pmatrix}.\end{aligned} \tag{A.19}$$

This corresponds explicitly to the two possibilities ($\varepsilon\equiv\sqrt{\xi\zeta-\eta^2}$)

$$\begin{aligned}=\begin{pmatrix}\kappa & \lambda\\ \mu & \nu\end{pmatrix}&=\begin{pmatrix}\sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\frac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}} & \xi\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}\sqrt{\frac{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}}\\ \zeta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}\sqrt{\frac{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} & \sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\frac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}}\end{pmatrix}\\ &\times\begin{pmatrix}\sqrt{\frac{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} & 0\\ 0 & \sqrt{\frac{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}}}\end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \sqrt{\dfrac{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} & 0 \\ 0 & \sqrt{\dfrac{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} \end{pmatrix} \\
&\times \begin{pmatrix} \sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}} & \xi\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\sqrt{\dfrac{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} \\ \zeta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\sqrt{\dfrac{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} & \sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}} \end{pmatrix}.
\end{aligned}
\tag{A.20}
$$

and if we introduce abbreviations

$$
\begin{aligned}
\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} &= \begin{pmatrix} \mathrm{ch}\left(\sqrt{\xi'\zeta'}\right) & \xi'\dfrac{\mathrm{sh}\left(\sqrt{\zeta'\zeta'}\right)}{\sqrt{\xi'\zeta'}} \\ \zeta'\dfrac{\mathrm{sh}\left(\sqrt{\zeta'\zeta'}\right)}{\sqrt{\xi'\zeta'}} & \mathrm{ch}\left(\sqrt{\xi'\zeta'}\right) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\eta_0} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\eta_0} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\eta_0} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\eta_0} \end{pmatrix} \begin{pmatrix} \mathrm{ch}\left(\sqrt{\xi''\zeta''}\right) & \xi''\dfrac{\mathrm{sh}\left(\sqrt{\zeta''\zeta''}\right)}{\sqrt{\xi''\zeta''}} \\ \zeta''\dfrac{\mathrm{sh}\left(\sqrt{\zeta''\zeta''}\right)}{\sqrt{\xi''\zeta''}} & ch\left(\sqrt{\xi''\zeta''}\right) \end{pmatrix}.
\end{aligned}
\tag{A.21}
$$

The rotation operators are the same in both cases and the squeezing operators distinguish themselves only by phase factors in the non-diagonal matrix elements. We have

$$
\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\eta_0} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\eta_0} \end{pmatrix}, \quad \Leftrightarrow \quad S(0,\eta_0,0) = \mathrm{e}^{\mathrm{i}\frac{\eta_0}{2}\left(aa^\dagger + a^\dagger a\right)}. \tag{A.22}
$$

The first possibility of the splitting in a squeezing and a rotation part is

$$
\begin{aligned}
\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} &= \begin{pmatrix} \mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon} & \xi\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon} \\ \zeta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon} & \mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}} & \xi\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\sqrt{\dfrac{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} \\ \zeta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\sqrt{\dfrac{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} & \sqrt{\mathrm{ch}^2(\zeta)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\times\begin{pmatrix} \sqrt{\dfrac{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} & 0 \\ 0 & \sqrt{\dfrac{\mathrm{ch}(\varepsilon)+\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)-\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{ch}\left(\sqrt{\xi'\zeta'}\right), & \xi'\dfrac{\mathrm{sh}\left(\sqrt{\zeta'\zeta'}\right)}{\sqrt{\xi'\zeta'}} \\ \zeta'\dfrac{\mathrm{sh}\left(\sqrt{\zeta'\zeta'}\right)}{\sqrt{\xi'\zeta'}}, & \mathrm{ch}\left(\sqrt{\xi'\zeta'}\right) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\eta_0}, & 0 \\ 0, & \mathrm{e}^{\mathrm{i}\eta_0} \end{pmatrix} \qquad \text{(A.23)} \\
&= \begin{pmatrix} \sqrt{\kappa\nu} & \sqrt{\dfrac{\kappa}{\nu}}\lambda \\ \sqrt{\dfrac{\nu}{\kappa}}\mu & \sqrt{\kappa\nu} \end{pmatrix} \begin{pmatrix} \sqrt{\dfrac{\kappa}{\nu}} & 0 \\ 0 & \sqrt{\dfrac{\nu}{\kappa}} \end{pmatrix},
\end{aligned}
$$

with correspondences

$$
\mathrm{ch}\left(\sqrt{\xi'\zeta'}\right) = \sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\frac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}} = \sqrt{1+\xi\zeta\frac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}},
$$

$$
\mathrm{sh}\left(\sqrt{\xi'\zeta'}\right) = \sqrt{\xi\zeta}\,\frac{\mathrm{sh}(\varepsilon)}{\varepsilon}, \quad \mathrm{th}\left(\sqrt{\xi'\zeta'}\right) = \frac{\sqrt{\xi\zeta}\,\mathrm{sh}(\varepsilon)}{\sqrt{1+\xi\zeta\,\mathrm{sh}^2(\varepsilon)}},
$$

