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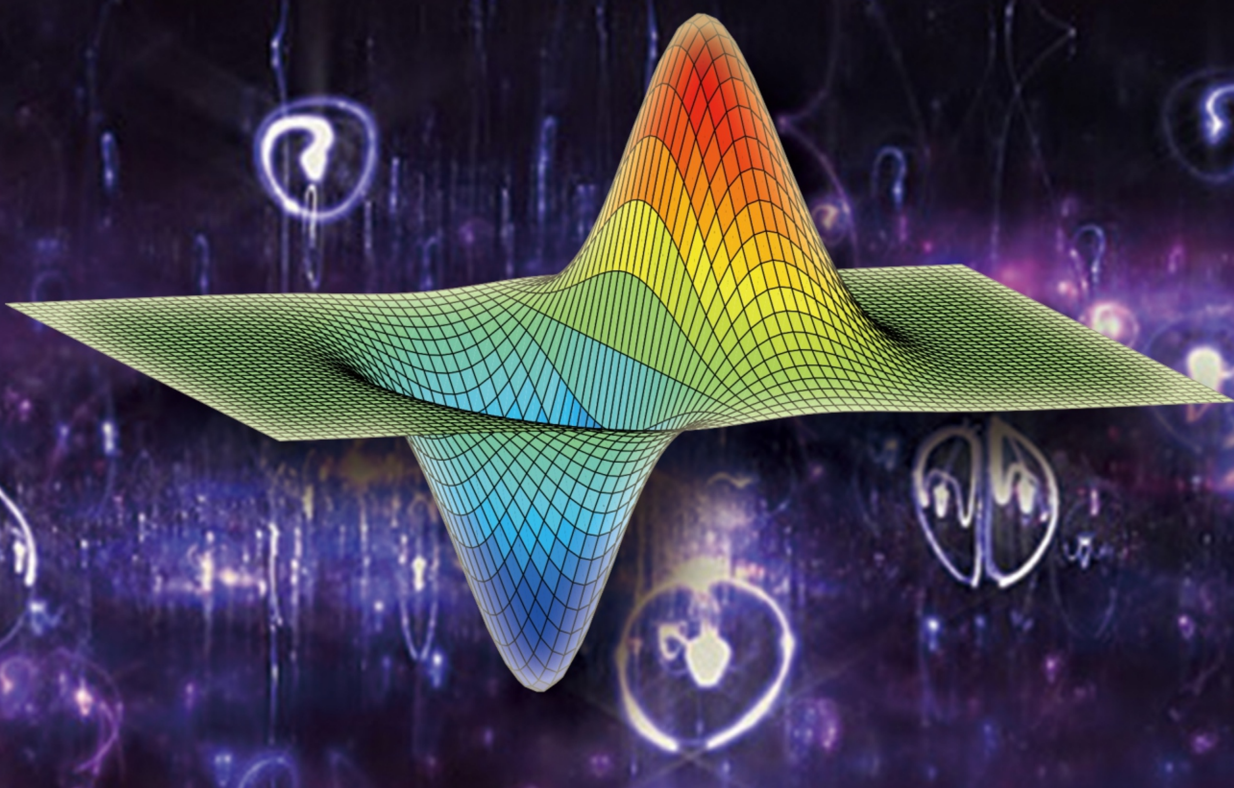

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Lattice Models of Finite Fields

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Abstract

Finite fields form an important chapter in abstract algebra, and mathematics in general, yet the traditional expositions, part of Abstract Algebra courses, focus on the axiomatic presentation, while Ramification Theory in Algebraic Number Theory, making a suited topic for their applications, is usually a separated course. We aim to provide a geometric and intuitive model for finite fields, involving algebraic numbers, in order to make them accessible and interesting to a much larger audience, and bridging the above mentioned gap. Such lattice models of finite fields provide a good basis for later developing their study in a more concrete way, including decomposition of primes in number fields, Frobenius elements, and Frobenius lifts, allowing to approach more advanced topics, such as Artin reciprocity law and Weil Conjectures, while keeping the exposition to the concrete level of familiar number systems. Examples are provided, intended for an undergraduate audience in the first place.

Keywords

Finite Fields, Algebraic Number Fields, Ramification Theory, Frobenius Element, Congruence Zeta Function, Weil Zero

1. Introduction

Finite fields are important mathematical structures, taking the learner from the familiar realm of congruence arithmetic to algebraic number theory territory, and providing new tools for mathematical physics and cryptography, for example.

We aim to highlight a pedagogical tool for the introduction of higher dimensional finite fields, which balances the traditional "axiomatic", top-down approach of Abstract Algebra, with a constructive, yet intuitive approach, using what we call *lattice models*.

The "standard" way in an Abstract Algebra course of introducing such higher dimensional finite fields, is to extends the primary finite field F_p, as a quotient of a polynomial ring $F_p[X]/\langle f(X)\rangle$ (e.g. MIT Modern Algebra Course, [1], Ch. 6 etc.). The lattice model approach *extends the lattice of integers first*, to place it in the context of complex numbers, followed by the quotient modulo a prime. In this way it mimics the elementary case of primary finite fields $F_p = Z/pZ$, providing also a geometric intuition accompanied by the corresponding analytic-topologic tools available.

$$
\begin{tikzcd}
\text{``Concrete''} & & Z[\xi]/<\pi> \arrow[r, "iso"] & F_p[X]/<f(X)> & \text{``Abstract''} \\
\mathbb{C} & Z[\xi] \arrow[l, "lattice"'] \arrow[ur, "mod\ \pi"] & & & F_p[X] \arrow[ul, "mod\ f(X)"'] \\
& & Z[X] \arrow[ul, "mod\ f(X)"] \arrow[urr, "mod\ p"'] & &
\end{tikzcd}
$$

Remark *Before comparing the two approaches, let us note that specialization (everywhere!) led to a fragmentation of the mathematical curriculum too, while the number of course a student may take remained essentially the same. This led teachers and researchers alike to advocate the need for a reintegration of Mathematics in various ways, for example in combination with teaching History of Mathematics (historical motivations)* [2], *Preface* 1*st ed., p.*12[1]: "*One of the disappointments experienced by most mathematics students is that they never get a course on mathematics*". *As a more modest goal of our paper, we believe that providing bridges between usually curricular separated ares in Mathematics, provides a* "*better circulation*" *of the underlying knowledge, and provides the student with more thinking power (links/synapses). The current tendency in Mathematics, and in fact in in general, is analytical specialization (well justified by the exponential growth of knowledge); there is a need for compensatory synthetic integration of symbiotic topics, supporting one another.*

Comparing with the concept of group, the "abstract way" is to define the algebraic structure with one binary operation, and then derive their properties from "axioms", perhaps too soon, before the student has enough examples to develop the "feeling" and intuition of what they are. The two dual, symbiotic types of groups, are the non-commutative groups of transformations, which always act on some space, and those we call Abelian, which in fact are "discrete vector spaces" on which the first kind act upon. The "unified" approach through generalization and abstraction has its price: treating alike the two becomes the norm, and the differences in interpretation neglected.

In this modern algebraic way of introducing algebraic structures abstractly, through general definitions, and then quickly deriving their properties, one would immediately ask the question of existence and uniqueness. The latter can be addressed in complete generality, without even knowing if they exist. Existence is proven, of course, by constructing finite fields explicitly.

[1]We further invite the reader to see the whole first paragraph.

For primary finite fields F_p, in characteristic *p*, this is easy: the well known Abelian groups Z/nZ, taught while doing congruence arithmetic, or rather viewed as rings Z/nZ, are easily shown to be fields, when $n=p$ is a prime number; but the other high dimensional finite fields are F_p^d are harder to construct, and the "future algebraist", the student, learns by heart the recipes for constructing field extensions.

Pedagogically, examples should be provided first, worked with them to the point the student begins to like them, and then "frame them" in the appropriate axiomatic context.

The lattice models of finite field presented in this paper represent construction of F_p^d, generalizing the above simple case of wrapping the 1-dimensional lattice *Z*, with period corresponding to the prime ideal pZ. By using higher dimensional lattice, instead of the standard adjunction of "roots" construction, we provide a geometric interpretation, with a corresponding graphical representation which brings geometry up-front, to enjoy and play with ... if time allows it!

Of course, there is a price to pay: some new number systems need to be introduced along the way, still extensions using the standard algebraic construction, but so important that they need to be made well known well before the theory of finite fields takes off: Gaussian and Eisenstein integers [3] [4], and their generalizations (cyclotomic extensions).

And yet here again, one can borrow the geometric interpretation of complex numbers as representing 2D-rotations, and still provide enough geometric intuition, to overcome the abstract "magical act" of adjoining new symbols; at least this is the opinion of one of the authors.

The article is organized as follows. The next section §0, introduces finite fields abstractedly, as in most textbooks of abstract algebra. Section §0 constructs finite fields as congruence rings of integers in number fields (algebraic extensions of the rationals). The geometric interpretation is emphasized. We conclude §0 discussing briefly some important topics at hand, like Frobenius elements and Weil zeros.

2. Finite Fields: The "Abstract Way"

We will recall the basic facts about finite fields, as introduced in most standard texts of abstract algebra. To keep it self-contained, and simple, we use a brief presentation available on the web [5]. See [6] for additional theoretical details and [7] for a computational approach.

Definition 2.1 *A finite field is a field which is finite!*

The additive order of the unit $1+\cdots+1=0$ is called the characteristic of the finite field. It is always a prime *p*. For example $F_3=Z/3Z$ has characteristic $p=3$.

Recalling some basic properties are in order.

Proposition 2.1 *A finite field F of characteristic p has* $q=p^n$ *elements. It is* *a vector space of dimension n over the primary field* F_p : $n=[F,F:p]$.

Theorem 1 (*i*) (*Existence and Uniqueness*) *For each p and n there exists a finite field of characteristic p with* p^n *elements.*

(ii) Two such finite fields with the same number of elements are isomorphic.

It is therefore natural to denote a *generic* finite field as F_q, with $q=p^n$, as if it is a specific one. By abuse of notation, yet well justified by the uniqueness modulo isomorphism, we write $F_p=Z/pZ$, without further comments (LHS is a "any" finite field of characteristic *p*, while the RHS is the preferred, specific construction of one such field).

The "standard" way to construct higher dimensional finite fields with a given number of elements p^n, and of course prescribed characteristic *p*, uses the standard algebraic construction of field extensions via polynomial rings and their quotients by ideal generated by irreducible polynomials.

We reproduce here the Example 1.88, from [6], p. 34.

Example 2.1 *Let the prime field be* F_3. *As an example of the formal process of root adjunction, consider the irreducible polynomial* $f(x)=x^2+x+2\in F_3[x]$. *Let* θ *be a "root" of f, that is,* θ *is the residue class* $x+(f)$ *in* $L=F_3[x]/(f)$. *The other root of f in L is* $2\theta+2$, *since*

$$f(2\theta+2)=(2\theta+2)^2+(2\theta+2)+2=\theta^2+\theta+2=0.$$

We obtain the algebraic extension $L=F_3(\theta)$ consisting of the nine elements $0,1,2,\theta,\theta+1,\theta+2,2\theta,2\theta+1,2\theta+2$, *i.e.* an instance of F_{3^2}.

3. What *Are* Number Fields?

The algebraic structure we call *field* was first introduced by Dedekind [8] ([2], Ch. 12). The usual number systems *Q*, *R* and *C* are the traditional examples of fields. When solving algebraic equations defined by polynomials, we are "forced" to extend our number system, and adjoin formal roots of polynomials as new "numbers". We can treat these either as new *symbols*, and *construct* the new number system, for example $C=\{x+iy \mid x,y\in R, i^2=-1\}$, as real linear combinations of 1 and the symbol *i* subject to the relation $i^2=-1$, or more formally, in the abstract (algebra), as quotients of polynomials modulo the ideal generated by the polynomial defining the relation:

$$C=R[X]/\langle X^2+1\rangle=\{a+bI \mid a,b\in R, I=[X]\},$$

Here $[X]$ denotes the congruence class of *X* modulo the ideal, satisfying the required relation: $I^2+1=[X^2+1]=0$ (since $X^2+1\cong 0 \; mod \; X^2+1$).

We will call this construction the *standard algebraic construction* of a field extension.

Now "integers" play a central role in arithmetic, in various rings, and they satisfy the structure of lattices. Initially we may call "integers" the subring of a field extension which emerges as a corresponding field of fractions, but field extensions require more care when defining the concept of *algebraic integer of a field extension*.

Definition 3.1 *A lattice* $\mathcal{L}$ *is a Z-submodule of a ring.*

In particular a lattice is a finitely generated abelian group, and can be interpreted and visualized as a "discrete (finite dimensional) vector space" (by abuse of language, when there are still relations among generators).

Two good examples of such lattices of algebraic integers are the Gaussian integers and Eisenstein integers [3] [4].

3.1. Gaussian Integers

Complex numbers are a familiar example of field extension of the reals. To keep the theory algebraic, and to investigate it from an arithmetic point of view, neglect the Cauchy reals as non-realistic numbers [9], and consider the quadratic extension $Q(\sqrt{-1})$ over the rationals *Q*. Even better, since these fields are fields of fractions, focus on the extension of integers: $Z[i]$[2].

The ring $Z[i]=\{m+in|m,n\in Z\}$ is called the ring of Gaussian integers. The rational primes *p* may factor in this larger arithmetic number system:

The prime 2 is special, and "ramifies" as $2=i^{-1}(1+i)^2$[3].

$p\cong 1\, mod\ 4$ splits into a product of conjugate primes, for example $5=(2+i)(2-i)$;

$p\cong -1\, mod\ 4$ is *inert*, *i.e.* it remains a prime in $Z[i]$; for example $p=3$.

For more facts about Gaussian integers see [3]. For a more technical account, including relations to Galois theory, see [10].

3.2. Eisenstein Integers

Similarly, taking a cubic root of unity ω instead of the 4-th root of unity *i*, we obtain the Eisenstein integers $Z[\omega]$, with its own primes and classes of rational primes ramifying ($p=3$), and splitting or being inert, according to a similar condition $p\cong \pm 1$, but this time modulo 3. Alternatively, one may look at the analog of Fermat's Two Squares Theorem, about representing primes $p=m^2+n^2$, except this time we use a different quadratic form (norm): x^2-xy+y^2, instead of the usual one $x^2+y^2=N(x+iy)$ in $Z[i]$.
For more details, see [4].

3.3. From Number Fields to Finite Fields

Now the idea for constructing higher dimensional finite fields, is to consider the congruence rings of algebraic integers, modulo a prime, the obvious analog of the construction of primary finite fields $F_p=Z/pZ$.
As a quick example, $Z[i]/3Z$ yields F_{3^2}, while $Z[i]/(2+i)Z\cong Z/5Z=F_5$.

Besides being a more natural construction, it provides the geometric background for a better understanding of finite fields as *Klein geometries* (Galois fields)[4].

[2]In general the extension of integers might not coincide with the algebraic integers of the corresponding field extension.

[3]Recall that we have more units here: $1,-1,i,-i$.

[4]...not to mention the connection with Galois Theory, splitting polynomials and Frobenius elements.

4. Lattice Models: The "Geometric Way"

We will proceed by way of example. Recall that the primary fields F_p can be constructed as ing quotients Z/pZ, where p is a prime number, the characteristic. geometrically, Z can be viewed as a 1-dimensional lattice, or as an infinite oriented graph[5].

The prime p defines a period, and the covering map $\phi(k) = k \; mode \; p$ is a discrete geometric analog of the familiar covering map of the circle $x \; mod \; 1$, sometimes used to define angles, sine and cosine. Algebraically, ϕ is a group (ring) homomorphism: the quotient map of the ring Z by the ideal pZ.

Now let's consider a 2D-example: the Gaussian integers, as a lattice, modulo a prime ideal $\mathcal{P}$.

Since $Z[i]$ is a principal ideal domain (PID), we need only consider $\mathcal{P} = Z[i]\pi$ with Gaussian prime π "sitting" over a rational prime *p*: $N(\pi) = p$.

For example $2+i$ is a Gaussian prime over 5, completely splitting it: $5 = (2+i)(2-i)$. Recall that other rational primes of the form $p \cong -1 \; mod \; 4$ are *inert*, *i.e.* are Gaussian primes too and $N(p) = p^2$.

There is also the special case of the *ramified prime* 2[6], which factors with multiplicity: $2 = (1+I)^2 \cdot (-i)$ [3]).

Remark *The factorization may also be written in an initially misleading way as* $2 = (1+i)(1-i)$, *but* $1+i$ *and its conjugate* $1-i$ *are the "same" prime, modulo a unit* $\pm 1, \pm i$.

Consider the same algebraic quotient map $\phi(z) = z \; mod \; \pi$. Since π is prime, the quotient ring $K = Z[i]/(\pi)$ is a field of characteristic *p*, *i.e.* F_{p^f}. The norm $N(\pi) = p^f$ gives the dimension $f = [K : F_p]$.

Excepting the case of the ramified prime $p = 2$, we have the following two cases. For inert (rational) primes $p \cong 3 \; mod \; 4$, $\pi = p$ is the only prime over *p*, and $f = 2$; otherwise $p = \pi\overline{\pi}$ splits and $N(\pi) = p$.

Example 4.1 *Let* $p = 5$ *and* $\pi = 2+i$. *Then* $K = Z[i]/(\pi)$ *is a lattice model of* F_5 (*the abstract finite field with* 5 *elements*). *We can see its canonical residue classes as the Gaussian integers in the fundamental region of the lattice* $\mathcal{L} = \{a\pi + b\overline{\pi} \mid a, b \in Z\}$, *for example with* a, b *non-negative integers, such that* $N(z) < p$ (*again considering the projection on the integers*).

Another example of lattice model, providing an alternative construction to the "standard" algebraic extension from Example 0.1, is the following.

Example 4.2 *Consider again* $Z[i]$ *as a quadratic extension and* $p = 3$ *the rational inert prime. Then the quotient lattice* $Z[i]/(3)$ *has* $q = 3^2$ *elements, representing the finite field* F_{3^2}.