$$
\frac{\mathrm{sh}\left(\sqrt{\xi'\zeta'}\right)}{\sqrt{\xi'\zeta'}} = \frac{\sqrt{\xi\zeta}\,\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{Arsh}\left(\sqrt{\xi\zeta}\,\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\right)},
$$

$$
\mathrm{e}^{\pm\mathrm{i}\eta_0} = \sqrt{\frac{\mathrm{ch}(\varepsilon)\pm\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\mathrm{ch}(\varepsilon)\mp\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}} = \frac{\mathrm{ch}(\varepsilon)\pm\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}}} = \frac{\sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}}}{\mathrm{ch}(\varepsilon)\mp\mathrm{i}\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}},
$$

$$
\mathrm{tg}(\eta_0) = \eta\frac{\mathrm{th}(\varepsilon)}{\varepsilon}, \quad \cos(\eta_0) = \frac{\mathrm{ch}(\varepsilon)}{\sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}}},
$$

$$
\sin(\eta_0) = \frac{\eta\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}{\sqrt{\mathrm{ch}^2(\varepsilon)+\eta^2\dfrac{\mathrm{sh}^2(\varepsilon)}{\varepsilon^2}}}
$$

$$
\xi' = \frac{\mathrm{Arsh}\left(\sqrt{\xi\zeta}\,\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\right)}{\sqrt{\xi\zeta}\,\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}\,\xi\mathrm{e}^{-\mathrm{i}\eta}, \quad \zeta' = \frac{\mathrm{Arsh}\left(\sqrt{\xi\zeta}\,\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}\right)}{\sqrt{\xi\zeta}\,\dfrac{\mathrm{sh}(\varepsilon)}{\varepsilon}}\,\zeta\mathrm{e}^{\mathrm{i}\eta}. \qquad \text{(A.24)}
$$

The parameters in the factorized matrices are (κ,λ,μ,ν are such as in (A.7))

$$
\begin{aligned}
&S(\xi,\eta,\zeta)\\
&=S\left(\frac{\operatorname{Arsh}\left(\sqrt{\lambda\mu}\right)}{\sqrt{\lambda\mu}}\sqrt{\frac{\kappa}{\nu}}\lambda,0,\frac{\operatorname{Arsh}\left(\sqrt{\lambda\mu}\right)}{\sqrt{\lambda\mu}}\sqrt{\frac{\nu}{\kappa}}\mu\right)S\left(0,\mathrm{i}\operatorname{Arsh}\left(\frac{1}{2}\left(\sqrt{\frac{\kappa}{\nu}}-\sqrt{\frac{\nu}{\kappa}}\right)\right),0\right)\\
&=S\left(\frac{\operatorname{Arsh}\left(\sqrt{\xi\zeta}\frac{sh(\varepsilon)}{\varepsilon}\right)}{\sqrt{\xi\zeta}}\frac{\operatorname{ch}(\varepsilon)-\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}{\sqrt{\operatorname{ch}^2(\varepsilon)+\eta^2\frac{\operatorname{sh}^2(\varepsilon)}{\varepsilon^2}}}\xi,0,\right.\\
&\qquad\left.\frac{\operatorname{Arsh}\left(\sqrt{\xi\zeta}\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}\right)}{\sqrt{\xi\zeta}}\frac{\operatorname{ch}(\varepsilon)+\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}{\sqrt{\operatorname{ch}^2(\varepsilon)+\eta^2\frac{\operatorname{sh}^2(\varepsilon)}{\varepsilon^2}}}\zeta\right)\\
&\quad\cdot S\left(0,\mathrm{i}\operatorname{Arsh}\left(-\frac{\mathrm{i}\eta\operatorname{sh}(\varepsilon)}{\sqrt{\varepsilon^2+\xi\zeta\operatorname{sh}^2(\varepsilon)}}\right),0\right).
\end{aligned}
\tag{A.25}
$$

The second possibility with interchanged order of the splitting in a squeezing and a rotation part is

$$
\begin{aligned}
\begin{pmatrix}\kappa & \lambda\\ \mu & \nu\end{pmatrix}
&=\begin{pmatrix}\operatorname{ch}(\varepsilon)-\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon} & \xi\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}\\ \zeta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon} & \operatorname{ch}(\varepsilon)+\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}\end{pmatrix}\\
&=\begin{pmatrix}\sqrt{\dfrac{\operatorname{ch}(\varepsilon)-\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}{\operatorname{ch}(\varepsilon)+\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}} & 0\\ 0 & \sqrt{\dfrac{\operatorname{ch}(\varepsilon)+\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}{\operatorname{ch}(\varepsilon)-\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}}\end{pmatrix}\\
&\quad\times\begin{pmatrix}\sqrt{\operatorname{ch}^2(\varepsilon)+\eta^2\frac{\operatorname{sh}^2(\varepsilon)}{\varepsilon^2}} & \xi\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}\sqrt{\dfrac{\operatorname{ch}(\varepsilon)+\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}{\operatorname{ch}(\varepsilon)-\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}}\\ \zeta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}\sqrt{\dfrac{\operatorname{ch}(\varepsilon)-\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}{\operatorname{ch}(\varepsilon)+\mathrm{i}\eta\frac{\operatorname{sh}(\varepsilon)}{\varepsilon}}} & \sqrt{\operatorname{ch}^2(\varepsilon)+\eta^2\frac{\operatorname{sh}^2(\varepsilon)}{\varepsilon^2}}\end{pmatrix}\\
&=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\eta_0} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\eta_0}\end{pmatrix}\begin{pmatrix}\operatorname{ch}\left(\sqrt{\xi''\zeta''}\right) & \xi''\frac{\operatorname{sh}\left(\sqrt{\zeta''\zeta''}\right)}{\sqrt{\xi''\zeta''}}\\ \zeta''\frac{\operatorname{sh}\left(\sqrt{\zeta''\zeta''}\right)}{\sqrt{\xi''\zeta''}} & ch\left(\sqrt{\xi''\zeta''}\right)\end{pmatrix}
\end{aligned}
$$