5. Applications to Weil Zeros

There are several topics of Algebraic Number Theory which may benefit from

[5]The interested reader may lookup partial ordered set, POSet for short, too, as a generalization.

[6]Divides the discriminant of the quadratic extension $[Z(i) : Z]$.

the introduction of finite fields as quotients of lattices of algebraic integers:

a) Ramification Theory, in the context of Galois Theory of such extensions;

b) The Frobenius element, as a generator of the Galois group of the corresponding extensions, controlling the factorization of prime ideals in extensions of number fields;

c) Quadratic Reciprocity using the connection between the Frobenius element and Legendre symbol in congruence rings of number fields; finally,

d) Applications to Weil Conjectures, and notably to the finite characteristic Riemann Hypothesis via the characteristic polynomial of a lift of the Frobenius element, having eigenvalues the Weil zeros of the Weil polynomial, *i.e.* the reciprocal of the numerator of the Hasse-Weil Congruence Zeta Function [11].

The first three applications are essentially described in [12]. In this article we will focus on this later important application to Algebraic Geometry, which can be accessed relatively easily, in a computational oriented way, using for example SAGE as a mathematical software. In this brief note, we will only point the way. For an exposition, see the classical texts, for example [13] [14]; additional explanations and computations can be found in the lecture notes of the first author [15].

5.1. Solving Algebraic Equations over Finite Fields

Quadratic equations were studied since ancient times, e.g. Appolonius' theory of conic sections. Replacing the usual number system with finite fields places the problem in the context of Algebraic Geometry.

Following [14], Ch. 8, consider the solution $X(F_q)$ of the equation $y^2 = x^d + D$ over the finite field F_q with $q = p^n$ elements. It is an algebraic *affine curve* of *degree d*. Denote the corresponding number of elements N_n, and the associated *congruence zeta function*[7]

$$Z_{X/Fp}(T) = \frac{P(T)}{(1-T)(1-pT)}, \quad P(T) = \prod_{i=1}^{2g}(1-\omega_i T),$$

where $g = (d-1)/2$ is called the *genus of the curve*, and w_i are algebraic numbers we will call the *Weil zeros* of the *Frobenius polynomial* $h(u) = u^{2g} P(1/u)$ [8]. We will not go in depth explaining the terminology, and just use it to exemplify the relation with factorization of primes and lattice models of finite fields.

Example 0.4 *The cubic* ($d = 3$) *curve* $X : y^2 = x^3 + D$, *is an elliptic curve of genus* $g = 1$, *which should be pictured topologically* (*over complex numbers*) *as a torus* (*when completed with the point at infinity*: *the projective curve*).

Regarding the fixed prime *p*, whether F_p has *m*-roots of unity or not decides the form of $P(T)$ and N_n. In what follows we will assume $m \mid p-1$, *i.e.* F_p has *m*-roots of unity[9] Then $P(T) = (1-wT)(1-\overline{w}T)$ is a quadratic polynomial and the number of affine points is $N_n = p^n - w^n - \overline{w}^n$, where $\overline{w}$

[7] Conform Weil Conjectures/Deligne Theorem.

[8] w_i are reciprocal of the zeros of the "Frobenius polynomial" $P(T)$.

[9] Cauchy Theorem for the multiplicative group $(F_q^\times, \cdot)$

denotes complex conjugation [13], Ch. 18§2, p. 302 (where the +1 stands for the point at infinity; see also [14], p. 292).

Remark *Later we will see how Weil zeros* w_i *are related to Gauss and Jacobi sums, which are valued in the cyclotomic numbers of roots of unity of order* $l(l-1)$ *and* $l-1$ *respectively, if* $l=m\mid p-1$ *is a prime.*

Part of *Weil Conjectures* [16] is that $w\overline{w}=p$, *i.e.* the *Riemann Hypothesis* holds in finite characteristic [14]. Moreover, introducing the *defect* $a_p=w+\overline{w}$, $P(T)=1-a_pT+pT^2$.

Remark *The coefficients of the Betti polynomial* $P(T)$ *are related to Weil zeros as a consequence of a deeper connection with the characteristic polynomial of the Frobenius element* $h(u)$: $a_p=Tr(Frob)$, $p=det(Frob)$.

Example 5.2 *The elliptic curve* $m=3$ [13], p. 306, *has* $N_1=p+w+\overline{w}$, *where the Weil zeros split the prime* $p=w\overline{w}$ *in the cyclotomic extension* $Z(\zeta_3)$ *of Eisenstein integers* (*assuming* $3\mid p-1$). *In terms of primary primes* $\pi,\overline{\pi}$, $\pi\cong 2 \bmod 3$ (*associated to* $w,\overline{w}$), *we have* [13], Th. 4, p. 305 (*affine points*; $6=l\cdot(l-1)$ *with* $l=3$):

$$N_p=p+2Re(\rho(4))\pi,\quad \rho(x)=\left(\frac{4D}{\pi}\right)_6\pi.$$

As a concrete example take $D=1$.

If $p=13$ then $\pi=-1+3\omega$ is a primitive prime, and together with $\overline{\pi}=-1+3\omega^2$ split *p*:

$$p:13=(-1+3\omega)(-1+3\omega^2):\pi\overline{\pi}.$$

Since $\rho(4)=\omega^2$, the Weil zero is $w=-\omega\pi$, associated to π (Units: $\{\pm1,\pm\omega,\pm\omega^2\}$).

Now the number of affine (finite) points in F_p is:

$$N_1=13+2Re(\omega\pi)=13+2(\omega^2+\omega)=13-2=11,$$

consistent with a direct check and counting argument:

$$X(F_{13})=\{(4,0),(10,0),(12,0),(0,\pm1),(2,\pm3),(5,\pm3),(6,\pm3)\}.$$

Example 5.3 *As another example consider* $D=5$ *and* $p=19$. *Note that* $\pi=5+3\omega$ *is primary and splits p*:

$$19=(5+3\omega)(5+3\omega^2), 5+3\omega\cong 2 \bmod 3.$$

From the above formula we obtain the number of points ([17], p. 8)[10]:

$$N_1=p+\pi+\overline{\pi}=19+5+\omega+5+\omega^2=26.$$

Then $a_p=-2Re(\pi)=-7$ and Weil zeros are $w=-\pi$ and its conjugate:

$$P(T)=(1-wT)(1-\overline{w}T)=1+7T+19T^2,\quad N_1^{proj}=P(1)=27.$$

Remark *One may use SAGE* (*recently renamed as CoCalc*) [18] *to conveniently compute Dirichlet characters* (*like* ρ *above*), *and Jacobi sums, which*

[10]We need Jacobi sums for this: see §5.3.

are instrumental in computing the number of points.

Well, what does this problem of counting the number of solutions, with its associated congruence zeta function, have to do with lattice models of finite fields!? For this we need to recall some facts about the Galois group of an extension, and the relation with the *Frobenius element*, which will turn out to be present as the numerator of the zeta function.

5.2. Frobenius Element

Following [19], consider a number field extension $Q(\xi)/Q$ which is Galois, and how a rational prime p decomposes in it, with π such a prime factor (assuming a principal ideal domain case for simplicity). Then the Galois group of the number field extension $Gal(Q(\xi)/Q)$ is related to the Galois extension of the corresponding (lattice models of) finite fields:

$$1 \to I(\pi/p) \to D(\pi/p) \to Gal(F_q/F_p) \to 1,$$

where $D(\pi/p)$, the *decomposition group* consists in Galois automorphisms preserving the ideal generated by π, each of its elements therefore inducing an automorphism of the corresponding finite fields extension $Gal(F_q/F_p)$, in a surjective manner. The kernel of this projection, the *inertia group*, will not be used in what follows.

If p is unramified, then the kernel (the *inertia group*) is trivial, and the above surjection becomes an isomorphism. Then one can "pull-back" the Frobenius automorphism $x \mapsto x^p$ of $Gal(F^q/F^p)$, where we recall that $F^q = Z[\xi]/\pi$ and $F_p = Z/p$ are lattice models of finite fields constructed in number fields *viewed (embedded) as subfields of the complex numbers.*

Definition 5.1 *In the context above (π prime in $Z[\xi]$ over the unramified rational prime p), the Frobenius element $Frob_p^\pi \in \mathrm{Aut}_Z(Z[\xi])$ is the unique Galois automorphism which induces the Frobenius automorphism $Fr(x) = x^p$ in the finite field extension F_q/F_p, of lattice models.*

At this stage the Frobenius elements may depend on the choice of prime π over *p*. But these Frobenius elements are conjugate to each other, so if the Galois group is Abelian, then the Frobenius element is unique, and will be denoted by $Frob_p$.

Example 5.4 *Consider $Q[i]$, with i a forth root of unity, and its Gaussian integers $Z[i]$. The only ramified prime is* 2; *otherwise $p \cong 1$, mod* 4 *or course, splits, or $p \cong -1$ is inert.*

The decomposition group $D(p)$ is trivial in the split and ramified cases, and equals $G = Gal(Q(i):Q) \cong Z_2$ (multiplicative group $\{-1,1\}$) otherwise.

Thus the Frobenius element is 1 when $p \equiv 1$ and -1 otherwise, *i.e.* $Frob_p = \left(\frac{-1}{p}\right)$ is given by the *Legendre symbol* (the unique multiplicative character of order 2).

Alternatively, we can compute the *lift to $Z[i]$ of the Frobenius* x^p, from the "abstract" setup, using our lattice model, to the ring of algebraic integers[11]:

$$(a+ib)^p \equiv a+bi^p \ mod \ pZ[i].$$

Since $i^p = \left(\frac{-1}{p}\right) i$ "on the nose", *i.e.* not just $mod\ p$, we conclude that the lift has the following closed formula in terms of the multiplicative quadratic residue character $\rho_2(z) = (z/p)$:

$$Frob_p(a+ib) = a + \left(\frac{-1}{p}\right) b, \quad Frob_p = \sigma^k, k = \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}.$$

The last form was written in terms of the generator of *G*, here complex conjugation.

Example 5.5 *The above example can be generalized to quadratic extensions* $Q(\sqrt{d})$, *where d is square free* ([19], p. 3)*. The Frobenius element, in* $Z/pZ^\times$, *is* $Frob_p = (d/p)$, *so that* $Frob_p(a+\sqrt{d}b) = a + (-1/p)\sqrt{d}b$, *i.e.* $Frob_p = \sigma^{ord(-1/p)}$ *as before.*

Example 5.6 *In the cyclotomic case* $Q(\xi_n)$, *the primes that ramify are those which divide n. The Galois group is isomorphic to the multiplicative group of roots of unity, and therefore isomorphic to* $(Z/nZ^\times, \cdot)$, *with a Galois element* $\sigma_m : \xi \mapsto \xi^m$, *with* $m \in Z/nZ^\times$ *relatively prime to n.*

In the non-ramified case $p \in Z/nZ^\times$, the Frobenius element is, again as expected:

$$Frob_p\left(\sum_{k=0,\cdots,n-2} c_k \xi_n^k\right) = \sum c_k Frob_p(\xi_n)^k, \quad Frob_p(\xi) = \xi_n^p.$$

As another quadratic extension example, consider $Q(\omega)$, corresponding to a cubic root of unity $\omega^3 = 1$, and its Eisenstien integers $Z[\omega]$. Then the corresponding Frobenius element is, similarly to the Gaussian integers case:

$$Frob_p(a+\omega b) = a + \rho_3(p)\omega b.$$

Remark *At this stage, one may further look into the correspondence between how the prime p factors into* $Z[\xi]$, *and how the primitive polynomial* $f(x)$, *of* ξ *factors in* $F_p[x]$, *reflecting the commutativity of the diamond diagram from the introduction.*

It is conceptually important to piece together these Frobenius elements as a map depending on the prime *p*, called the *Artin map*: $Frob : p \mapsto Frob_p$[12]. For cyclotomic extensions, If we identify the Galois group $Gal(Q(\xi_n)/Q)$ with $(Z_n^\times, \cdot)$, then the Artin map is simply the "identity" map:

$$p \mapsto Frob_p = (p \ mod \ n), \ p \in Z_n^\times.$$

For example, with $m = 4$ and *p* an odd prime, the Galois group is generated

[11]It is enough to consider the extension of *Z*, and not the full algebraic closure in $Q(i)$, which incidenltally, here coincide.

[12]In a more general setup [12], Ch. 5, $Frob(p) = \left(\frac{L/K}{P}\right)$ is called the *Artin symbol*.

by conjugation $G=\langle\sigma\rangle$, since $Z_4^\times \cong Z_2$, and $Frob(p)=p\ mod\ 4$ as an element of $Z_4^\times$. This is essentially the Legendre symbol $\frac{p}{4}$, when identifying Z_4 with the 4-th roots of unity, via exponentiation (the Galois group identification).

Once we know the Frobenius element, its characteristic polynomial can be computed easily:

$$P\left(Frob_p\right)(t)=det\left(Frob_p-tI\right).$$

For example, in the cyclotomic setup, with $m=4$ (Gaussian integers), the matrix of $Frob_p=\sigma^{(-1/p)}$ in the basis $1,i$ is:

$$Split: p\equiv 1\ mod\ 4:\quad Frob_p=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$Inert: p\equiv -1\ mod\ 4:\quad Frob_p=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

and the characteristic polynomials are, respectively:

$$P(T)=(1-T)^2,\quad P(T)=1-T^2.$$

Similarly, for a quadratic extension, for example $m=3$ (Eisenstein integers), the matrices of the Frobenius elements I and σ, and their matrices are essentially the same (but computed in a different basis $1,\omega$).

Now let's see how the Frobenius element, or rather its lift and the corresponding characteristic polynomial is related to the Hasse-Weil congruence zeta function.

5.3. Weil Zeros and Jacobi Sums

We will only document the facts with an example, following [13] [14] [20] [21], and leave the general case for a separate study.

Let $y^2=x\left(x^2+1\right)$ define an elliptic curve over F_q. Since the RHS of its defining equation $f(x)$, splits in $Z[i]$, we will work with Gaussian integers in the number fields side of the "picture".

For $p\cong 3\ mod\ 4$, the prime is inert in $Z[i]$, which corresponds to the factor x^2+1 being irreducible in F_p and the Frobenius element complex conjugation.

Theorem 5, [13], p. 307, with $D=-1$, yields the number of *projective* points[13], according to the type of prime:

$$Inert: N_p=1+p,\quad Split: N_p=p+1-2Re\left(\rho_4(-1)\pi\right),$$

where π is a primary prime splitting p and ρ_4 is a character of order 4.

We will focus on the split case $p\cong 1\ mod\ 4$ (Ramification Theory parameters: $g=2$, $e=1$, $f=1$).

To have a "nice" description of the lift of Frobenius Fr_p on C preserving our

[13]The +1 stands for the point at infinity.

curve, and not some "deformation" of identity (the Frobenius element) $(x,y)\mapsto(x^q+f(x,y),y^q+g(x,y))$ [20], p. 10, we use Weierstrass coordinates. The elliptic curve is then the quotient of $\mathbf{C}$ by our lattice $\Lambda=Z[i]$ of (Gaussian) algebraic integers:

$$e(\tau z):(\mathbf{C},+)\to(\mathbf{C}^{\times},\cdot),\quad \mathbf{C}/Z[i]\cong E(\mathbf{C}),$$

where here $\tau=2\pi i$. Then the Frobenius lift $Fr_p(z)=z\cdot c$ is multiplication by some lattice element $c=a+ib\in Z[i]$ [20] [21].

Remark *Alternatively, we could lift the Frobenius to the p-adic completion, and taking advantage of Hasse principle for finding the above "perturbations"* $f(x,y)$ *and* $g(x,y)$.

If the curve is defined by a polynomial in the powers of the variables (Weil curves), e.g. Riemann surfaces $y^2=x^s+D$ ([14] p. 292) and Fermat curves $x^m+y^m=z^m$ ([13] [22]), then Jacobi sums provide a powerful tool to compute the number of points.

Then $N_p=1+p-a_p$, with the *defect* given by the Jacobi sum $a_p=2ReJ(c_2,c_4)$, which also yields the Weill zeros $w,\bar{w}$ of the (reciprocal of) "Betti polynomial"[14]:

$$Z_p(T)=\frac{L_p(T)}{(1-T)(1-pT)},$$
$$L_p(T)=(1-wT)(1-\bar{w}T),\quad w\bar{w}=q.$$

Then $w=-J(c_2,c_4)$ is primary [13] and our lift of Frobenius is given by $c=w=\pi$, conform with [21], with $a_p=Tr(Fr_p)$ and $p=det(Fr_p)$ (Riemann Hypothesis, part of the Weil Conjectures; see also [11], Lecture #8, Hasse's Theorem):

$$CharPoly(Fr_p):\quad det[Fr_p-u\,Id]=u^2-a_pu+p,u=1/T.$$

Rewriting the number of points as in [20] $N=q-2d\sqrt{q}+1$, and the Weil zero as $w=\mathrm{e}^{i\theta}\sqrt{q}$, one may interpret the "Betti coefficient" $d=Re(\mathrm{e}^{i\theta})$ via a comparison with the Jacobi sum, as phase of the 2-cocycle of the Fourier coefficients of the Dirichlet characters (Gauss sums)... but this is another story!

Remark *A similar discussion applies to our previous example of elliptic curve* $y^2=x^3-D$ ([13], p. 304; [17], p. 7), *with Eisenstein integers replacing Gaussian integers.*

Remark *For higher dimensional extensions* F_{p^f}, *needed when the genus of the curve excedes* $g=1$, *can be implemented via cyclotomic extensions* $Z[\xi_m]$, *such that the dimension* $n=\phi(m)$ *factors as* $g\cdot f$ *with* $f=\mathrm{ord}(p)$ *is the order of multiplication by p in* $Z/mZ^{\times}$ *and g the ramification genus of the prime p.*

6. Conclusions and Further Developments

There are various styles of teaching (and designing) Abstract Algebra. We

[14]The numerator of the Zeta function is a local L-function having a cohomological interpretation.

attempted to plead that, in the case of finite fields, the abstract approach to the introduction of the algebraic structure ("axiomatic"/top-down design), can be supplemented by the specific construction we call *lattice models*, which introduces the number fields first, as more "familiar" to the student used to solve polynomial equations, and presenting F_{p^n} as a congruence ring, in perfect analogy to the way we introduce the primary finite fields $F_p = Z/pZ$.

6.1. Motivation, Goals and Contributions

One motivation of this article for emphasizing lattice models of finite fields is the need for a pedagogical introduction to finite fields as part of an Abstract Algebra course, with strong ties with number fields ("number systems" in their natural habitat of complex numbers), with best results after covering Galois Theory, of course. As mentioned in the introduction, a discussion of the residue fields in the context of decomposition of primes in algebraic number fields, implies a long wait in providing such concrete examples of extensions of finite fields [23]. Bridging mathematical topics in general, is a much needed way to balance specialization [2], p. 12.

In our case at hand, the bonus is some extra intuition, but more importantly, a rich *geometric framework* for bridging and interpreting other abstract algebra concepts, like Galois Groups, Frobenius elements, paving the road towards understanding *General Reciprocity Laws* the "right way" [24].

The second goal, which in fact started the current project, was to provide a direct approach to Weil Conjectures, to be understood not in their natural "habitat" of abstract Algebraic Geometry, but in the more geometric and topological context of complex manifolds, by using lattice models of finite fields. Then the Frobenius element of the number can be related to a Frobenius lift of the Frobenius automorphism. Then the numerator of the Weil Congruence zeta function is the characteristic polynomial of the lift of the Frobenius element, allowing to count numbers of solutions without the use of a Weil cohomology (e.g. Grothendieck's approach via l-adic cohomology).

Our main contribution in this open effort for bridging the modern abstract ("axiomatic") exposition of finite fields and traditional concrete, by example, approach using the familiar "number systems", is the emphasis on the concrete examples of finite fields we call *lattice models*, with the early benefit of learning of how primes decompose: Ramification Theory. As a "bonus", as mentioned before, this bridge may constitute a shortcut to understanding Frobenius lifts, via Frobenius elements, in a more familiar context, towards understanding more advanced topics like congruence zeta function and Weil Conjectures, without having to "cross-over" to p-adic analysis and etale cohomology.

Regarding other studies in this direction, we noticed only highly specialized articles and presentations, either focusing on ramification theory, or in attempting the construction of a lift of Frobenius directly, by not so accessible to students [20]. This provided additional motivation for starting this project.

6.2. Towards Other Applications: Algebraic Topology/Geometry

Once "in motion", if one wishes, Lefshetz formula, as well as algebraic topology/ geometry technics, such as Riemann-Roch/Hurwitz Th., may be used on this characteristic zero "side" of number fields and lattice models of finite fields, in the natural and familiar framework of the complex numbers. This direction of continuing the combined study of finite fields in concrete applications to Algebraic Topology and Algebraic geometry, benefitting even more from an intuitive understanding via graphical representations, will be the subject of future faculty-student research projects.

On the concrete complementary side, *SAGE/CoCalc programs* [25] were specifically designed to allow for computer explorations of the presented topics of Algebraic Number Theory [18], which, in our opinion, constitute interesting studies, accessible for undergraduate research. Further programs for representing graphically the corresponding lattices, Frobenius orbits etc., are envisaged as further developments, with student help.

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Appendix. SAGE/CoCalc Programs

Programs for computing the Weil zeros of $EC: y^2 = x^3 + D$ are available from [18]. They can be easily adapted to other cases, for example to Riemann Surfaces or Fermat Curves. The programs can also be used to compute Jacobi sums, and for other Algebraic Number Theory studies using technology.

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Generalization of the Pecaric-Rajic Inequality in a Quasi-Banach Space

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Abstract

In the present paper, we shall give an extension of the well known Pecaric-Rajic inequality in a quasi-Banach space, we establish the generalized inequality for an arbitrary number of finitely many nonzero elements of a quasi-Banach space, and obtain the corresponding upper and lower bounds. As a result, we get some more general inequalities.

Keywords

Pecaric-Rajic Inequality, Dunkl-Williams Inequality, Triangle Inequality, Quasi-Banach Space

1. Introduction

Let us first recall some basic facts concerning quasi-Banach spaces and some preliminary results. For more information about quasi-Banach spaces, the readers can refer to [1].

Definition 1 *Let* X *be a linear space. A quasi-norm is a real-valued function on* X *satisfying the following:*

1. $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;
3. There is a constant $K \ge 1$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X.

A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p*-norm $(0 < p \le 1)$ if

$$\|x + y\|^p \le \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p*-Banach space.

Let X be a normed linear space. The following is the well known Dunkl-Williams inequality (see [2]), which states that the for any two nonzero elements $a,b\in X$,

$$\left\|\frac{a}{\|a\|}-\frac{y}{\|b\|}\right\|\le\frac{4\|a-b\|}{\|a\|+\|b\|}. \tag{1}$$

Many authors have studied this inequality over the years, and various refinements of this inequality (1) have been obtained (see e.g [3] [4] [5]). Pecaric and Rajic [6] got the following inequality in a normed linear space.

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|\le\min_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|a_i\|}\left(\left\|\sum_{j=1}^{n}a_j\right\|+\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|\right)\right\}, \tag{2}$$

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|\ge\max_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|a_i\|}\left(\left\|\sum_{j=1}^{n}a_j\right\|-\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|\right)\right\}. \tag{3}$$

Furthermore, the authors [6] also showed that these inequalities imply some refinements of the generalized triangle inequalities obtained by some authors. For generalized triangle inequalities, note that, some authors have also got many related results (see [7] [8]). In this paper, we shall discuss some extensions of the inequalities (2) and (3) for an arbitrary number of finitely many nonzero elements of a quasi-Banach space.

2. Main Results

Note that, given a p-norm, the formula $d(x,y):=\|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [9] (see also [1]), each quasi-norm is equivalent to some p-norm. Henceforth we can get similar results with p-norm. In the following, we first generalize the inequalities (2) and (3) with p-norm a p-Banach space.

Theorem 2 *Let X be a p-Banach space and $a_1,\cdots,a_n$ nonzero elements of X. Then we have*

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|^p\le\min_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|a_i\|^p}\left(\left\|\sum_{j=1}^{n}a_j\right\|^p+\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|^p\right)\right\}, \tag{4}$$

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|^p\ge\max_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|a_i\|^p}\left(\left\|\sum_{j=1}^{n}a_j\right\|^p-\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|^p\right)\right\}. \tag{5}$$

Proof. First, let us prove the inequality (4): for a fixed $i\in\{1,\cdots,n\}$, we have

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|^p=\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}+\sum_{j=1}^{n}\left(\frac{1}{\|a_j\|}-\frac{1}{\|a_i\|}\right)a_j\right\|^p$$

$$\le\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|^p+\sum_{j=1}^{n}\left|\frac{1}{\|a_i\|}-\frac{1}{\|a_j\|}\right|^p\|a_j\|^p=\frac{1}{\|a_i\|^p}\left(\left\|\sum_{j=1}^{n}a_j\right\|^p+\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|^p\right).$$

from this it follows that

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|^p \le \min_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|a_i\|^p}\left(\left\|\sum_{j=1}^{n}a_j\right\|^p+\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|^p\right)\right\},$$

which is the inequality (4). The second inequality (5) follows likewise and the details are omitted.

Now, we generalize the inequalities (2) and (3) with quasi-norm in a quasi-Banach space.

Theorem 3 *Let* X *be a quasi-Banach space and* $a_1,\cdots,a_n$ *nonzero elements of* X. *Then we have*

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\| \le \min_{i\in\{1,\cdots,n\}}\left\{\frac{C}{\|a_i\|}\left(\left\|\sum_{j=1}^{n}a_j\right\|+\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|\right)\right\}, \tag{6}$$

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\| \ge \max_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|a_i\|}\left(\left\|\sum_{j=1}^{n}\frac{a_j}{C}\right\|-\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|\right)\right\}. \tag{7}$$

where C is a constant and $C\ge 1$.

Proof. First, let us prove the inequality (6): for a fixed $i\in\{1,\cdots,n\}$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\| &= \left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}+\sum_{j=1}^{n}\left(\frac{1}{\|a_j\|}-\frac{1}{\|a_i\|}\right)a_j\right\| \\ &\le C_1\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|+C_1\left\|\sum_{j=1}^{n}\left(\frac{1}{\|a_j\|}-\frac{1}{\|a_i\|}\right)a_j\right\| \\ &\le C_1\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|+C_1C_2\left\|\left(\frac{1}{\|a_1\|}-\frac{1}{\|a_i\|}\right)a_j\right\|+C_1C_2\left\|\sum_{j=2}^{n}\left(\frac{1}{\|a_j\|}-\frac{1}{\|a_i\|}\right)a_j\right\|.\end{aligned}$$

where $C_i\ge 1, i=1,2$. Hence, in order to get the inequality (6), let us set $C=\prod_{j=1}^{n}C_j$, where $C_j\ge 1$ for all $1\le j\le n$. Thus, from the above inequality it follows that

$$\begin{aligned}\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\| &\le C\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|+C\sum_{j=1}^{n}\left|\frac{1}{\|a_i\|}-\frac{1}{\|a_j\|}\right|\|a_j\| \\ &= C\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|+C\sum_{j=1}^{n}\left|\frac{\|a_j\|}{\|a_i\|}-1\right| = \frac{C}{\|a_i\|}\left(\left\|\sum_{j=1}^{n}a_j\right\|+\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|\right).\end{aligned}$$

From this it follows that

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\| \le \min_{i\in\{1,\cdots,n\}}\left\{\frac{C}{\|a_i\|}\left(\left\|\sum_{j=1}^{n}a_j\right\|+\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|\right)\right\},$$

which is the inequality (6).

In order to proof the second inequality (7), we proceed in a similar way. For a fixed $i\in\{1,\cdots,n\}$, we get,

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|x_j\|}\right\|=\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}-\sum_{j=1}^{n}\left(\frac{1}{\|a_i\|}-\frac{1}{\|a_j\|}\right)a_j\right\|$$
$$\geq\frac{1}{C_1}\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|-\left\|\sum_{j=1}^{n}\left(\frac{1}{\|a_i\|}-\frac{1}{\|a_j\|}\right)a_j\right\|.$$

where $C_1\geq 1$. From this it follows that

$$C_1\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|\geq\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|-C_1\left\|\sum_{j=1}^{n}\left(\frac{1}{\|a_i\|}-\frac{1}{\|a_j\|}\right)a_j\right\|$$
$$\geq\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|-C_1C_2\left\|\left(\frac{1}{\|a_1\|}-\frac{1}{\|a_i\|}\right)a_j\right\|$$
$$-C_1C_2\left\|\sum_{j=2}^{n}\left(\frac{1}{\|a_j\|}-\frac{1}{\|a_i\|}\right)a_j\right\|.$$

where $C_i\geq 1, i=1,2$. Hence, in order to proof the inequality (7), let us set $C=\prod_{j=1}^{n}C_j$, where $C_j\geq 1$ for all $1\leq j\leq n$. Thus, from the above inequality it follows that

$$C\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|\geq\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_i\|}\right\|-C\sum_{j=1}^{n}\left|\frac{1}{\|a_j\|}-\frac{1}{\|a_i\|}\right|\|a_j\|$$
$$=\frac{1}{\|a_i\|}\left\|\sum_{j=1}^{n}a_j\right\|-\frac{C}{\|a_i\|}\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|.$$

Thus, from the above inequality we can get

$$\left\|\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}\right\|\geq\frac{1}{\|a_i\|}\left\|\sum_{j=1}^{n}\frac{a_j}{C}\right\|-\frac{1}{\|a_i\|}\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|$$
$$\geq\max_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|a_i\|}\left(\left\|\sum_{j=1}^{n}\frac{a_j}{C}\right\|-\sum_{j=1}^{n}\left|\|a_j\|-\|a_i\|\right|\right)\right\}.$$

This completes the proof.

3. Conclusion

In this paper we establish a generalisation of the so-called Pecaric-Rajic inequality by providing upper and lower bounds for the norm of the linear combination $\sum_{j=1}^{n}\frac{a_j}{\|a_j\|}$, where $a_1,\cdots,a_n$ nonzero elements of X. Furthermore, we also obtain the corresponding inequalities in a *p*-Banach space with *p*-norm. We should also indicate that when $C=1$ in Theorem 3, the inequalities (2) and (3) can be obtained as a particular case of the results established in Theorem 3. Thus, we get some more general inequalities.

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Factorization of Cyclotomic Polynomials with Quadratic Radicals in the Coefficients

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Abstract

In this article we continue the consideration of geometrical constructions of regular n-gons for odd n by rhombic bicompasses and ruler used in [1] for the construction of the regular heptagon ($n=7$). We discuss the possible factorization of the cyclotomic polynomial in polynomial factors which contain not higher than quadratic radicals in the coefficients whereas usually the factorization of the cyclotomic polynomials is considered in products of irreducible factors with integer coefficients. In considering the regular heptagon we find a modified variant of its construction by rhombic bicompasses and ruler. In detail, supported by figures, we investigate the case of the regular tridecagon ($n=13$) which in addition to $n=7$ is the only candidate with low n (the next to this is $n=769$) for which such a construction by rhombic bicompasses and ruler seems to be possible. Besides the coordinate origin we find here two points to fix for the possible application of two bicompasses (or even four with the addition of the complex conjugate points to be fixed). With only one bicompass one has in addition the problem of the trisection of an angle which can be solved by a neusis construction that, however, is not in the spirit of constructions by compass and ruler and is difficult to realize during the action of bicompasses. As discussed it seems that to finish the construction by bicompasses the correlated action of two rhombic bicompasses must be applied in this case which avoids the disadvantages of the neusis construction. Single rhombic bicompasses allow to draw at once two circles around two fixed points in such correlated way that the position of one of the rotating points on one circle determines the positions of all the other points on the second circle in unique way. The known case $n=17$ embedded in our method is discussed in detail.

Keywords

Geometrical Constructions by Compass and Ruler, Bicompasses, Cyclotomic Polynomials, Chebyshev Polynomials, 7-Gon, 13-Gon, 17-Gon, Fermat Numbers

1. Introduction

From ancient time on it was a problem of serious and of recreational mathematics which of the regular *n*-gons may be constructed by compass and ruler (straightedge without marks) and all "simple" constructions were known without a proof of the completeness of their possibilities up to the appearance of Gauss (in German: Gauß) on the scene at the very beginning of the 19-th century. Gauss showed that the basic numbers n for such constructions are the prime Fermat numbers $F_l = 2^{2^l} + 1, (l = 0,1,2,\cdots)$ with the long known cases corresponding to $n = F_0 = 3$ and $n = F_1 = 5$ and with the first unknown and surprising case at this time $n = F_2 = 2^{2^2} + 1 = 17$ (e.g., [2]-[13] and the more popular articles of Gardner [14] [15]). This results from the solution of the cyclotomic equations for these cases. A little later the general theory was developed for the solvability of polynomial equations with integer or rational coefficients in radicals (now called Galois theory) to which the cyclotomic equation is a special case. Some prehistory to this connected with names such as Lagrange, Ruffini and Abel is told by Stewart [5] (chap. 8). The construction by compass and ruler requires not higher than quadratic radicals. In [1] it was shown that the regular heptagon ($n = 7$) can be constructed by rhombic bicompasses and ruler. The rhombic bicompasses are two correlated compasses with, at least, 3 connected arms of equal length which can be fixed in two different points and which allow then the motion of the arms in two correlated circles around the fixed points with one degree of freedom. The addition of such bicompasses as device for geometric constructions is, in our persuasion, certainly in the spirit of the ancient geometers and extends our possibilities for constructions. Exact constructions with rhombic bicompasses are possible if the fixed points are determined by not higher than quadratic radicals (nested square roots) and, therefore, are constructible by compass and ruler. To apply rhombic bicompasses and ruler for the construction of regular *n*-gons it is necessary that the cyclotomic polynomials $\frac{z^n - 1}{z - 1}$ can be factorized into products of polynomial equations of 3-rd degree with not higher than quadratic radicals in their coefficients. In Sections 3, 9 and 11 we suggest arguments that this restricts the possible applications for odd n to prime numbers n equal to $n = G_l = 3 \cdot 2^{2^l} + 1$ in analogy to the Fermat numbers F_l that means to $n = 7$ for $l = 0$ and next to $n = 13$ for $l = 1$ and discuss these cases in detail.

In present article we investigate the factorizations of the cyclotomic polynomials for low odd n (up to $n = 19$) in polynomial factors of 3-cycles (polynomial equations of 3-rd degree with 3 involved roots) which contain not higher than quadratic radicals in the coefficients and explain how this can be obtained in explicit form. We also give in explicit form for low n the factorization with only quadratic radicals in the coefficients but with other than 3-cycles and determine some general rules for this. For odd order

$n = 2m+1, (m = 1, 2, \cdots)$ in a first step the factorization of the cyclotomic polynomial in a product of two polynomials of m-th degree with not higher than quadratic radicals in the coefficients is generally possible. Concerning the regular heptagon ($n = 7$) we add a further modification of its construction by rhombic bicompasses and ruler and we discuss in detail the interesting case $n = 13$ where some problems remain open. For the well-known case $n = 17$ we find in fully explicit form the factorization of the cyclotomic polynomial in 8- 4- and 2-cycles with nested quadratic radicals in the coefficients. For odd and even n we express the the polynomials for the determination of the Cosines of the angles of the circle-division problem by Chebyshev polynomials of first and second kind and derive more information about this in an appendix.