$$
= \begin{pmatrix} \sqrt{\frac{\kappa}{\nu}} & 0 \\ 0 & \sqrt{\frac{\nu}{\kappa}} \end{pmatrix} \begin{pmatrix} \sqrt{\kappa\nu} & \sqrt{\frac{\nu}{\kappa}}\lambda \\ \sqrt{\frac{\kappa}{\nu}}\mu & \sqrt{\kappa\nu} \end{pmatrix}. \tag{A.26}
$$

The diagonal matrix of a rotation with parameter η_0 is the stable part in the two factorizations (A.23) and (A.26). These considerations show that without disadvantage for the generality we may use the special squeezing operators $S(\xi', 0, \zeta')$ which are equivalent to the general operators $S(\xi, \eta, \zeta)$ after splitting rotation factors. However, these special squeezing operators alone do not form a group.

Appendix B

Influence of Displacement and Squeezing of States onto the Wigner Quasi-Probability

The displacement of states and the squeezing make transformations of the variables in the Wigner quasiprobability which can be given in a general form. For the derivation we use the representation of the Wigner quasiprobability by complex variables (α, α^*) in the following form [22] [29] which is equivalent to the definition given by Wigner [27] [28]

$$
W(\alpha, \alpha^*) = \left\langle \rho \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*} \right) \right\rangle \delta(\alpha, \alpha^*), \tag{B.1}
$$

where ρ is the density operator of the state. First we investigate the displacement of a state with the density operator ρ_0 and the Wigner quasiprobability $W_0(\alpha, \alpha^*)$ according to

$$
\rho = D(\beta, \beta^*) \rho_0 \left(D(\beta, \beta^*) \right)^\dagger. \tag{B.2}
$$

Then one finds for $W(\alpha, \alpha^*)$

$$
\begin{aligned}
W(\alpha, \alpha^*) &= \left\langle D(\beta, \beta^*) \rho_0 \left(D(\beta, \beta^*) \right)^\dagger \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*} \right) \right\rangle \delta(\alpha, \alpha^*) \\
&= \left\langle \rho_0 \left(D(\beta, \beta^*) \right)^\dagger \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*} \right) D(\beta, \beta^*) \right\rangle \delta(\alpha, \alpha^*) \\
&= \left\langle \rho_0 \left(D(\beta, \beta^*) \right)^\dagger \exp\left(-(a + \beta I) \frac{\partial}{\partial \alpha} - (a^\dagger + \beta^* I) \frac{\partial}{\partial \alpha^*} \right) \right\rangle \delta(\alpha, \alpha^*) \quad \text{(B.3)} \\
&= \left\langle \rho_0 \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*} \right) \right\rangle \exp\left(-\beta \frac{\partial}{\partial \alpha} - \beta^* \frac{\partial}{\partial \alpha^*} \right) \delta(\alpha, \alpha^*) \\
&= \left\langle \rho_0 \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*} \right) \right\rangle \delta(\alpha - \beta, \alpha^* - \beta^*).
\end{aligned}
$$

Therefore we obtained

$$
W(\alpha, \alpha^*) = W_0(\alpha - \beta, \alpha^* - \beta^*). \tag{B.4}
$$

This means that the Wigner quasiprobability $W(\alpha, \alpha^*)$ for the displaced state is equal to the Wigner quasiprobability $W_0\left(\alpha,\alpha^*\right)$ for the undisplaced state with displaced arguments corresponding to the displacement parameters $\left(\beta,\beta^*\right)$. This displacement property is one of the minimal requirements for a phase-space function $F\left(\alpha,\alpha^*\right)$ in quantum theory to be called a quasiprobability.