The cyclotomic polynomials $p_n(z)$ for the complex corner points of a regular n-gon to circumradius $r = 1$ at the coordinate origin as it is well known are

$$p_n(z) = z^n - 1, \quad \left(z \equiv x + \mathrm{i}y \equiv (x, y),\, z^* \equiv x - \mathrm{i}y \equiv (x, -y)\right). \tag{1.1}$$

The n complex solutions $z = z_k = \exp\left(\mathrm{i}k\frac{2\pi}{n}\right), \left(k = 0, 1, 2, \cdots, \text{modulo } n\right)$ of the cyclotomic equation

$$0 = z^n - 1 = (z-1)\left(z^{n-1} + z^{n-2} + \cdots + z + 1\right), \quad \left(z_0 \equiv z_n = 1\right), \tag{1.2}$$

solve the problem of the circle division into n equal sectors and determine the corner points

$$\begin{aligned} z_k &\equiv \exp\left(\mathrm{i}k\frac{2\pi}{n}\right), \quad z_k^{-1} = z_k^*, \\ z_k z_l &= z_{k+l} = z_1^{k+l}, \quad \left(k = 0, 1, 2, \cdots, (\operatorname{mod} n)\right), \end{aligned} \tag{1.3}$$

of the regular n-gon in the complex plane.

In the following we describe the procedure to obtain factorizations of the cyclotomic equation with not higher than quadratic radicals in the coefficients and give the explicit results for odd n up to $n = 19$. In particular, we discuss in detail the cases $n = 7$ and $n = 13$ which possess a relation to the application of bicompasses and ruler. In the case $n = 17$ which we also discuss in some detail we demonstrate how our method acts in a case known since Gauss. The results for all corner points of the regular 17-gon are given in an explicit form (see and compare also [9] [11] [12]).

2. The Cosine of the Angles for the Cyclotomic Polynomials in Odd Case $n = 2m+1$

We consider in this Section the case of the Cosine of the angles for the odd case $n = 2m+1, (m = 1, 2, \cdots)$ of the regular n-gon and introduce the Cosine of angles for z on the unit circle by

$$x = \frac{z + z^{-1}}{2} = \frac{z + z^*}{2} \equiv \cos(\theta). \tag{2.1}$$

The cyclotomic polynomials $p_{2m+1}(z)$ can be transformed to

$$\begin{aligned}\frac{p_{2m+1}(z)}{(z-1)z^m} &= z_0 + \sum_{k=1}^{m}\left(z^k + z^{-k}\right) = 1 + 2\sum_{k=1}^{m}\cos(k\theta) = 1 + 2\sum_{k=1}^{m}\mathrm{T}_k\left(\cos(\theta)\right) \\ &= 1 + 2\sum_{k=1}^{m}\mathrm{T}_k(x) = \mathrm{U}_m(x) + \mathrm{U}_{m-1}(x),\end{aligned} \tag{2.2}$$

where $\mathrm{T}_n(x)$ are the Chebyshev polynomials of first kind and $\mathrm{U}_n(x)$ the Chebyshev polynomials of second kind (e.g., [16] [17] [18]). The well-known property of the Chebyshev polynomials

$$\mathrm{T}_n\left(\cos(\theta)\right) = \cos(n\theta), \quad \mathrm{U}_n\left(\cos(\theta)\right) = \frac{\sin\left((n+1)\theta\right)}{\sin(\theta)}, \tag{2.3}$$

is used. The relation between $\mathrm{U}_n(x)$ and $\mathrm{T}_n(x)$ in the second line of (2.2) together with many other relations for the Chebyshev polynomials may be proved by complete induction using the addition theorems for the trigonometric functions.

For a few first polynomials $\mathrm{U}_m(x) + \mathrm{U}_{m-1}(x)$ concerning the odd regular *n*-gons ($n = 2m+1$) one finds explicitly together with possible factorizations with integer or rational coefficients (*i.e.* in $\mathbb{Z}$ or $\mathbb{Q}$) in case of prime or composite *n*:

One peculiarity is that the polynomials $\mathrm{U}_m(x) + \mathrm{U}_{m-1}(x)$ take on their simplest form with lowest integer coefficients by the substitution $x = u/2$ (similarly to the polynomials $\mathrm{U}_m(x)$ themselves; but not $\mathrm{T}_n(x)$). These polynomials possess factorizations with integer coefficients for composite numbers $n = 2m+1$. The polynomials $\mathrm{U}_n(z) + \mathrm{U}_{n-1}(z)$ are sometimes denoted by $\mathrm{V}_n(z)$ (see Appendix A).

The Cosines $\cos(k\theta_1) = \cos\left(k\frac{2\pi}{n}\right)$ of the angles $k\theta_1$ are obtained from the solutions $x = \cos(\theta)$ of the equation

Table 1. Cyclotomic polynomials for odd *n* and variable $x = \cos(\theta) \equiv \frac{u}{2}$ and integer factorization.

n	m	$\mathrm{U}_m\left(\frac{u}{2}\right) + \mathrm{U}_{m-1}\left(\frac{u}{2}\right);\quad u \equiv 2x = 2\cos(\theta)$
1	0	1
3	1	$u+1$
5	2	u^2+u-1
7	3	u^3+u^2-2u-1
9	4	$u^4+u^3-3u^2-2u+1$
		$=(u+1)(u^3-3u+1)$
11	5	$u^5+u^4-4u^3-3u^2+3u+1$
13	6	$u^6+u^5-5u^4-4u^3+6u^2+3u-1$
15	7	$u^7+u^6-6u^5-5u^4+10u^3+6u^2-4u-1$
		$=(u+1)(u^2+u-1)(u^4-u^3-4u^2+4u+1)$
17	8	$u^8+u^7-7u^6-6u^5+15u^4+10u^3-10u^2-4u+1$
19	9	$u^9+u^8-8u^7-7u^6+21u^5+15u^4-20u^3-10u^2+5u+1$

(2.4)

$$\frac{p_{2m+1}(z)}{(z-1)z^m} = \mathrm{U}_m(x) + \mathrm{U}_{m-1}(x) = 0, \quad \left(x \equiv \frac{z+z^*}{2} \equiv \cos(\theta)\right), \quad (2.5)$$

with exclusion of the solution $z_0 = x_0 = 1$. From the geometrical meaning of the polynomials as providing the doubled Cosines of the angle division problem as roots for odd $n = 2m+1$ within the unit circle it is fully obvious that all roots of all polynomials in (2.4) possess only m real solutions within the limits $-1 \le x = u/2 \le 1$.

Without further going into details we mention that in case of even numbers $n = 2m+2$ the cyclotomic polynomials for the Cosines of the angles $k\theta$ can be represented in the form (we use $\mathrm{T}_{-k}(x) = \mathrm{T}_k(x)$)

$$\begin{aligned} \frac{p_{2m+2}(z)}{(z^2-1)z^m} &= \sum_{l=0}^{m} z^{m-2l} = \sum_{l=0}^{m} \frac{z^{m-2l}+z^{2l-m}}{2} = \sum_{l=0}^{m} \cos((m-2l)\theta) \\ &= \sum_{l=0}^{m} \mathrm{T}_{m-2l}(\cos(\theta)) = \sum_{l=0}^{m} \mathrm{T}_{m-2l}(x) = \mathrm{U}_m(x), \quad \left(x \equiv \frac{z+z^*}{2} \equiv \cos(\theta)\right). \end{aligned} \quad (2.6)$$

The vanishing of these polynomials provides as solutions the possible Cosines of the angles to the corners of the n-gons with even $n = 2m+2$ with exclusion of the already eliminated Cosines $x_0 = 1, x_{m+1} = -1$.

We mention here that as (irreducible) cyclotomic polynomials $\Phi_n(z)$ are mostly understood the polynomials $p_n(z) = z^n - 1$ divided by all products of (irreducible) polynomials $\Phi_d(z)$ where $d < n$ runs over all divisors of n (*i.e.*, irreducible in the sense of coefficients in $\mathbb{Q}$ or in $\mathbb{Z}$ but likely here already in $\mathbb{Z}$) (e.g., van der Waerden [3], Stillwell [7], Shkolnik [11] (p. 40)). They are for $n \ge 2$ palindromic polynomials (only $\Phi_1(z) = z-1$ is not palindromic) with real coefficients[1]. General explicit formulae for the polynomials $\Phi_n(z)$ seem to be possible for different divisibility classes and for prime $n = p$ it is $\Phi_n(z) = \frac{p_n(z)}{z-1} = \sum_{k=0}^{p-1} z^k$ that means they are then of degree $p-1$. Stillwell [7] (p. 70) mentions as a curious property of the polynomials $\Phi_n(z)$ that the first polynomial $\Phi_n(z)$ with coefficients of modulus 2 besides 1 (and 0) is $\Phi_{105}(z)$ of degree 48 whereas all polynomials $\Phi_n(z)$ with $n < 105$ possess coefficients only of modulus 1 or equal to 0[2].

In connection with constructions by rhombic bicompasses and ruler we are mainly interested in factorizations of the polynomials $\frac{p_n(z)}{z-1}$ with not higher than (in general, nested) quadratic radicals in the coefficients. A general explicit

[1]By definition, a polynomial $P_m(z) = \sum_{k=0}^{m} a_k z^k$ is palindromic if for all coefficients holds $a_k = a_{m-k}$.

[2]With a PC and program Mathematica one may astonishingly easily and quickly calculate the polynomials $\Phi_n(z)$ for "high" n in explicit form by factorization of $p_n(z) = z^n - 1$. It seems that such polynomials with coefficients of modulus $\neq 1$ preferably appear if the composite numbers n are products of primes 3, 5, 7, 11, ... (e.g, $n = 105 = 3\times5\times7$) but not for all such products (not besides for $n < 105$, e.g., for $n = 231$ but for, e.g., $n = 165, 195, 385$). For $n = 1155 = 3\times5\times7\times11$ the polynomial $\Phi_{1155}(z)$ contains a lot of coefficients of modulus 1, 2, 3 or equal to 0 and is of degree 480.

formula for the factorized polynomials could not be obtained but a procedure will be described how such factorization in given cases leads to the result.

3. Cycles in the Circle-Division Problem

We explain in this Section the factorization of the cyclotomic equation for odd $n=2m+1$ and concentrate us to the case $n=p$ where p is a prime number larger than $p=2$. A solution $z_k=\exp\left(\mathrm{i}k\frac{2\pi}{n}\right)$ of the cyclotomic equation $z^n-1=0$ is called a primitive root if there does not exist a positive integer $m<n$ for which $z_k^m=1$. It is clear that in each case $z_1=\exp\left(\mathrm{i}\frac{2\pi}{n}\right)$ is a primitive root and that for prime numbers $n=p$ all solutions z_k with $k=1,2,\cdots,n-1,(\mathrm{mod}\, n)$ are primitive roots (only $z_0=1$ is never primitive). For prime $n=p$ the cyclotomic polynomial in the form $\frac{p_n(z)}{z-1}$ cannot be factorized into polynomials with integer coefficients. It is said that it is irreducible in $\mathbb{Q}$ and therefore also in $\mathbb{Z}$ with coefficient of the highest power equal to 1. Obviously, this does not mean that it is not factorizable into polynomials with radicals in the coefficient from which such are interesting for us which contain not higher than quadratic radicals since such radicals are constructible by compass and ruler. To obtain such factorizations one may apply a procedure using the little theorem of Fermat. We explain this in the following.

According to the little theorem of Fermat (e.g., [3] [8] [9]) for prime numbers p and natural numbers $g=1,2,\cdots,(\mathrm{mod}\, p)$ holds (symbol $\equiv$ stands here for congruences modulo p)

$$g^{p-1}\equiv 1,\quad (\mathrm{mod}\, p). \tag{3.1}$$

This means that for prime numbers $n=p$ and for the primitive root z_1 we have

$$z_1^{g^{n-1}}=z_{g^{n-1}}=z_1,\quad \Rightarrow z_k^{g^{n-1}}=z_1^{kg^{n-1}}=z_1^k=z_k. \tag{3.2}$$

We choose positive integers $g=1,2,\cdots,n-1$ and form first with the solution z_1 the sequences $\left(z_1^{g^0}=z_{g^0}=z_1,z_1^{g^1}=z_{g^1},z_1^{g^2}=z_{g^2},\cdots,z_1^{g^l}=z_{g^l}\right)$ up to the case when $z_1^{g^{l+1}}=z_{g^{l+1}}=z_1,(\mathrm{mod}\, n)$. Each such sequence we call a cycle. The sequence $\left(z_1,z_{g^1},z_{g^2},\cdots,z_{g^{n-1}}\right)$, latest after the next step to z_{g^n}, leads back to the initial solution z_1. If l is a divisor of $n-1=2m$ then, depending on the choice of g, the mentioned sequence may lead back to z_1 already after $l+1$ steps. For odd prime $n=2m+1$ the numbers $l_1=2,l=2m$ and $l_2=m$ are divisors of $n-1=2m$ and there are sequences with cycle lengths 2 belonging to the choice $g=2m$ and such with cycle lengths m and $2m$. If one does not begin with the element z_1 in a cycle but with another element of this cycle then one obtains by the described procedure the same cycle with rotated order of the elements. If a root z_k is not contained in a certain cycle then we form the sequences $\left(z_k^{g^0} = z_{kg^0} = z_k, z_k^{g^1} = z_{kg}, z_k^{g^2} = z_{kg^2}, \cdots, z_k^{g^l} = z_{kg^l}\right)$ and obtain for prime n an equivalent cycle of the same length and so we may continue up to the case when all numbers $z_k \neq z_0$, $(\bmod n$ are comprised.

We illustrate the factorization in cycles for our two most interesting prime cases $n=7$ and 13:

1. Case $n=7$, basic cycles (all equalities are modulo 7 in the indices)

$$
\begin{gathered}
g=1: z_1,\left(z_1=z_1\right), \\
g=2: z_1, z_2, z_4,\left(z_8=z_1\right), \\
g=3: z_1, z_3, z_9=z_2, z_6, z_{18}=z_4, z_{12}=z_5,\left(z_{15}=z_1\right), \\
g=4: z_1, z_4, z_{16}=z_2,\left(z_8=z_1\right), \\
g=5: z_1, z_5, z_{25}=z_4, z_{20}=z_6, z_{30}=z_2, z_{10}=z_3,\left(z_{15}=z_1\right), \\
g=6: z_1, z_6,\left(z_{36}=z_1\right).
\end{gathered}
\tag{3.3}
$$

For $g=3$ and $g=5$ the sequences are 6-cycles which comprise all $n-1$ solutions $z_k \neq z_0=1$ in different order of the elements. For $g=2$ and $g=4$ we find equivalent 3-cycles in different order of the elements $\left(z_1, z_2, z_4\right)$ which can be complemented by the 3-cycle $\left(z_3, z_6, z_5=z_{12}\right)$ to comprise all elements $z_k \neq z_0$. For $g=n-1=6$ we find the 2-cycle $\left(z_1, z_6\right)$ which can be complemented by the other possible 2-cycles $\left(z_2, z_5\right)$ and $\left(z_3, z_4\right)$ for which one does not find a factorization in polynomials of 2-nd degree with only quadratic radicals in the coefficients. In the trivial case $g=1$ one finds in every case only the 1-cycle with element $\left(z_1\right)$ which can be complemented by 1-cycles $\left(z_k\right)$ of the other roots.

2. Case $n=13$, basic cycles (all equalities are modulo 13 in the indices)

$$
\begin{gathered}
g=1: z_1,\left(z_1=z_1\right), \\
g=2: z_1, z_2, z_4, z_8, z_{16}=z_3, z_6, z_{12}, z_{24}=z_{11}, z_{22}=z_9, \\
z_{18}=z_5, z_{10}, z_{20}=z_7,\left(z_{14}=z_1\right), \\
g=3: z_1, z_3, z_9,\left(z_{27}=z_1\right), \\
g=4: z_1, z_4, z_{16}=z_3, z_{12}, z_{48}=z_9, z_{36}=z_{10},\left(z_{40}=z_1\right), \\
g=5: z_1, z_5, z_{25}=z_{12}, z_{60}=z_8,\left(z_{40}=z_1\right), \\
g=6: z_1, z_6, z_{36}=z_{10}, z_{60}=z_8, z_{48}=z_9, z_{54}=z_2, z_{12}, z_{72}=z_7, \\
z_{42}=z_3, z_{18}=z_5, z_{30}=z_4, z_{24}=z_{11},\left(z_{66}=z_1\right), \\
g=7: z_1, z_7, z_{49}=z_{10}, z_{70}=z_5, z_{35}=z_9, z_{63}=z_{11}, z_{77}=z_{12}, z_{84}=z_6, \\
z_{42}=z_3, z_{21}=z_8, z_{56}=z_4, z_{28}=z_2,\left(z_{14}=z_1\right), \\
g=8: z_1, z_8, z_{64}=z_{12}, z_{96}=z_5,\left(z_{40}=z_1\right), \\
g=9: z_1, z_9, z_{81}=z_3,\left(z_{27}=z_1\right), \\
g=10: z_1, z_{10}, z_{100}=z_9, z_{90}=z_{12}, z_{120}=z_3, z_{30}=z_4,\left(z_{40}=z_1\right),
\end{gathered}
$$

$$g=11: z_1, z_{11}, z_{121}=z_4, z_{44}=z_5, z_{55}=z_3, z_{33}=z_7, z_{77}=z_{12}, z_{132}=z_2,$$
$$z_{22}=z_9, z_{99}=z_8, z_{88}=z_{10}, z_{110}=z_6, (z_{66}=z_1),$$
$$g=12: z_1, z_{12}, (z_{144}=z_1). \tag{3.4}$$

We have here cycles of lengths $1,2,3,4,6,12$ which all are divisors of $n-1=2m=12$. For example, we find the following four 3-cycles $\{(z_1,z_3,z_9),(z_4,z_{10},z_{12})\},\{(z_2,z_5,z_6),(z_7,z_8,z_{11})\}$ covering all primitive roots where the two pairs of 3-cycles in braces form two 6-cycles. The 3-cycles follow from the subdivision of the 12-cycles in two step by division of 2 leading first to two 6-cycles and then in last step by division of 2 to four 3-cycles. The subdivision of the cyclotomic equation of 12-th degree in a product of six 2-cycles with quadratic equations containing 6 paired roots $\left(z_k, z_{-k}=z_k^* \equiv z_{13-k}, (k=1,2,\cdots,6)\right)$ does not lead to the explicit form of the quadratic equations since the resolution of the 12-degree cyclotomic polynomial in one step by division of 3 is not possible with coefficients in form of quadratic radicals independently from the order in which the division by 3 is made, from $12\to 4\to 2$ or $12\to 6\to 2$. Therefore the three 4-cycles $(z_1,z_5,z_8,z_{12}),(z_2,z_3,z_{10},z_{11}),(z_4,z_6,z_7,z_9)$ obtained from choice $g=5$ and $g=8$ are also not to find in form of polynomial equations with only quadratic radicals as coefficient. Each of these three 4-cycles contains only one of the roots of the four 3-cycles.