In widely analogous way one may treat the squeezing of a state with density operator ρ_0 according to

$$\rho = S\left(\zeta^*,\eta=\eta^*,\zeta\right)\rho_0\left(S\left(\zeta^*,\eta=\eta^*,\zeta\right)\right)^{\dagger}. \tag{B.5}$$

Instead of the unitary squeezing operators $S\left(\zeta^*,\eta=\eta^*,\zeta\right)$ we will calculate the transformations first a little more general with the operators $S(\xi,\eta,\zeta)$ and at the end we will specialize the result to $\xi=\zeta^*$ and $\eta=\eta^*$. Therefore, we have first to substitute $\left(S\left(\zeta^*,\eta=\eta^*,\zeta\right)\right)^{\dagger} \rightarrow \left(S(\xi,\eta,\zeta)\right)^{-1} = S(-\xi,-\eta,-\zeta)$ and find

$$\begin{aligned}
W\left(\alpha,\alpha^*\right) &= \left\langle S(\xi,\eta,\zeta)\rho_0\left(S(\xi,\eta,\zeta)\right)^{-1}\exp\left(-a\frac{\partial}{\partial\alpha}-a^{?}\frac{\partial}{\partial\alpha^*}\right)\right\rangle\delta\left(\alpha,\alpha\ \right)\\
&= \left\langle \rho_0\left(S(\xi,\eta,\zeta)\right)^{-1}\exp\left(-a\frac{\partial}{\partial\alpha}-a^{\dagger}\frac{\partial}{\partial\alpha^*}\right)S(\xi,\eta,\zeta)\right\rangle\delta\left(\alpha,\alpha^*\right)\\
&= \left\langle \rho_0\exp\left(-\left(\nu a-\mu a^{\dagger}\right)\frac{\partial}{\partial\alpha}-\left(-\lambda a+\kappa a^{\dagger}\right)\frac{\partial}{\partial\alpha^*}\right)\right\rangle\delta\left(\alpha,\alpha^*\right)\\
&= \left\langle \rho_0\exp\left(-a\left(\nu\frac{\partial}{\partial\alpha}-\lambda\frac{\partial}{\partial\alpha^*}\right)-a^{\dagger}\left(-\mu\frac{\partial}{\partial\alpha}+\kappa\frac{\partial}{\partial\alpha^*}\right)\right)\right\rangle\delta\left(\alpha,\alpha^*\right)\\
&= \left\langle \rho_0\exp\left(-a\frac{\partial}{\partial\gamma}-a^{\dagger}\frac{\partial}{\partial\gamma'}\right)\right\rangle\delta\left(\nu\gamma-\mu\gamma',-\lambda\gamma+\kappa\gamma'\right)\\
&= \left\langle \rho_0\exp\left(-a\frac{\partial}{\partial\gamma}-a^{\dagger}\frac{\partial}{\partial\gamma'}\right)\right\rangle\delta\left(\gamma,\gamma'\right).
\end{aligned} \tag{B.6}$$

Formally this can be written

$$W\left(\alpha,\alpha^*\right) \equiv W_0\left(\gamma,\gamma'\right), \tag{B.7}$$

with the matrix elements in explicit form given in (A.7). This corresponds to the transformation

$$\begin{aligned}
&\gamma=\kappa\alpha+\mu\alpha^*, \quad \alpha=\nu\gamma-\mu\gamma', \quad \frac{\partial}{\partial\gamma}=\nu\frac{\partial}{\partial\alpha}-\lambda\frac{\partial}{\partial\alpha^*},\\
&\gamma'=\lambda\alpha+\nu\alpha^*, \quad \alpha^*=-\lambda\gamma+\kappa\gamma', \quad \frac{\partial}{\partial\gamma'}=-\mu\frac{\partial}{\partial\alpha}+\kappa\frac{\partial}{\partial\alpha^*}.
\end{aligned} \tag{B.8}$$

In the last step in (B.6) was used that the two-dimensional delta function $\delta\left(\gamma,\gamma'\right)$ is invariant with respect to a (complex) unimodular transformation of the variables. This can be proved, for example, by transition to real variables and transformation to principal axes (see below).

In the special case of unitary transformations with the operator $S\left(\zeta^*,\eta=\eta^*,\zeta\right)$ we can substitute $\kappa\rightarrow\nu^*,\lambda\rightarrow\mu^*$ and $\gamma'\rightarrow\gamma^*$ and (B.7) using (B.8) can be specialized to [35]

$$W\left(\alpha,\alpha^{*}\right)=W_{0}\left(\nu^{*}\alpha+\mu\alpha^{*},\mu^{*}\alpha+\nu\alpha^{*}\right)\equiv W_{0}\left(\alpha',\alpha'^{*}\right). \tag{B.9}$$

We mention that a displacement relation of the form (B.4) is true for all quasiprobabilities (e.g., $Q\left(\alpha,\alpha^{*}\right)$ and $P\left(\alpha,\alpha^{*}\right)$) whereas for squeezing of states the form (B.9) is only true for the Wigner quasiprobability. The reason is that we do not have in other cases the operator $-a\frac{\partial}{\partial\alpha}-a^{\dagger}\frac{\partial}{\partial\alpha^{*}}$ as a whole in the exponent of only one exponential function.