If we look to the cycles in (3.3) and (3.4) we find in case of $g\neq 1$ for the sum of the powers of g within the cycle

$$\sum_{j=0}^{l} g^j = \frac{g^{l+1}-1}{g-1} \equiv n, \bmod n, \quad (g\neq 1). \tag{3.5}$$

This is a general property which we will prove now. According to the definition of a cycle of length l the power g^l is the last in the cycle before the next power g^{l+1} leads back the root z_1 to $z_{g^{l+1}}=z_1$ and g^{l+1} is congruent to 1 modulo *n*. Since the sum on the left-hand side in (3.5) is a positive integer and $g-1\neq 0$ is also a positive integer the right-hand side in (3.5) is a positive integer and due to the given congruence a multiple of *n*. From this follows for the product of roots within a cycle with the primitive root z_1

$$\prod_{j=0}^{l} z_{g^j} = z_1 z_g z_{g^2} \cdots z_{g^l} = z_1^{\sum_{j=0}^{l} g^j} = z_1^{\frac{g^{l+1}-1}{g-1}} = z_{\frac{g^{l+1}-1}{g-1}} = z_0 = 1. \tag{3.6}$$

The same is the case with each cycle of the length l containing an arbitrary primitive root z_k. A consequence is that we know at once the constant term in the factorized cyclotomic polynomials that is of importance when we begin from behind (low-order *k*-terms proportional to z^k) to find the factor polynomials in factorizations.

The question about the cyclotomic polynomials $\frac{p_{2m+1}(z)}{z-1}$ of degree $2m$

Table 2. Fermat numbers F_l and related numbers G_l and factorization into prime numbers.

l	0	1	2	3	4	5	
$F_l \equiv 2^{2^l}+1$	3	5	17	257	65537	4294967297	(3.7)
prime factors						$641 \cdot 6700417$	
$G_l \equiv 3 \cdot 2^{2^l}+1$	7	13	49	769	196609	12884901889	
prime factors			7^2		$7 \cdot 28087$	$19^2 \cdot 35692249$	

which can be split into products of polynomials of 3-rd degree (3-cycles) leads basically, analogously to Fermat numbers $F_l = 2^{2^l}+1$, to numbers $G_l = 3 \cdot 2^{2^l}+1$ which have to be prime numbers. Table 2 shows the few initial possibilities up to $l = 5$.

For the next three cases $l = 6,7,8$ the numbers F_l and G_l are composite numbers as the computer shows but these numbers grow very fast and my PC (with program "Mathematica 10") did not provide a result for the next case $l = 9$ of both numbers in acceptable time. However, it is now known that all numbers F_l from $l = 9$ up to $l = 19$ are composite without knowing all prime factors in all these cases (see [9], end of chap. 5). Since the number $G_2 = 49$ is composite it is not a possible candidate for construction of the regular 49-gon by rhombic bicompasses and ruler.

We mention here that if one admits angle trisection by a neusis construction attributed to Archimedes [2] [9] as an additional element of constructions which in our persuasion is not in the spirit of ancient constructions by compass and ruler then one comes to possible numbers for the solubility of the circle division problem of the form $P_{k,l} = 2^k 3^l + 1, (k,l = 0,1,2,\cdots)$ if they are prime numbers and which are called Pierpont numbers (from 1895, see [19] [20]). These numbers are more general ones than the Fermat numbers F_l and also than the numbers G_l in Table 2 (see also end of Section 11).

4. Factorization of Cyclotomic Polynomial for *n* = 3 with Real Coefficients

In case of $n = 3$ the cyclotomic polynomial $\frac{p_3(z)}{z-1}$ can be represented

$$\frac{z^3-1}{z-1} = z^2+z+1 = \left(z+\frac{1-\mathrm{i}\sqrt{3}}{2}\right)\left(z+\frac{1+\mathrm{i}\sqrt{3}}{2}\right) = (z-z_1)(z-z_2). \quad (4.1)$$

It is written down here for the analogy to higher less trivial cases.

5. Factorization of Cyclotomic Polynomial for *n* = 5 with Quadratic Radicals

In case of $n = 5$ the cyclotomic polynomial $\frac{p_5(z)}{z-1}$ possesses the factorization with two 2-cycles which has the form

$$\begin{aligned}\frac{z^5-1}{z-1} &= z^4+z^3+z^2+z+1\\ &= \left(z^2+\frac{1-\sqrt{5}}{2}z+1\right)\left(z^2+\frac{1+\sqrt{5}}{2}z+1\right)\\ &= \big((z-z_1)(z-z_4)\big)\big((z-z_2)(z-z_3)\big).\end{aligned}\tag{5.1}$$

The complex conjugate roots (z_1, z_4) and (z_2, z_3) modulo 5 are here paired in one of the two 2-cycles and therefore the coefficients in form of quadratic radicals possess real values. The case of the regular pentagon is also commonly known and we do not consider it in detail.

6. Factorization of Cyclotomic Polynomial for $n = 7$ with Quadratic Radicals

In case of $n=7$ the cyclotomic polynomial $\frac{p_7(z)}{z-1}$ possesses the following factorization with two 3-cycles with only quadratic radicals in the coefficients

$$\begin{aligned}\frac{z^7-1}{z-1} &= z^6+z^5+z^4+z^3+z^2+z+1\\ &= \left(z^3+\frac{1-\mathrm{i}\sqrt{7}}{2}z^2-\frac{1+\mathrm{i}\sqrt{7}}{2}z-1\right)\left(z^3+\frac{1+\mathrm{i}\sqrt{7}}{2}z^2-\frac{1-\mathrm{i}\sqrt{7}}{2}z-1\right)\\ &= \big((z-z_1)(z-z_2)(z-z_4)\big)\big((z-z_3)(z-z_5)(z-z_6)\big).\end{aligned}\tag{6.1}$$

The factorization in this case with only quadratic radicals in the coefficients and concerning the corners of the regular heptagon was discussed in [1]. It is easy to determine this factorization with quadratic radicals in the coefficients from $z_1+z_2+z_4=-\frac{1}{2}+\mathrm{i}y$ and $z_6+z_5+z_3=z_1^*+z_2^*+z_4^*=-\frac{1}{2}-\mathrm{i}y$ with imaginary part y to find from the polynomial using $\sum_{k=1}^{6} z_k=-1$ and $z_1z_2z_4=z_1^{1+2+4}=1$. The "primitive" roots $z_k, (k=1,\cdots,6)$ of the polynomials (6.1) form two 3-cycles (z_1,z_2,z_4) and $(z_6, z_{12}=z_5, z_{24}=z_{10}=z_3)$ modulo 7 in the indices. The quadratic radical for the sum $z_1+z_2+z_4$ together with the circle division in 7 equal parts is shown in **Figure 1**.

In **Figure 2** we illustrate possibilities for the construction of the regular heptagon by bicompasses and ruler and draw additionally a circle with radius $r_2=\sqrt{2}$ around the coordinate origin $(0,0)$ which is equal to the distance from the coordinate origin to the fixed point $\left(-\frac{1}{2},\frac{\sqrt{7}}{2}\right)$ of the bicompasses. This circle possesses intersection points with the circle of radius $r_1=1$ around the mentioned second fixpoint $\frac{-1+\mathrm{i}\sqrt{7}}{2}$ at $z=\frac{1+\mathrm{i}\sqrt{7}}{2}=1+z_1+z_2+z_4$ and at $z=\frac{-5+\mathrm{i}\sqrt{7}}{4}$ in the complex plane as one easily calculates. In this way we see that the intersection point $\frac{1+\mathrm{i}\sqrt{7}}{2}$ lies on the second arm of the absolute mini-

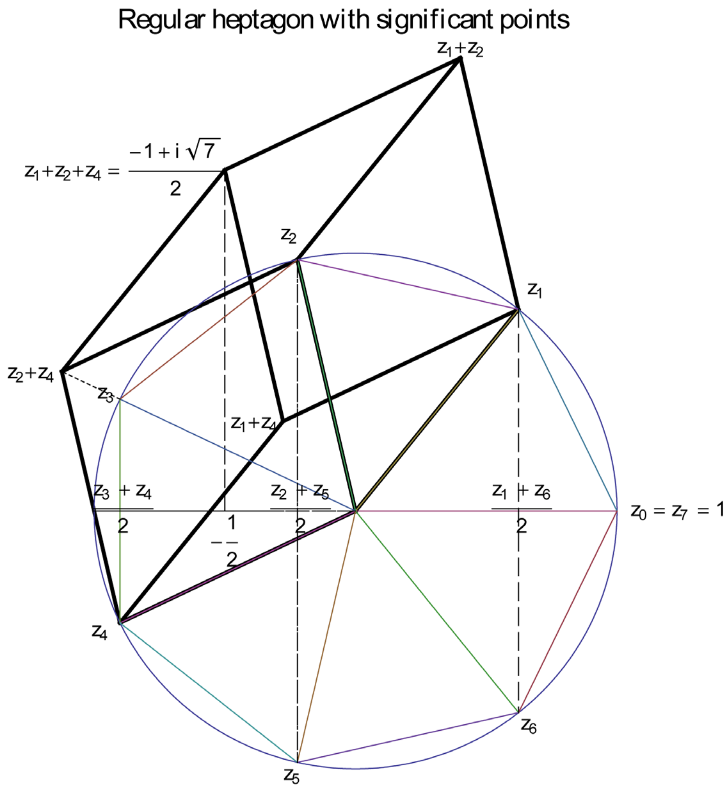

Figure 1. Regular heptagon with axes projection of corners in complex z-plane. Besides the coordinate origin $z=0$ the second fixed point for the rhombic bicompasses is $z=\frac{-1+\mathrm{i}\sqrt{7}}{2}$ (figure from [1] made by "Mathematica 6").

mum 3-arm bicompasses (see [1]) in the right position for the construction of the regular heptagon that means on the line between z_1 and z_1+z_2. Clearly, one has not only to believe to the optical impression but have to prove it.

The line between z_1 and z_1+z_2 can be parameterized by

$$z=z_1+rz_2,\ \left(0\le r\le 1,\, r=r^*\right), \tag{6.2}$$

with real parameter r. For the parameter value $r=\frac{1+z_2+z_4}{z_2}$ (numerically $r\approx 0.554958$) one finds that the value z on the line between z_1 and z_1+z_2 coincides with the point $\frac{1+\mathrm{i}\sqrt{7}}{2}$ according to

$$\begin{aligned} r&=\frac{1+z_2+z_4}{z_2}=z_2^{-1}+1+z_2=z_2^*+1+z_2=r^*,\\ \Rightarrow z&=z_1+\frac{1+z_2+z_4}{z_2}z_2=1+z_1+z_2+z_4=\frac{1+\mathrm{i}\sqrt{7}}{2}. \end{aligned} \tag{6.3}$$

Thus we have to bring the second arm of the absolute minimum 3-arm bicompasses in the position that it intersects the point $\left(\frac{1}{2},\frac{\sqrt{7}}{2}\right)$ which last can be constructed by the intersection of the two mentioned circles and an alternative method of construction of the regular heptagon is described.

In addition, Figure 2 shows that the point $z=\frac{1+\mathrm{i}\sqrt{7}}{2}$ lies also on the prolongation of the line between z_2+z_4 and z_2. This line can be parameterized with real parameter s by

$$z=z_2+sz_4,\quad \left(0\le s\le 1,\ s=s^*\right). \tag{6.4}$$

For the parameter value $s=\frac{1+z_1+z_4}{z_4}$ (numerically $s\approx -0.801938$) one finds that the value z on the prolongation of the line between z_2+z_4 and z_2

Regular heptagon with additional circle of radius $r_2=\sqrt{2}$

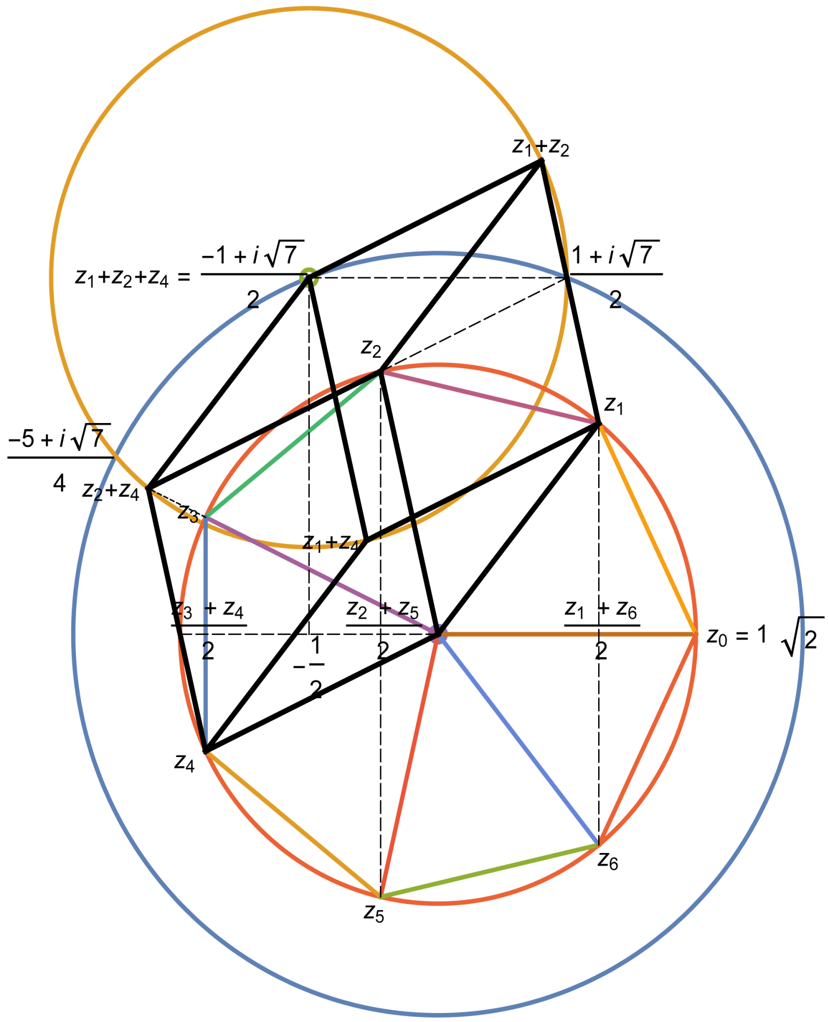

Figure 2. Regular heptagon and construction with rhombic bicompasses and ruler in complex z-plane. Additionally to the fixed points for the bicompasses and corresponding circles of radius $r_1=1$ we have drawn a circle around the coordinate origin with radius $r_2=\sqrt{2}$ and obtain in this way a modified construction by absolute minimum bicompasses with 3 arms (figure made as all following figures by "Mathematica 10").

coincides with the point $\frac{1+\mathrm{i}\sqrt{7}}{2}$ according to

$$s=\frac{1+z_1+z_4}{z_4}=z_4^{-1}+z_4+1=z_4^*+z_4+1=s^*,$$

$$\Rightarrow z=z_2+\frac{1+z_1+z_4}{z_4}z_4=1+z_1+z_2+z_4=\frac{1+\mathrm{i}\sqrt{7}}{2}, \tag{6.5}$$

that affirms the mentioned intersection.

In [1] it was already shown that the point $z=-1$ lies on the line between z_4 and z_2+z_4 which can be parameterized by

$$z=z_4+tz_2, \quad \left(0\le t\le 1, t=t^*\right), \tag{6.6}$$

with real parameter *t*. With the parameter value $t=-\frac{1+z_4}{z_2}$ (numerically $t=0.445042$) follows

$$t=-\frac{1+z_4}{z_2}=-\left(z_2^*+z_2\right)=t^*,$$

$$\Rightarrow z=z_4-\frac{1+z_4}{z_2}z_2=z_4-1-z_4=-1, \tag{6.7}$$

that proves the statement. This, alternatively, can be also used for the construction of the regular heptagon by rhombic bicompasses and ruler. We mention that the parameters with the notation *r* and *t* are essentially the same since $r+t=1$.

The equation for the Cosines $u=z+z^*=2x=2\cos(\theta)$ (equation for $n=7$ in **Table 1**, Equation (2.4))

$$\mathrm{U}_3\left(\frac{u}{2}\right)+\mathrm{U}_2\left(\frac{u}{2}\right)=u^3+u^2-2u-1=0, \tag{6.8}$$

is a 3-rd degree equation which cannot be solved only in quadratic radicals as it is known and its solution involves (complex) cubic radicals. Therefore, this does not help for the construction by compass and ruler. However, with other means of construction (e.g., neusis construction of angle trisection [2] [9] [14] and Gleason [13] (angle *p*-section)) this becomes possible.

Thus the regular heptagon loses a little its horror as not constructible by compass and ruler between the cases of the regular trigon $n=3$ and the regular octagon $n=8$ since it is constructible by bicompasses and ruler.