The transformation of the variables $\left(\alpha',\alpha'^{*}\right)\leftrightarrow\left(\alpha,\alpha^{*}\right)$ possesses the property

$$\begin{aligned}\mathrm{d}\alpha\wedge\mathrm{d}\alpha^{*}&=\left(\nu^{*}\mathrm{d}\alpha'+\mu\mathrm{d}\alpha'^{*}\right)\wedge\left(\mu^{*}\mathrm{d}\alpha+\nu\mathrm{d}\alpha'^{*}\right)\\&=\left(\nu\nu^{*}-\mu\mu^{*}\right)\mathrm{d}\alpha'\wedge\mathrm{d}\alpha'^{*}=\mathrm{d}\alpha'\wedge\mathrm{d}\alpha'^{*},\end{aligned} \tag{B.10}$$

and therefore

$$\frac{\mathrm{d}q\wedge\mathrm{d}p}{2\hbar}=\frac{\mathrm{i}}{2}\mathrm{d}\alpha\wedge\mathrm{d}\alpha^{*}=\frac{\mathrm{i}}{2}\mathrm{d}\alpha'\wedge\mathrm{d}\alpha'^{*}=\frac{\mathrm{d}q'\wedge\mathrm{d}p'}{2\hbar}. \tag{B.11}$$

This means that each area element of the Wigner quasiprobability $W_{0}\left(\alpha,\alpha^{*}\right)$ is mapped in a new area element of $W\left(\alpha,\alpha^{*}\right)$ of the same area with preservation of the topology but without preservation of angles such as for transformations in classical mechanics. Therefore, for example, if the Wigner quasiprobability possesses regions of negativity (squeezed coherent states do not possess such regions) then after the transformation the area of negativity remains the same as before the transformation. The similar property is true for displacements and is here obvious.

A simple proof of the invariance of the two-dimensional delta function under (in general, complex) unimodular transformations of the variables by a matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ with $ad-bc=1$ can be given by extension of coordinates (x,y) to their own complex planes according to

$$\begin{aligned}&\delta(ax+by)\delta(cx+dy)\\&=\delta\left(\frac{(ad-bc)}{d}x\right)\delta\left(d\left(\frac{c}{d}x+y\right)\right)=\delta\left(\frac{x}{d}\right)\delta\left(d\left(\frac{c}{d}x+y\right)\right)\\&=\delta\left(\frac{x}{d}\right)\delta(dy)=\delta(x)\delta(y),\quad(ad-bc=1).\end{aligned} \tag{B.12}$$

This is then also correct for the representation of the two-dimensional delta function by complex variables.

Appendix C

An Identity between Jacobi Polynomials with Different Arguments

We derive here an identity between Jacobi polynomials which we used in Section 5 in specialized form.

Starting from the right-hand side of (5.1) we make a transformation of the following more general expression

$$\begin{aligned}
&\left(1-y^2\right)^{\alpha+\frac{1}{2}}\frac{\partial^n}{\partial y^n}\frac{1}{\left(1-y^2\right)^{\alpha+\frac{1}{2}}}\\
&=\left(\frac{\partial}{\partial y}+\frac{(2\alpha+1)y}{1-y^2}\right)^n\\
&=\left(\frac{\partial}{\partial y}-\frac{\alpha+\frac{1}{2}}{1-y}-\frac{\alpha+\frac{1}{2}}{1+y}\right)^n\\
&=\sum_{j=0}^{n}\left(-\frac{2}{1-y^2}\right)^j \mathrm{P}_j^{\left(-\alpha-\frac{1}{2}-j,-\alpha-\frac{1}{2}-j\right)}(y)\frac{n!}{(n-j)!}\frac{\partial^{n-j}}{\partial^{n-j}}.
\end{aligned}\tag{C.1}$$

This is an operator identity which in the last line is written in the entangled form derived in [32] and was given there in the following here slightly changed form

$$\begin{aligned}
&\left(\frac{\partial}{\partial y}-\frac{\alpha'}{1-y}+\frac{\beta'}{1+y}\right)^n\\
&=\sum_{j=0}^{j}\left(-\frac{2}{1-y^2}\right)^j \mathrm{P}_j^{(\alpha'-j,\beta'-j)}(y)\frac{n!}{(n-j)!}\frac{\partial^{n-j}}{\partial y^{n-j}},
\end{aligned}\tag{C.2}$$

and can be proved by complete induction. With $\mathrm{P}_n^{(\alpha,\beta)}(u)$ are denoted the Jacobi polynomials.

The operator identity (C.1) can be applied to arbitrary functions of the variable *y*. If we apply it to the function $f(y)=1$ we find

$$\begin{aligned}
&\left(1-y^2\right)^{\alpha+\frac{1}{2}}\frac{\partial^n}{\partial y^n}\frac{1}{\left(1-y^2\right)^{\alpha+\frac{1}{2}}}1\\
&=\sum_{j=0}^{n}\left(-\frac{2}{1-y^2}\right)^j \mathrm{P}_j^{\left(-\alpha-\frac{1}{2}-j,-\alpha-\frac{1}{2}-j\right)}(y)\frac{n!}{(n-j)!}\frac{\partial^{n-j}}{\partial y^{n-j}}1\\
&=n!\left(-\frac{2}{1-y^2}\right)^n \mathrm{P}_n^{\left(-\alpha-\frac{1}{2}-n,-\alpha-\frac{1}{2}-n\right)}(y).
\end{aligned}\tag{C.3}$$

Up to this point it may be considered as a transformation of the (Rodrigues-kind) definition of specialized Jacobi polynomials given by Szegö [30] (see also [31]).

The special case of Jacobi polynomials with equal upper indices $\mathrm{P}_n^{(\beta,\beta)}(y)$ is called Ultraspherical polynomials and they are related to the Gegenbauer polynomials $\mathrm{C}_n^{\nu}(y)$ in the following simple way

$$\mathrm{P}_n^{(\beta,\beta)}(y)=\frac{(2\beta)!(n+\beta)!}{\beta!(n+2\beta)!}\mathrm{C}_n^{\beta+\frac{1}{2}}(y).\tag{C.4}$$

They possess the following for us interesting expansions (Equations (4.5) and

(5.2) in [32])

$$\mathrm{P}_n^{(\beta,\beta)}(y)=\frac{2^{2\beta}(n+\beta)!}{(n+2\beta)!}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(n+\beta-k-\frac{1}{2}\right)!}{k!(n-2k)!\left(-\frac{1}{2}\right)!}(2y)^{n-2k}$$
$$=\sum_{j=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^j(n+\beta)!}{j!(n-2j)!(j+\beta)!2^{2j}}\left(1-y^2\right)^j y^{n-2j}. \tag{C.5}$$

Substituting in (C.5) $\beta=-\alpha-\frac{1}{2}-n$ and using transformation relations for the factorials one finds the following representation of the Ultraspherical polynomials involved in (C.3)

$$\mathrm{P}_n^{\left(-\alpha-\frac{1}{2}-n,-\alpha-\frac{1}{2}-n\right)}(y)$$
$$=\frac{(-1)^n\alpha!(n+2\alpha)!}{2^{2n}(2\alpha)!(n+\alpha)!}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n+\alpha)!}{k!(n-2k)!(k+\alpha)!}(2y)^{n-2k}. \tag{C.6}$$

In [32] (Equation (5.4)) it was derived an identity which can be written

$$\left(2\sqrt{1+x^2}\right)^n\mathrm{P}_n^{(\alpha,\alpha)}\left(\frac{x}{\sqrt{1+x^2}}\right)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(n+\alpha)!}{k!(n-2k)!(k+\alpha)!}(2x)^{n-2k}, \tag{C.7}$$

or after the substitution $x=\mathrm{i}y$

$$\left(-\mathrm{i}2\sqrt{1-y^2}\right)^n\mathrm{P}_n^{(\alpha,\alpha)}\left(\frac{\mathrm{i}y}{\sqrt{1-y^2}}\right)$$
$$=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n+\alpha)!}{k!(n-2k)!(k+\alpha)!}(2y)^{n-2k}. \tag{C.8}$$

An identity for these polynomials and their explicit representation is

$$\left(-\mathrm{i}2\sqrt{1-y^2}\right)^n\mathrm{P}_n^{(\alpha,\alpha)}\left(\frac{\mathrm{i}y}{\sqrt{1-y^2}}\right)$$
$$=\frac{(-1)^n 2^{2n}(2\alpha)!(n+\alpha)!}{\alpha!(n+2\alpha)!}\mathrm{P}_n^{\left(-\alpha-\frac{1}{2}-n,-\alpha-\frac{1}{2}-n\right)}(y). \tag{C.9}$$

These relations were applied in Section 5 to get the representation of $\overline{a^{\dagger k}a^k}$ by known polynomials.

Appendix D

Generating Functions for Hermite Polynomials

For easy use we collect here the most important generating functions for Hermite polynomials. The basic generating function for Hermite polynomials is

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_n(z)=\exp\left(2tz-t^2\right). \tag{D.1}$$

The basic generating function for products of two Hermite polynomials with different arguments is the Mehler formula [31] (chap. 10.13, Equation (22))

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{2}\right)^n \mathrm{H}_n(x)\mathrm{H}_n(y)=\frac{1}{\left(1-t^2\right)^{\frac{1}{2}}}\exp\left(\frac{2txy-t^2\left(x^2+y^2\right)}{1-t^2}\right). \tag{D.2}$$

By setting $y=0$ and using $\mathrm{H}_{2m}(0)=\frac{(-1)^m(2m)!}{m!}$ and $\mathrm{H}_{2m+1}(0)=0$ one obtains for even Hermite polynomials and by its differentiation with respect to variable z for odd Hermite polynomials the following non-trivial generating functions

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{t}{2}\right)^{2m}\mathrm{H}_{2m}(z)=\frac{1}{\left(1-t^2\right)^{\frac{1}{2}}}\exp\left(-\frac{(tz)^2}{1-t^2}\right),$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{t}{2}\right)^{2m+1}\mathrm{H}_{2m+1}(z)=\frac{tz}{\left(1-t^2\right)^{\frac{3}{2}}}\exp\left(-\frac{(tz)^2}{1-t^2}\right). \tag{D.3}$$

The generating functions for even and odd Hermite polynomials which may be easily derived from (D.1) are different from the generating functions (D.3).

Appendix E

Expectation Values for Squeezed Vacuum States

We compile here expectation values for ordered operators calculated from the formulae derived in the main text of Section 5.