7. Factorization of Cyclotomic Polynomial for *n* = 9 in Different Ways

Since $n=9$ is a composite number we find different favorable factorizations of the cyclotomic polynomials. As special roots the circular division of the unit circle in 9 equal sectors contains the third roots of unity and we have the factorization

$$\begin{aligned}\frac{z^9-1}{z-1} &= z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1\\ &= \left(z^2+z+1\right)\left\{z^6+z^3+1\right\}\\ &= \left(z+\frac{1-\mathrm{i}\sqrt{3}}{2}\right)\left(z+\frac{1+\mathrm{i}\sqrt{3}}{2}\right)\left\{\left(z^3+\frac{1-\mathrm{i}\sqrt{3}}{2}\right)\left(z^3+\frac{1+\mathrm{i}\sqrt{3}}{2}\right)\right\}\\ &= \left(z-z_3\right)\left(z-z_6\right)\left\{\left(\left(z-z_1\right)\left(z-z_4\right)\left(z-z_7\right)\right)\left(\left(z-z_2\right)\left(z-z_5\right)\left(z-z_8\right)\right)\right\},\end{aligned} \tag{7.1}$$

where the second factor in braces is of the form of the first factor with the substitution $z \rightarrow z^3$. It requires the angle trisections of the angles to the roots $z_3 = z_1^3 = \frac{-1+\mathrm{i}\sqrt{3}}{2}$ and $z_6 = z_2^3 = \frac{-1-\mathrm{i}\sqrt{3}}{2} = z_3^*$. This is possible by the neusis construction known from ancient time [2] [9] which, however, is not in the spirit of constructions by compass and ruler. The last is impossible for almost all arbitrary angles including the angle $\frac{2\pi}{3}$. Therefore, as known, the circle division problem in case of $n=9$ cannot be solved by compass and ruler since the third root of an arbitrary complex number (here of $z_{3,6} = \frac{-1\pm\mathrm{i}\sqrt{3}}{2}$) cannot be constructed in this way. This can be also seen from the equation of **Table 1** (Equation (2.4)) for the doubled Cosines $u_k = 2\cos\left(k\frac{2\pi}{9}\right)$, $\left(k=1,2,\cdots,8\right)$ as the solutions of the following polynomial equation in factorized form

$$\mathrm{U}_4\left(\frac{u}{2}\right)+\mathrm{U}_3\left(\frac{u}{2}\right) = \left(u+1\right)\left(u^3-3u+1\right) = 0. \tag{7.2}$$

From the coefficients of the vanishing cubic polynomial follows

$$\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)=0,\quad \cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)\cos\left(\frac{8\pi}{9}\right)=-\frac{1}{8}, \tag{7.3}$$

where the second relation does not provide independent in formation in comparison to the first. The cubic equation $u^3-3u+1=0$ can be solved by cubic but not by quadratic radicals alone.

The 'standard' factorization into two polynomials of 4-th degree is

$$\begin{aligned}\frac{z^9-1}{z-1} &= \left\{z^4+\frac{1-\mathrm{i}\sqrt{3}}{2}z^3+\frac{1+\mathrm{i}\sqrt{3}}{2}z+1\right\}\left\{z^4+\frac{1+\mathrm{i}\sqrt{3}}{2}z^3+\frac{1-\mathrm{i}\sqrt{3}}{2}z+1\right\}\\ &= \left\{\left(z-z_6\right)\left(\left(z-z_1\right)\left(z-z_4\right)\left(z-z_7\right)\right)\right\}\left\{\left(z-z_3\right)\left(\left(z-z_2\right)\left(z-z_5\right)\left(z-z_8\right)\right)\right\}.\end{aligned} \tag{7.4}$$

In the factor polynomials of 4-th degree are contained two 3-cycles $\left(z_1, z_4, z_7\right)$ and $\left(z_2, z_8, z_{32} \equiv z_5\right)$ paired with one of the third roots z_3 and z_6 of unity. Other genuine than 3- and 6-cycles do not exist in case of $n=9$ but it happens that the root $z_0=1$ appears in the determination of the cycles according to the general procedure (it is then no more a cycle) that for prime n is impossible.

A similar interestingly simple factorization by two polynomials of 4-th degree follows directly from (7.1) by the product

$$\frac{z^9-1}{z-1}=\left\{z^4+\frac{1-\mathrm{i}\sqrt{3}}{2}\left(z^3+z\right)-\frac{1+\mathrm{i}\sqrt{3}}{2}\right\}\left\{z^4+\frac{1+\mathrm{i}\sqrt{3}}{2}\left(z^3+z\right)-\frac{1-\mathrm{i}\sqrt{3}}{2}\right\} \\ =\left\{\left(z-z_3\right)\left(\left(z-z_1\right)\left(z-z_4\right)\left(z-z_7\right)\right)\right\}\left\{\left(z-z_6\right)\left(\left(z-z_2\right)\left(z-z_5\right)\left(z-z_8\right)\right)\right\}. \tag{7.5}$$

and means the exchange of the factors $z-z_3$ and $z-z_6$. It is possible due to $n=9$ as a composite number.

8. Factorization of Cyclotomic Polynomial for *n* = 11 with Quadratic Radicals

In case of $n=11$ one has only the following factorization by 5-cycles which leads to polynomials with quadratic radicals in the coefficients

$$\begin{aligned}\frac{z^{11}-1}{z-1}&=z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1\\&=\left(z^5+\frac{1-\mathrm{i}\sqrt{11}}{2}z^4-z^3+z^2-\frac{1+\mathrm{i}\sqrt{11}}{2}z-1\right)\\&\quad\cdot\left(z^5+\frac{1+\mathrm{i}\sqrt{11}}{2}z^4-z^3+z^2-\frac{1-\mathrm{i}\sqrt{11}}{2}z-1\right)\\&=\left(\left(z-z_1\right)\left(z-z_3\right)\left(z-z_4\right)\left(z-z_5\right)\left(z-z_9\right)\right)\\&\quad\cdot\left(\left(z-z_2\right)\left(z-z_6\right)\left(z-z_7\right)\left(z-z_8\right)\left(z-z_{10}\right)\right).\end{aligned} \tag{8.1}$$

The 5-cycle in the first factor is formed by the roots $\left(z_1,z_3,z_9,z_{27}=z_5,z_{15}=z_4,\left(z_{12}=z_1\right)\right)$ and the second factor by the complex conjugate roots $\left(z_2,z_6,z_{18}=z_7,z_{21}=z_{10},z_{30}=z_8,\left(z_{24}=z_2\right)\right)$, all (mod 11).

The equation for the doubled Cosines $u=z+z^*=2x=2\cos(\theta)$ (case $n=11$ in Table 1 (Equation (2.4)) as a genuine 5-th order equation without special symmetries is not possible to solve in radicals as it is known.

9. Factorizations of Cyclotomic Polynomial for *n* = 13 with Quadratic Radicals

The case $n=13$ is very interesting due to factorization of the cyclotomic equation by polynomials of 3-rd degree with only quadratic radicals in the coefficients in 3-cycles. This makes it possible for the application of the rhombic bicompasses and ruler for the construction of the regular tridecagon (13-gon).

The first factorization by two 6-cycles provides two factor polynomials of 6-th degree with real coefficients

$$\begin{aligned}\frac{z^{13}-1}{z-1}=\sum_{k=0}^{12}z^k&=\left\{z^6+\frac{1-\sqrt{13}}{2}z^5+2z^4-\frac{1+\sqrt{13}}{2}z^3+2z^2+\frac{1-\sqrt{13}}{2}z+1\right\}\\&\quad\cdot\left\{z^6+\frac{1+\sqrt{13}}{2}z^5+2z^4-\frac{1-\sqrt{13}}{2}z^3+2z^2+\frac{1+\sqrt{13}}{2}z+1\right\}\\&=\left\{\left(z-z_1\right)\left(z-z_3\right)\left(z-z_4\right)\left(z-z_9\right)\left(z-z_{10}\right)\left(z-z_{12}\right)\right\}\\&\quad\cdot\left\{\left(z-z_2\right)\left(z-z_5\right)\left(z-z_6\right)\left(z-z_7\right)\left(z-z_8\right)\left(z-z_{11}\right)\right\}.\end{aligned} \tag{9.1}$$

Each of the two factor polynomials of 6-th degree can be again factorized in two polynomials of 3-rd degree with only quadratical radicals in the coefficients and the involved zeros we find from the four 3-cycles with the first involving the element z_1 explicitly given in (3.4)

$$\begin{aligned}\frac{z^{13}-1}{z-1} &= \left\{\left(z^3+\frac{1-\sqrt{13}-\mathrm{i}\sqrt{2\left(13-3\sqrt{13}\right)}}{4}z^2-\frac{1-\sqrt{13}+\mathrm{i}\sqrt{2\left(13-3\sqrt{13}\right)}}{4}z-1\right)\right.\\ &\quad\left.\cdot\left(z^3+\frac{1-\sqrt{13}+\mathrm{i}\sqrt{2\left(13-3\sqrt{13}\right)}}{4}z^2-\frac{1-\sqrt{13}-\mathrm{i}\sqrt{2\left(13-3\sqrt{13}\right)}}{4}z-1\right)\right\}\\ &\quad\cdot\left\{\left(z^3+\frac{1+\sqrt{13}-\mathrm{i}\sqrt{2\left(13+3\sqrt{13}\right)}}{4}z^2-\frac{1+\sqrt{13}+\mathrm{i}\sqrt{2\left(13+3\sqrt{13}\right)}}{4}z-1\right)\right.\\ &\quad\left.\cdot\left(z^3+\frac{1+\sqrt{13}+\mathrm{i}\sqrt{2\left(13+3\sqrt{13}\right)}}{4}z^2-\frac{1+\sqrt{13}-\mathrm{i}\sqrt{2\left(13+3\sqrt{13}\right)}}{4}z-1\right)\right\}\\ &= \left\{\left((z-z_1)(z-z_3)(z-z_9)\right)\left((z-z_4)(z-z_{10})(z-z_{12})\right)\right\}\\ &\quad\cdot\left\{\left((z-z_2)(z-z_5)(z-z_6)\right)\left((z-z_7)(z-z_8)(z-z_{11})\right)\right\}.\end{aligned} \tag{9.2}$$

This means that the fixed points of the two rhombic bicompasses besides the coordinate origin $z=0$ are the following quadratic radicals $z=z_1+z_3+z_9$ and $z=z_2+z_5+z_6$ or their complex conjugates

$$\begin{aligned} z_1+z_3+z_9 &= \frac{\sqrt{13}-1+\mathrm{i}\sqrt{2\left(13-3\sqrt{13}\right)}}{4} = z_{12}^*+z_{10}^*+z_4^*\\ &\approx 0.651388+\mathrm{i}0.522416,\\ \left|z_1+z_3+z_9\right|^2 &= \frac{5-\sqrt{13}}{2}\approx 0.697224,\quad \left|z_1+z_3+z_9\right|\approx 0.835000,\\ z_2+z_5+z_6 &= \frac{-\sqrt{13}-1+\mathrm{i}\sqrt{2\left(13+3\sqrt{13}\right)}}{4} = z_{11}^*+z_8^*+z_9^*\\ &\approx -1.15139+\mathrm{i}1.72542,\\ \left|z_2+z_5+z_6\right|^2 &= \frac{5+\sqrt{13}}{2}\approx 4.30278,\quad \left|z_2+z_5+z_6\right|\approx 2.07431. \end{aligned} \tag{9.3}$$

Together with the circle division problem the two principally possible fixed points (without the complex conjugate ones) for the bicompasses are shown in Figure 3.

The sums and differences of the fixed points are also expressible by only quadratic radicals. We find for the sums algebraically and numerically

$$\left(z_1+z_3+z_9\right)+\left(z_2+z_5+z_6\right)=\frac{-1+\mathrm{i}\sqrt{13+2\sqrt{13}}}{2}\approx -0.5+\mathrm{i}2.24784,$$

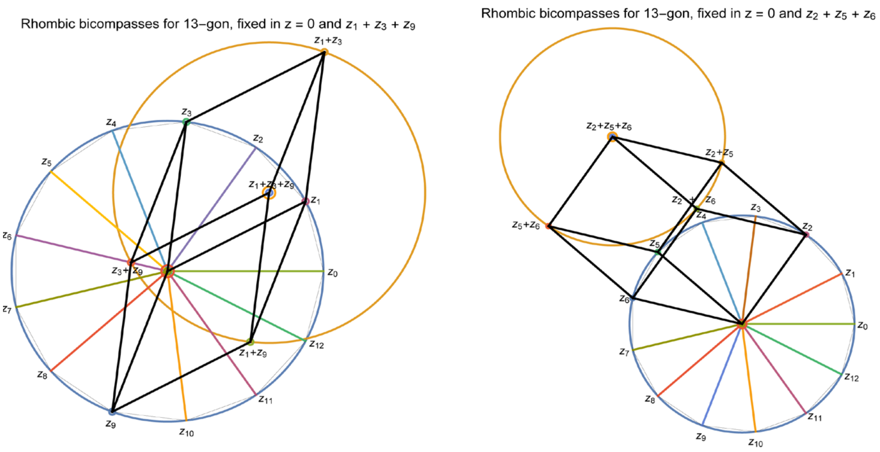

Figure 3. Two variants of rhombic bicompasses in case of a regular 13-gon. The two fixed points of the rhombic bicompasses are $z=0$ and $z=z_1+z_3+z_9$ in first picture and $z=0$ and $z=z_2+z_5+z_6$ in second picture (see text).

$$\begin{aligned}
\left|\left(z_1+z_3+z_9\right)+\left(z_2+z_5+z_6\right)\right| &= \frac{\sqrt{13}+1}{2} \approx 2.30278,\\
\left(z_1+z_3+z_9\right)+\left(z_7+z_8+z_{11}\right) &= \frac{-1-\mathrm{i}\sqrt{13-2\sqrt{13}}}{2} \approx -0.5-\mathrm{i}1.20301,\\
\left|\left(z_1+z_3+z_9\right)+\left(z_7+z_8+z_{11}\right)\right| &= \frac{\sqrt{13}-1}{2} \approx 1.30278,
\end{aligned} \tag{9.4}$$

and for the differences

$$\begin{aligned}
\left(z_1+z_3+z_9\right)-\left(z_2+z_5+z_6\right) &= \frac{\sqrt{13}-\mathrm{i}\sqrt{13-2\sqrt{13}}}{2} \approx 1.80278-\mathrm{i}1.20301,\\
\left|\left(z_1+z_3+z_9\right)-\left(z_2+z_5+z_6\right)\right| &= \sqrt{\frac{13-\sqrt{13}}{2}} \approx 2.16731,\\
\left(z_1+z_3+z_9\right)-\left(z_7+z_8+z_{11}\right) &= \frac{\sqrt{13}+\mathrm{i}\sqrt{13+2\sqrt{13}}}{2} \approx 1.80278+\mathrm{i}2.24784,\\
\left|\left(z_1+z_3+z_9\right)-\left(z_7+z_8+z_{11}\right)\right| &= \sqrt{\frac{13+\sqrt{13}}{2}} \approx 2.88145.
\end{aligned} \tag{9.5}$$

All these numbers and radiuses are constructible by compass and ruler. The points $z_k, (k \neq 0 \bmod 13)$ alone and combinations of only two such points in form of $z_k + z_l$ are not constructible by compass and ruler since they contain cubic radicals.

This is illustrated in **Figure 4** and **Figure 5** where the two additional possibilities with the complex conjugate fixed points are obtained by reflection of each partial picture at the horizontal line through the coordinate origin. This provides 4 possibilities for the common combination of the two rhombic bicompasses where two of them are obtained again by reflection on the horizontal line through the coordinate origin. The two essentially different ones

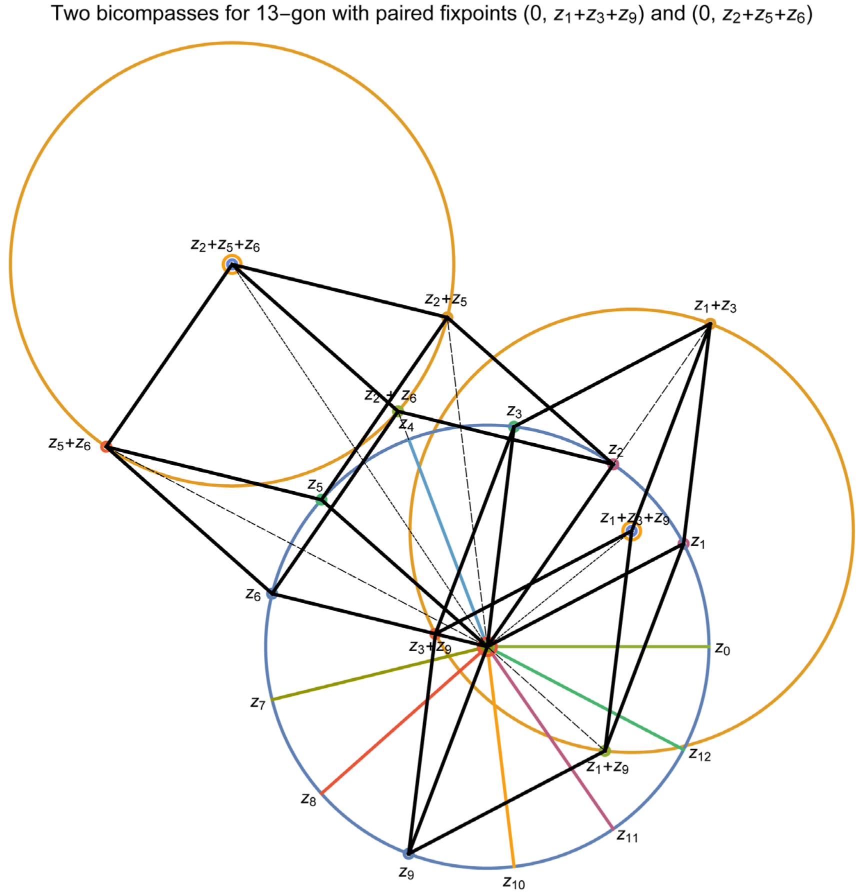

Figure 4. First of two variants of correlated rhombic bicompasses in case of a regular 13-gon.

are illustrated in **Figure 4** and **Figure 5**.