For a few initial numbers of expectation values $\overline{a^{\dagger k}a^k}$ we calculated by (5.2) and (5.3) or from (6.7)-(6.11)

$$\begin{aligned}
\overline{a^{\dagger 0}a^0}&=1,\\
\overline{a^{\dagger 1}a^1}&=\frac{|\zeta|^2}{1-|\zeta|^2},\\
\overline{a^{\dagger 2}a^2}&=\frac{|\zeta|^2\left(1+2|\zeta|^2\right)}{\left(1-|\zeta|^2\right)^2},\\
\overline{a^{\dagger 3}a^3}&=\frac{3|\zeta|^4\left(3+2|\zeta|^2\right)}{\left(1-|\zeta|^2\right)^3},\\
\overline{a^{\dagger 4}a^4}&=\frac{3|\zeta|^4\left(3+24|\zeta|^2+8|\zeta|^4\right)}{\left(1-|\zeta|^2\right)^4},\\
\overline{a^{\dagger 5}a^5}&=\frac{15|\zeta|^6\left(15+40|\zeta|^2+8|\zeta|^4\right)}{\left(1-|\zeta|^2\right)^5}.
\end{aligned} \tag{E.1}$$

The expectation values of symmetrically (Weyl) ordered operators calculated according to (5.4) are

$$\begin{aligned}
\overline{\mathcal{S}\left\{a^{\dagger 0}a^{0}\right\}} &= 1,\\
\overline{\mathcal{S}\left\{a^{\dagger 1}a^{1}\right\}} &= \frac{1}{2}\frac{1+|\zeta|^2}{1-|\zeta|^2},\\
\overline{\mathcal{S}\left\{a^{\dagger 2}a^{2}\right\}} &= \frac{1}{2}\frac{1+4|\zeta|^2+|\zeta|^4}{\left(1-|\zeta|^2\right)^2},\\
\overline{\mathcal{S}\left\{a^{\dagger 3}a^{3}\right\}} &= \frac{3}{4}\frac{1+9|\zeta|^2+9|\zeta|^4+|\zeta|^6}{\left(1-|\zeta|^2\right)^3},\\
\overline{\mathcal{S}\left\{a^{\dagger 4}a^{4}\right\}} &= \frac{3}{2}\frac{1+16|\zeta|^2+36|\zeta|^4+16|\zeta|^6+|\zeta|^8}{\left(1-|\zeta|^2\right)^4},\\
\overline{\mathcal{S}\left\{a^{\dagger 5}a^{5}\right\}} &= \frac{15}{4}\frac{1+25|\zeta|^2+100|\zeta|^4+100|\zeta|^6+25|\zeta|^8+|\zeta|^{10}}{\left(1-|\zeta|^2\right)^5}.
\end{aligned} \tag{E.2}$$

The coefficients in front of powers of $|\zeta|$ in the numerators can be obtained forming the squares of the numbers in the Pascal triangle and, therefore, the polynomials in the numerator are palindromic ones.

For the corresponding expectation values of powers $\overline{N^l}$ of the number operator N we found from (5.7)

$$\overline{N^0} = 1,$$

$$\overline{N^1} = \frac{|\zeta|^2}{1-|\zeta|^2}, \quad \Leftrightarrow \quad |\zeta|^2 = \frac{\overline{N}}{1+\overline{N}} < 1,$$

$$\overline{N^2} = \frac{|\zeta|^2\left(2+|\zeta|^2\right)}{\left(1-|\zeta|^2\right)^2}, \quad \Rightarrow \quad \overline{(\Delta N)^2} = \frac{2|\zeta|^2}{\left(1-|\zeta|^2\right)^2} = 2\overline{N}\left(1+\overline{N}\right),$$

$$\overline{N^3} = \frac{|\zeta|^2\left(4+10|\zeta|^2+|\zeta|^4\right)}{\left(1-|\zeta|^2\right)^3},$$

$$\Rightarrow \quad \overline{(\Delta N)^3} = \frac{4|\zeta|^2\left(1+|\zeta|^2\right)}{\left(1-|\zeta|^2\right)^3} = 4\overline{N}\left(1+\overline{N}\right)\left(1+2\overline{N}\right),$$

$$\overline{N^4} = \frac{|\zeta|^2\left(8+60|\zeta|^2+36|\zeta|^4+|\zeta|^6\right)}{\left(1-|\zeta|^2\right)^4},$$

$$\Rightarrow \quad \overline{(\Delta N)^4} = \frac{4|\zeta|^2\left(2+11|\zeta|^2+2|\zeta|^4\right)}{\left(1-|\zeta|^2\right)^4},$$

$$\overline{N^5} = \frac{|\zeta|^2\left(16+296|\zeta|^2+516|\zeta|^4+116|\zeta|^6+|\zeta|^8\right)}{\left(1-|\zeta|^2\right)^5},$$
$$\Rightarrow \quad \overline{(\Delta N)^5} = \frac{16|\zeta|^2\left(1+16|\zeta|^2+16|\zeta|^4+|\zeta|^6\right)}{\left(1-|\zeta|^2\right)^5}. \tag{E.3}$$

The photon distribution p_n of squeezed vacuum states (2.8) is highly asymmetric to the mean value $\overline{N}$ with a longer tail to higher values of n and, therefore, $\overline{(\Delta N)^k}$ is nonnegative for all odd k (in addition to all even k for which this is trivial).

We give yet the initial members of the sequences of expectation values $\overline{a^{2l}}$ and $\overline{a^\dagger a^{2l-1}}$ for squeezed vacuum states calculated by computer from the more general formulae in Section 6. We found

$$\begin{aligned}
&\overline{a^0} = 1,\\
&\overline{a^2} = -\frac{\zeta}{1-\zeta\zeta^*}, \quad \overline{a^\dagger a} = \frac{\zeta^*\zeta}{1-\zeta\zeta^*},\\
&\overline{a^4} = \frac{3\zeta^2}{\left(1-\zeta\zeta^*\right)^2}, \quad \overline{a^\dagger a^3} = -\frac{3\zeta^*\zeta^2}{\left(1-\zeta\zeta^*\right)^2},\\
&\overline{a^6} = -\frac{15\zeta^3}{\left(1-\zeta\zeta^*\right)^3}, \quad \overline{a^\dagger a^5} = \frac{15\zeta^*\zeta^3}{\left(1-\zeta\zeta^*\right)^3},\\
&\overline{a^8} = \frac{105\zeta^4}{\left(1-\zeta\zeta^*\right)^4}, \quad \overline{a^\dagger a^7} = -\frac{105\zeta^*\zeta^4}{\left(1-\zeta\zeta^*\right)^4},\\
&\overline{a^{10}} = -\frac{945\zeta^5}{\left(1-\zeta\zeta^*\right)^5}, \quad \overline{a^\dagger a^9} = \frac{945\zeta^*\zeta^5}{\left(1-\zeta\zeta^*\right)^5}.
\end{aligned} \tag{E.4}$$

The expectation values $\overline{a^{\dagger 2k}}$ and $\overline{a^{\dagger 2k-1}a}$ follow from (E.4) using the general relation $\overline{a^{\dagger k}a^l} = \overline{a^{\dagger l}a^k}^*$. The obvious relation seen from (E.4)

$$\begin{aligned}
&\overline{a^\dagger a^{2l-1}} \equiv \langle 0,\zeta|a^\dagger a^{2l-1}|0,\zeta\rangle = -\zeta^*\langle 0,\zeta|a^{2l}|0,\zeta\rangle \equiv -\zeta^*\overline{a^{2l}},\\
&\overline{a^{2l}} = \frac{(-1)^l(2l-1)!!\zeta^l}{\left(1-\zeta\zeta^*\right)^l},
\end{aligned} \tag{E.5}$$

results from the eigenvalue Equation (2.12) written for the left-hand squeezed vacuum states $\langle 0,\zeta|$ as follows

$$\langle 0,\zeta|\left(a^\dagger+\zeta^* a\right) = 0. \tag{E.6}$$

We calculated some of the members in (E.4) in independent alternative ways and did not find contradictions.

Notations

A pair of boson annihilation and creation operators $\left(a, a^\dagger\right)$ is connected with the Hermitean canonical operators (Q, P) by ($\hbar \equiv \frac{h}{2\pi}$ with h Planck's action quantum)

$$\left(a, a^{\dagger}\right) = \left(\frac{Q + \mathrm{i}P}{\sqrt{2\hbar}}, \frac{Q - \mathrm{i}P}{\sqrt{2\hbar}}\right), \quad (Q, P) = \sqrt{\frac{\hbar}{2}}\left(a + a^{\dagger}, -\mathrm{i}\left(a - a^{\dagger}\right)\right),$$
$$\left[a, a^{\dagger}\right] = I, \quad [Q, P] = \mathrm{i}\hbar I. \tag{1}$$

Corresponding pairs of complex conjugate variables $\left(\alpha, \alpha^{*}\right)$ and real canonical variables (q, p) are related by

$$\left(\alpha, \alpha^{*}\right) = \left(\frac{q + \mathrm{i}p}{\sqrt{2\hbar}}, \frac{q - \mathrm{i}p}{\sqrt{2\hbar}}\right), \quad (q, p) = \sqrt{\frac{\hbar}{2}}\left(\alpha + \alpha^{*}, -\mathrm{i}\left(\alpha - \alpha^{*}\right)\right),$$
$$\frac{\mathrm{i}}{2}\mathrm{d}\alpha \wedge \mathrm{d}\alpha^{*} = \frac{\mathrm{d}q \wedge \mathrm{d}p}{2\hbar}. \tag{2}$$

Furthermore: $[A, B]$ denotes the commutator of operators A and B and $\langle A \rangle$ the trace of an operator A (usually in Hilbert space) and $[A]$ the determinant of two-dimensional operators A. Expectation values of operators to pure states $|\psi\rangle$ or density operators ρ are denoted by overlining, i.e. $\overline{A} \equiv \langle \psi | A | \psi \rangle = \langle \rho A \rangle$.