According to **Table 1** (Equation (2.4)) the doubled Cosines $u = 2\cos\left(k\frac{2\pi}{13}\right) = 2\cos\left((13-k)\frac{2\pi}{13}\right)$, $(k = 1, 2, \cdots, 6)$ are the roots of the 6-th degree polynomial equation with 6 real-valued solutions

$$\mathrm{U}_6\left(\frac{u}{2}\right) + \mathrm{U}_5\left(\frac{u}{2}\right) = u^6 + u^5 - 5u^4 - 4u^3 + 6u^2 + 3u - 1 = 0. \tag{9.6}$$

The polynomial on the left-hand side of this equation can be factorized in two polynomials of 3-rd degree with only quadratic radicals in the coefficients in the following way [13]

$$\begin{aligned} &u^6 + u^5 - 5u^4 - 4u^3 + 6u^2 + 3u - 1 \\ &= \left(u^3 - \frac{\sqrt{13}-1}{2}u^2 - u + \frac{\sqrt{13}-3}{2}\right)\left(u^3 + \frac{\sqrt{13}+1}{2}u^2 - u - \frac{\sqrt{13}+3}{2}\right). \end{aligned} \tag{9.7}$$

This corresponds to the following relations for sums of the Cosines

$$\cos\left(k\frac{2\pi}{13}\right) = \cos\left((13-k)\frac{2\pi}{13}\right)$$

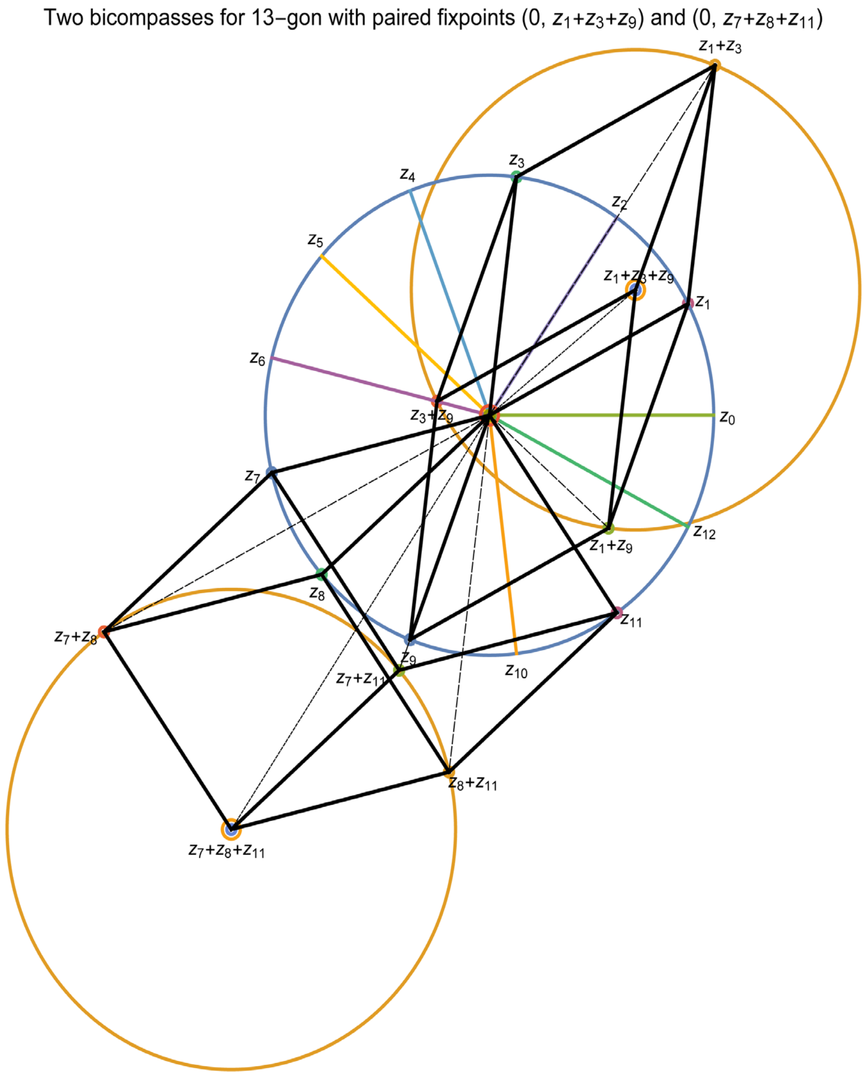

Figure 5. Second of two variants of correlated rhombic bicompasses in case of a regular 13-gon.

$$\cos\left(\frac{2\pi}{13}\right)+\cos\left(\frac{6\pi}{13}\right)+\cos\left(\frac{8\pi}{13}\right)=\frac{\sqrt{13}-1}{4},$$
$$\cos\left(\frac{4\pi}{13}\right)+\cos\left(\frac{10\pi}{13}\right)+\cos\left(\frac{12\pi}{13}\right)=-\frac{\sqrt{13}+1}{4}, \tag{9.8}$$

and for products of the same Cosines

$$\cos\left(\frac{2\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right)\cos\left(\frac{8\pi}{13}\right)=-\frac{\sqrt{13}-3}{16},$$
$$\cos\left(\frac{4\pi}{13}\right)\cos\left(\frac{10\pi}{13}\right)\cos\left(\frac{12\pi}{13}\right)=\frac{\sqrt{13}+3}{16}. \tag{9.9}$$

One may obtain these relations from relations (9.3) for the fixed points of the rhombic bicompasses. Relations (9.9) do not give additional independent information to relations (9.8). This can be seen if one transforms the products in (9.9) into sums using addition theorems for trigonometric functions. The same is true for the relations obtainable from the coefficients in front of u in (9.7).

Equation (9.6) together with (9.7) means that the cyclotomic equation for $n=13$ can be exactly solved by not higher than cubic radicals. This, however, is not appropriate for the construction by compass and ruler which allows only quadratic radicals. The same conclusion can be drawn for the full solutions z_k of the corner points of the regular 13-gon by setting equal to zero the four 3-rd degree polynomials in (9.2) of the factored cyclotomic polynomial (9.1).

The real construction of the regular 13-gon by rhombic bicompasses and ruler seems to be rather complicated. One may distinguish two principal cases:

1. The use of only one of the two possible bicompasses shown in the two pictures in **Figure 3**.

2. The use of the two rhombic bicompasses in correlated way in one of the two principally different variants as illustrated in **Figure 4** and **Figure 5**.

The problem in first case is that during the application of the bicompasses, for example, to points z_1 and z_3 in first partial picture in **Figure 3** one has to find the position when the angle between z_1 and z_3 is the doubled angle between $z_0=1$ and z_1 that is equivalent to solve the trisection of the angle between z_0 and z_3. The same problem arises if we use the points z_1 and z_9 or z_3 and z_9 instead of z_1 and z_3. In principal, an angle trisection can be made by a neusis construction attributed as already said to Archimedes [2] [9] but it is not in the spirit of construction by compass and ruler and, furthermore, it is unclear how it could be combined at the same time with the action of the bicompasses. We could not find a possibility also in case if we use in addition to one of the bicompasses the corresponding bicompasses with the complex conjugate fixed point.

The second case with correlated bicompasses seems to be, in principal, possible. In case of the regular heptagon we found points from the pictures on the arms of the bicompasses or on their prolongation which can be expressed by not higher than quadratic radicals and proved this property then algebraically. In case of the 13-gon we go a similar way but start from an opposite point of view. We look for possibilities of points on the lines between corners of bicompasses which can be represented by not higher than quadratic radicals. We consider the bicompasses which in the right position determine the corner points (z_1,z_3,z_9) of the regular 13-gon (see **Figure 4**). A first possibility is then to look on the line between z_1 and z_1+z_6 with z_6 on the other bicompasses which can be parameterized with real parameter r by

$$z=z_1+rz_6,\quad (r=r^*). \tag{9.10}$$

With the choice of a real parameter $r=\frac{z_3+z_9+\lambda z_6}{z_6}$ with real parameter λ we find

$$r=\frac{z_3+z_9+\lambda z_6}{z_6}=z_3^*+z_3+\lambda=r^*,\quad \left(\lambda=\lambda^*\right),$$

$$\Rightarrow z=z_1+\frac{z_3+z_9+\lambda z_6}{z_6}z_6=z_1+z_3+z_9+\lambda z_6=\frac{\sqrt{13}-1+\mathrm{i}\sqrt{2\left(13-3\sqrt{13}\right)}}{4}+\lambda z_6, \quad (9.11)$$

$$\Rightarrow \lambda=0.$$

The points λz_6 with real $\lambda\neq 0$ are, in general, and contrary to $z_1+z_3+z_9$ not representable by only quadratic radicals and, therefore, we set $\lambda=0$ in (9.11).

An analogous second possibility to the described one is to look on the line between z_3 of the first bicompasses and z_3+z_5 with z_5 of the second bicompasses which can be parameterized with real parameter s by

$$z=z_3+sz_5,\quad \left(s=s^*\right). \quad (9.12)$$

Then with the choice of parameter $s=\frac{z_1+z_9}{z_5}$ we find

$$s=\frac{z_1+z_9}{z_5}=z_4^*+z_4=s^*,$$

$$\Rightarrow z=z_3+\frac{z_1+z_9}{z_5}z_5=z_3+z_1+z_9. \quad (9.13)$$

A third equivalent possibility is to look on the line between z_9 on the first bicompasses and z_9+z_2 with z_2 on the second bicompasses. It can be parameterized with real parameter t by

$$z=z_9+tz_2,\quad \left(t=t^*\right). \quad (9.14)$$

With the choice $t=\frac{z_1+z_3}{z_2}$ we obtain

$$t=\frac{z_1+z_3}{z_2}=z_1^*+z_1=t^*,$$

$$\Rightarrow z=z_9+\frac{z_1+z_3}{z_2}z_2=z_9+z_1+z_3. \quad (9.15)$$

All three possibilities lead to the fixed point $z_1+z_3+z_9$ of the first bicompasses expressible by quadratic radical through which or through their prolongations the considered parameterized lines have to go. There are also further equivalent possibilities to use instead of the constructible points
$z_1+z_3+z_9$ and $z_2+z_5+z_6$ their sums and differences which we did not investigate up to now in detail.

For the real construction one has to establish a correlation between the two bicompasses with fixed points $\left(0,z_1+z_3+z_9\right)$ and $\left(0,z_2+z_5+z_6\right)$ in such a way that when one of these bicompasses is in the right position the second at the same time has also to be in the right position for the angle division in thirteen equal parts. This means that we have to guarantee that the second of the bicompasses acts in concerted way with the first. It seems that one can use for such a coupling, for example, that the point $z_3 + z_9$ of the first bicompasses lies exactly on the line (arm) between $z = 0$ and z_6 of the second bicompasses (see **Figure 4**). This is already clear from symmetry and does not need to be proved. One has to guarantee that the point equivalent to $z_3 + z_9$ in the right position of the bicompasses may glide along the line from $z = 0$ to z_6 in the right position during the action of the two bicompasses. Then the point z_2 of the second bicompasses makes a bisection of the angle formed by the points z_1 and z_3 of the first bicompasses during their action when they arrive the right position. To determine this bisection seems to be possible during the action of the bicompasses. There are equivalent possibilities to realize the construction by the two bicompasses.

A weak point is the gliding of a point (here $z_3 + z_9$) on a line (here between $z = 0$ and z_6) as an admissible action in the spirit of geometric constructions from ancient time on. This is somehow problematic and requires more discussion in future as we are able to give here at this time.

There are also some not exact coincidences looking onto the figures one could think to be exact coincidences and which in the study of languages would be called 'false friends'. They may be used for approximate constructions of the 13-gon but it is not our intention to find and discuss them in detail. We mention only a few ones which are evident from the figures. The point z_{12} lies on the circle with radius equal to 1 around the coordinate origin but not at the same time on the circle with radius equal to 1 around $z_1 + z_3 + z_9$ that is already optically to sense in **Figure 4** and **Figure 5**. The projection of the point $z_1 + z_3$ onto the real *x*-axis is not exactly equal to 1 as it seems to be from **Figure 4** and numerically we find $z_1 + z_3 \approx 1.005993 + \mathrm{i}1.457432$ where the deviation of the real part from 1 is widely above the numerical errors of calculation and it is nothing more to prove in this case. This is also clear since $z_1 + z_3$ is not representable by quadratic radicals. The line from coordinate origin 0 to fixed point $z_2 + z_5 + z_6$ does not bisect the angle between z_4 and z_5 as from parameterized line set equal to the bisected angle $t(z_2 + z_5 + z_6) = \exp\left(\mathrm{i}\frac{9\pi}{13}\right)$ with numerically calculated complex (but not real) $t = 0.482028 + \mathrm{i}0.00757137$ follows.

10. Factorization of Cyclotomic Polynomial for *n* = 15 with Quadratic Radicals

According to the composite number $n = 15$ the cyclotomic equation contains the cases of the cyclotomic equations for $n_1 = 3$ and $n_2 = 5$ as factors and possesses the form

$$\begin{aligned}\frac{z^{15}-1}{z-1} &= \sum_{k=0}^{14} z^k = \left(z^2+z+1\right)\left(z^4+z^3+z^2+z+1\right)\left(z^8-z^7+z^5-z^4+z^3-z+1\right)\\ &= \left(\left(z-z_5\right)\left(z-z_{10}\right)\right)\left(\left(z-z_3\right)\left(z-z_6\right)\left(z-z_9\right)\left(z-z_{12}\right)\right)\\ &\quad\cdot\left(\left(z-z_1\right)\left(z-z_2\right)\left(z-z_4\right)\left(z-z_7\right)\left(z-z_8\right)\left(z-z_{11}\right)\left(z-z_{13}\right)\left(z-z_{14}\right)\right).\end{aligned} \tag{10.1}$$

The last factor is the irreducible part of the cyclotomic polynomial $p_n(z)$ (with coefficients in $\mathbb{Z}$) and is denoted by $\Phi_{15}(z)$ can be decomposed in two complex conjugate factors according to

$$\begin{aligned}\Phi_{15}(z) &= z^8 - z^7 + z^5 - z^4 + z^3 - z + 1 \\ &= \left(z^4 - \frac{1+\mathrm{i}\sqrt{15}}{2}z^3 - 2z^2 - \frac{1-\mathrm{i}\sqrt{15}}{2}z + 1\right) \\ &\quad \cdot\left(z^4 - \frac{1-\mathrm{i}\sqrt{15}}{2}z^3 - 2z^2 - \frac{1+\mathrm{i}\sqrt{15}}{2}z + 1\right) \\ &= \left((z-z_1)(z-z_2)(z-z_4)(z-z_8)\right) \\ &\quad \cdot\left((z-z_7)(z-z_{11})(z-z_{13})(z-z_{14})\right).\end{aligned} \tag{10.2}$$

In this factorizations we have two 4-cycles with the zeros (z_1, z_2, z_4, z_8) and $(z_{14}, z_{13}, z_{11}, z_7)$ with $z_1z_2z_4z_8 = z_{14}z_{13}z_{11}z_7 = 1$. From the given factorizations and from other considerations follow some known possibilities to solve the circle division problem for $n = 15$ with compass and ruler and we do not further discuss this.

11. Factorizations of Cyclotomic Polynomial for *n* = 17 with Quadratic Radicals

The possibility of the solution of the circle division problem in case $n = 17$ together with the solution of the general problem from ancient time for which n it can be solved at all by compass and ruler was discovered by the young Gauss and is represented in numerous books (e.g., [7] [9] [11] and others). The representation here of this known case illustrates our approach and facilitates its understanding in the unknown cases, in particular for $n = 13$.

A first factorization of the cyclotomic polynomial which is palindromic in two palindromic polynomial factors with quadratic radicals in the coefficients is

$$\begin{aligned}\frac{z^{17}-1}{z-1} &= \sum_{k=0}^{16} z^k \\ &= \left\{\left(z^8+1\right) + \frac{1-\sqrt{17}}{2}\left(z^7+z\right) + \frac{5-\sqrt{17}}{2}\left(z^6+z^2\right)\right. \\ &\quad \left. + \frac{7-\sqrt{17}}{2}\left(z^5+z^3\right) + \left(2-\sqrt{17}\right)z^4\right\} \\ &\quad \cdot\left\{\left(z^8+1\right) + \frac{1+\sqrt{17}}{2}\left(z^7+z\right) + \frac{5+\sqrt{17}}{2}\left(z^6+z^2\right)\right. \\ &\quad \left. + \frac{7+\sqrt{17}}{2}\left(z^5+z^3\right) + \left(2+\sqrt{17}\right)z^4\right\} \\ &= \left\{(z-z_1)(z-z_2)(z-z_4)(z-z_8)(z-z_9)(z-z_{13})(z-z_{15})(z-z_{16})\right\} \\ &\quad \cdot\left\{(z-z_3)(z-z_5)(z-z_6)(z-z_7)(z-z_{10})(z-z_{11})(z-z_{12})(z-z_{14})\right\}.\end{aligned} \tag{11.1}$$

The first factor, for example, involves the zeros

$z_1, z_2, z_4, z_8, z_{16}, z_{32} \equiv z_{15}, z_{64} \equiv z_{30} \equiv z_{13}$ modulo 17. The two palindromic factor polynomials of 8-th degree can be decomposed each in the product of two polynomials of 4-th degree which are also palindromic in the following way:

first factor

$$\begin{aligned}
&\left(z^8+1\right)+\frac{1-\sqrt{17}}{2}\left(z^7+z\right)+\frac{5-\sqrt{17}}{2}\left(z^6+z^2\right)+\frac{7-\sqrt{17}}{2}\left(z^5+z^3\right)+\left(2-\sqrt{17}\right)z^4 \\
&=\left(\left(z^4+1\right)+\frac{1-\sqrt{17}-\sqrt{2\left(17-\sqrt{17}\right)}}{4}\left(z^3+z\right)+\frac{7-\sqrt{17}+\sqrt{2\left(17+\sqrt{17}\right)}}{4}z^2\right) \\
&\quad\cdot\left(\left(z^4+1\right)+\frac{1-\sqrt{17}+\sqrt{2\left(17-\sqrt{17}\right)}}{4}\left(z^3+z\right)+\frac{7-\sqrt{17}-\sqrt{2\left(17+\sqrt{17}\right)}}{4}z^2\right) \\
&=\left((z-z_1)(z-z_4)(z-z_{13})(z-z_{16})\right)\left((z-z_2)(z-z_8)(z-z_9)(z-z_{15})\right),
\end{aligned} \tag{11.2}$$

second factor

$$\begin{aligned}
&\left(z^8+1\right)+\frac{1+\sqrt{17}}{2}\left(z^7+z\right)+\frac{5+\sqrt{17}}{2}\left(z^6+z^2\right)+\frac{7+\sqrt{17}}{2}\left(z^5+z^3\right)+\left(2+\sqrt{17}\right)z^4 \\
&=\left(\left(z^4+1\right)+\frac{1+\sqrt{17}-\sqrt{2\left(17+\sqrt{17}\right)}}{4}\left(z^3+z\right)+\frac{7+\sqrt{17}-\sqrt{2\left(17-\sqrt{17}\right)}}{4}z^2\right) \\
&\quad\cdot\left(\left(z^4+1\right)+\frac{1+\sqrt{17}+\sqrt{2\left(17+\sqrt{17}\right)}}{4}\left(z^3+z\right)+\frac{7+\sqrt{17}+\sqrt{2\left(17-\sqrt{17}\right)}}{4}z^2\right) \\
&=\left((z-z_3)(z-z_5)(z-z_{12})(z-z_{14})\right)\left((z-z_6)(z-z_7)(z-z_{10})(z-z_{11})\right).
\end{aligned} \tag{11.3}$$

In the factorization we have a product of 4 palindromic polynomials of 4-th degree each of which can be decomposed again in a product of 2 palindromic polynomials of now 2-nd degree with only quadratic radicals in the coefficients according to the general formula[3]

$$\begin{aligned}
z^4+az^3+bz^2+az+1 &= \left(z^2+\frac{a+\sqrt{a^2-4b+8}}{2}z+1\right) \\
&\quad\cdot\left(z^2+\frac{a-\sqrt{a^2-4b+8}}{2}z+1\right).
\end{aligned} \tag{11.4}$$

This provides the further splitting in products of pairs of two palindromic polynomials of 2-nd degree as follows:

first factor

[3]A palindromic polynomial of 8-th degree cannot be decomposed, in general, in a product of 2 palindromic polynomials of 4-th degree with not higher than quadratic radicals in the coefficients such as in formulae (11.2) and (11.3) and this is only possible under certain restrictions to the coefficients of the 8-th degree palindromic polynomial. There are, however, also some interesting relatives to palindromic polynomials of n-th degree without special names. These are, in particular, polynomials where the coefficients in front of z^{n-k} are equal to $(\pm 1)^k$ to that of z^k or where they are complex conjugate to the coefficients in front of z^k with possible additional factors $(\pm 1)^k$.

$$
\begin{aligned}
&(z-z_1)(z-z_4)(z-z_{13})(z-z_{16}) = \big((z-z_1)(z-z_{16})\big)\big((z-z_4)(z-z_{13})\big) \\
&= \left(z^2 + \frac{1-\sqrt{17}-\sqrt{2\left(17-\sqrt{17}\right)}-2\sqrt{17+3\sqrt{17}-\sqrt{2\left(85+19\sqrt{17}\right)}}}{8}z+1\right) \\
&\quad\cdot\left(z^2 + \frac{1-\sqrt{17}-\sqrt{2\left(17-\sqrt{17}\right)}+2\sqrt{17+3\sqrt{17}-\sqrt{2\left(85+19\sqrt{17}\right)}}}{8}z+1\right),
\end{aligned} \tag{11.5}
$$

second factor

$$
\begin{aligned}
&(z-z_2)(z-z_8)(z-z_9)(z-z_{15}) = \big((z-z_2)(z-z_{15})\big)\big((z-z_8)(z-z_9)\big) \\
&= \left(z^2 + \frac{1-\sqrt{17}+\sqrt{2\left(17-\sqrt{17}\right)}-2\sqrt{17+3\sqrt{17}+\sqrt{2\left(85+19\sqrt{17}\right)}}}{8}z+1\right) \\
&\quad\cdot\left(z^2 + \frac{1-\sqrt{17}+\sqrt{2\left(17-\sqrt{17}\right)}+2\sqrt{17+3\sqrt{17}+\sqrt{2\left(85+19\sqrt{17}\right)}}}{8}z+1\right),
\end{aligned} \tag{11.6}
$$

third factor

$$
\begin{aligned}
&(z-z_3)(z-z_5)(z-z_{12})(z-z_{14}) = \big((z-z_3)(z-z_{14})\big)\big((z-z_5)(z-z_{12})\big) \\
&= \left(z^2 + \frac{1+\sqrt{17}-\sqrt{2\left(17+\sqrt{17}\right)}-2\sqrt{17-3\sqrt{17}+\sqrt{2\left(85-19\sqrt{17}\right)}}}{8}z+1\right) \\
&\quad\cdot\left(z^2 + \frac{1+\sqrt{17}-\sqrt{2\left(17+\sqrt{17}\right)}+2\sqrt{17-3\sqrt{17}+\sqrt{2\left(85-19\sqrt{17}\right)}}}{8}z+1\right),
\end{aligned} \tag{11.7}
$$

fourth factor

$$
\begin{aligned}
&(z-z_6)(z-z_7)(z-z_{10})(z-z_{11}) = \big((z-z_6)(z-z_{11})\big)\big((z-z_7)(z-z_{10})\big) \\
&= \left(z^2 + \frac{1+\sqrt{17}+\sqrt{2\left(17+\sqrt{17}\right)}-2\sqrt{17-3\sqrt{17}-\sqrt{2\left(85-19\sqrt{17}\right)}}}{8}z+1\right) \\
&\quad\cdot\left(z^2 + \frac{1+\sqrt{17}+\sqrt{2\left(17+\sqrt{17}\right)}+2\sqrt{17-3\sqrt{17}-\sqrt{2\left(85-19\sqrt{17}\right)}}}{8}z+1\right).
\end{aligned} \tag{11.8}
$$

The bigger brackets in the short and in the explicit expressions correspond to each other in their ordering in the written form of the formulae.

The coefficients in all the polynomials of second degree in (11.5)-(11.8) in front of z are real ones and the half of the negatively taken value provides the

Cosine of the corresponding angle. From first factor in first polynomial in (11.5), for example, one find $\cos\left(\frac{2\pi}{17}\right)$, that means the Cosine of the angle to the first corner of the 17-gon which is explicitly

$$\cos\left(\frac{2\pi}{17}\right)=\frac{-1+\sqrt{17}+\sqrt{2\left(17-\sqrt{17}\right)}+2\sqrt{17+3\sqrt{17}-\sqrt{2\left(85+19\sqrt{17}\right)}}}{16}. \quad (11.9)$$

This is identical in content (but not in form) with the expression derived by Gauss in Section 365 of *Disquisitiones Arithmetica* as given in the citation by Edwards [6] on p. 32 and here cited according to him (see also Stewart [5] (p. 232)). All values for $\cos\left(k\frac{2\pi}{17}\right),(k=1,2,\cdots,8)$ can be taken from the corresponding coefficients in the quadratic factor polynomials in (11.5)-(11.8).

Let us give for completeness and for the factorization of the polynomial $\mathrm{U}_8\left(\frac{u}{2}\right)+\mathrm{U}_7\left(\frac{u}{2}\right)$ for $n=17$ in **Table 1** (Equation (11.4)) with coefficients in form of quadratic radicals the complete expressions for the doubled Cosines for all primitive roots u_k. From (11.5) and (11.6) follows for

$$u_k\equiv 2\cos\left(k\frac{2\pi}{17}\right)=2\cos\left(\left(17-k\right)\frac{2\pi}{17}\right)$$

$$u_{1,4}=\frac{-1+\sqrt{17}+\sqrt{2\left(17-\sqrt{17}\right)}\pm 2\sqrt{17+3\sqrt{17}-\sqrt{2\left(85+19\sqrt{17}\right)}}}{8},$$

$$u_{2,8}=\frac{-1+\sqrt{17}-\sqrt{2\left(17-\sqrt{17}\right)}\pm 2\sqrt{17+3\sqrt{17}+\sqrt{2\left(85+19\sqrt{17}\right)}}}{8}, \quad (11.10)$$

where the first of the two indices in $u_{k,l}$ corresponds to the upper sign and the second to the lower one in "$\pm$" and, analogously, from (11.7) and (11.8)

$$u_{3,5}=\frac{-1-\sqrt{17}+\sqrt{2\left(17+\sqrt{17}\right)}\pm 2\sqrt{17-3\sqrt{17}+\sqrt{2\left(85-19\sqrt{17}\right)}}}{8},$$

$$u_{6,7}=\frac{-1-\sqrt{17}-\sqrt{2\left(17+\sqrt{17}\right)}\pm 2\sqrt{17-3\sqrt{17}-\sqrt{2\left(85-19\sqrt{17}\right)}}}{8}. \quad (11.11)$$

The factorization of $\sum_{l=0}^{16} z^l$ taken together with (11.5)-(11.8) can be formally written

$$\begin{aligned}&\frac{z^{17}-1}{z-1}=\sum_{l=0}^{16} z^l=\prod_{k=1}^{8}\left(z^2-u_k z+1\right),\\ &u_k\equiv 2\cos\left(k\frac{2\pi}{17}\right),\quad \left(k\in\mathbb{Z}\left(\mathrm{mod}\,17\right)\right),\end{aligned} \quad (11.12)$$

where the explicit form of the coefficients $u_k,\left(k=1,2,\cdots,m=8\right)$ given in (11.10) and (11.11) are taken from (11.5) and (11.8). If we insert the expressions u_k for u into the cyclotomic Equation (2.5) for $m=8, n=2m+1=17$ using **Table 1**

(Equation (2.4))

$$
\begin{aligned}
\mathrm{U}_8\left(\frac{u}{2}\right)+\mathrm{U}_7\left(\frac{u}{2}\right) &= u^8+u^7-7u^6-6u^5+15u^4+10u^3-10u^2-4u+1 \\
&= \frac{p_{17}(z)}{(z-1)z^8}, \quad \left(u \equiv z+z^{-1}=z+z^{*}\right),
\end{aligned}
\tag{11.13}
$$

then its vanishing is identically satisfied. We checked this algebraically by computer with "Mathematica"[4] and we checked also numerically by computer that the order of the braces and big brackets in different parts of the Formulae (11.2) and (11.3) and in Formulae (11.5)-(11.8) remains preserved.

Since in case of the regular 17-gon all Cosines of the angles are expressed by quadratic radicals the multiplication of arbitrary factors $(u-u_k)(u-u_l)$ leads again to polynomials with only quadratic radicals in the coefficients. Therefore, the factorization of cyclotomic polynomials such as (11.13) with quadratic radicals in the coefficients admits many possibilities but only a few lead to simple expressions. In a first step one may obtain the following factorization of the polynomial of 8-th degree on the right-hand side of (11.13) into the product of two polynomials of 4-th degree with quadratic radicals as coefficients

$$
\begin{aligned}
& u^8+u^7-7u^6-6u^5+15u^4+10u^3-10u^2-4u+1 \\
&= \left(u^4+\frac{1-\sqrt{17}}{2}u^3-\frac{3+\sqrt{17}}{2}u^2+\left(2+\sqrt{17}\right)u-1\right) \\
&\quad \cdot\left(u^4+\frac{1+\sqrt{17}}{2}u^3-\frac{3-\sqrt{17}}{2}u^2+\left(2-\sqrt{17}\right)u-1\right).
\end{aligned}
\tag{11.14}
$$

This is certainly the simplest of the possible factorizations of the 8-th degree polynomial in two polynomials with quadratic radicals as coefficients. The first factor polynomial contains the factors $(u-u_1, u-u_4, u-u_2, u-u_8)$ and the second the remaining factors $(u-u_3, u-u_5, u-u_6, u-u_7)$. Further factorizations of the polynomials of 4-th degree into products of polynomials of 2-nd degree with quadratic radicals in the coefficients are to obtain in analogous way using Formulae (11.2)-(11.3) and since each factorization in factors $u-u_k$ contains only quadratic radicals this is even possible in different ways. We do not write down all this.

The general possibilities for the construction of regular n-gons by compass and ruler according to the rules of ancient time which young Gauss finally found and proved and which the problem solved forever are the following (e.g., [7] [9]). The basis form such Fermat numbers $F_l \equiv 2^{2^l}+1$ which are prime numbers p_i. We denote them by $p_1, p_2, \cdots$. An n-gon is constructible by compass and ruler if it is a power 2^k (for angle bisections) multiplied by an arbitrary product of distinct prime Fermat numbers p_i that means

$$
n = 2^k p_{i_1} p_{i_2} \cdots. \tag{11.15}
$$

[4]The used program "Mathematica 10" does not automatically provide the solutions of (11.13) in the explicit form of the u_k but affirms immediately the satisfaction of the equation if we insert there separately their explicit values.

As it is well known the first 5 Fermat numbers which are $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$ and are all prime numbers but the next Fermat number F_5 is composite (e.g., [7] [9]; see also **Table 2** in Section 3). The analogous case of numbers $G_l = 3 \cdot 2^{2^l} + 1$ for geometric constructions by bicompasses and ruler are given also in **Table 2**.

By combinations of constructions by compass and ruler with constructions by bicompasses and ruler one finds further possibilities to solve the circle division problem geometrically. For example, by combination of the constructions of the angles $\frac{2\pi}{3}$ or $\frac{2\pi}{5}$ with the angles $\frac{2\pi}{7}$ or $\frac{2\pi}{13}$ by bicompasses one may construct

$$\frac{2\pi}{3} \mp \frac{2\pi}{7} = \frac{(7 \mp 3)2\pi}{21} = \left\{ \begin{matrix} 2^2 \frac{2\pi}{21} \\ 10 \frac{2\pi}{21} \end{matrix} \right\}, \quad \frac{2\pi}{5} \mp \frac{2\pi}{7} = \frac{(7 \mp 5)2\pi}{35} = \left\{ \begin{matrix} 2^1 \frac{2\pi}{35} \\ 12 \frac{2\pi}{35} \end{matrix} \right\},$$

$$\frac{2\pi}{3} \mp \frac{2\pi}{13} = \frac{(13 \mp 3)2\pi}{39} = \left\{ \begin{matrix} 10 \frac{2\pi}{39} \\ 2^4 \frac{2\pi}{39} \end{matrix} \right\}, \quad \frac{2\pi}{5} \mp \frac{2\pi}{13} = \frac{(13 \mp 5)2\pi}{65} = \left\{ \begin{matrix} 2^3 \frac{2\pi}{65} \\ 18 \frac{2\pi}{35} \end{matrix} \right\}. \quad (11.16)$$

We see that in connection with (repeated) angle bisections admitted by powers of 2 in front of $\frac{2\pi}{n}$ the circle division problem can be principally also solved, for example, for $n = 21, 35, 39, 65$. More generally, if the angles $\frac{2\pi}{m}$ and $\frac{2\pi}{n}$ are constructible, one may consider the combinations $k\frac{2\pi}{m} + l\frac{2\pi}{n}$ with $k \bmod m$ and $l \bmod n$ in analogous way.

12. Is Factorization of Cyclotomic Polynomial for *n* = 19 with 3-Cycles and with Only Quadratic Radicals Possible?

In case of $n = 2m + 1 = 19$ we find as a first factorization of the cyclotomic polynomial

$$\begin{aligned} \frac{z^{19}-1}{z-1} = \sum_{k=0}^{18} z^k &= \left\{ z^9 + \frac{1 - \mathrm{i}\sqrt{19}}{2} z^8 - 2z^7 + \frac{3 + \mathrm{i}\sqrt{19}}{2} z^6 + \frac{5 - \mathrm{i}\sqrt{19}}{2} z^5 \right. \\ &\quad \left. - \frac{5 + \mathrm{i}\sqrt{19}}{2} z^4 - \frac{3 - \mathrm{i}\sqrt{19}}{2} z^3 + 2z^2 - \frac{1 + \mathrm{i}\sqrt{19}}{2} z - 1 \right\} \\ &\quad \cdot \left\{ z^9 + \frac{1 + \mathrm{i}\sqrt{19}}{2} z^8 - 2z^7 + \frac{3 - \mathrm{i}\sqrt{19}}{2} z^6 + \frac{5 + \mathrm{i}\sqrt{19}}{2} z^5 \right. \\ &\quad \left. - \frac{5 - \mathrm{i}\sqrt{19}}{2} z^4 - \frac{3 + \mathrm{i}\sqrt{19}}{2} z^3 + 2z^2 - \frac{1 - \mathrm{i}\sqrt{19}}{2} z - 1 \right\} \\ &= \left\{ (z - z_1)(z - z_4)(z - z_5)(z - z_6)(z - z_7)(z - z_9)(z - z_{11})(z - z_{16})(z - z_{17}) \right\} \\ &\quad \cdot \left\{ (z - z_2)(z - z_3)(z - z_8)(z - z_{10})(z - z_{12})(z - z_{13})(z - z_{14})(z - z_{15})(z - z_{18}) \right\}. \end{aligned} \quad (12.1)$$

A further factorization of the two polynomial factors in three 3-cycles each with only quadratic radicals in the coefficients is not possible since such an ansatz leads to cubic equations for the coefficients. Therefore the construction of the 19-gon is not possible by compass and ruler as well as it is not possible by rhombic bicompasses and ruler.

13. Some General Rules for the Factorization of the Special Palindromic Polynomials $\sum_{k=0}^{2m} z^{2m-k}$ in Two Factors with Quadratic Radicals in the Coefficients

If one looks through the first factorizations of the cyclotomic polynomials $\frac{p_{2m+1}(z)}{z-1} = \sum_{k=0}^{2m} z^{2m-k}, \quad (m = 0,1,2,\cdots)$ for odd $n = 2m+1$ into the product of two 'similar' polynomials of m-th degree with only quadratic radicals in their coefficients as explicitly given up to $n = 19, m = 9$ one finds at once some regularities. First, one may distinguish two categories for odd n first where the coefficients in the both factor polynomials possess real coefficients and second where they possess complex conjugate coefficients. It depends on the two cycles of length m and on their content of roots. If each such factor involves together with each root also the conjugate root modulo n then all coefficients are real ones. This is the case up to $n = 19$ only for $n = 5$ and $n = 17$ which are prime Fermat numbers and for $n = 13$. In all other cases up to $n = 19$ as one can see from the given explicit factorization the coefficients in the two factor polynomials are complex conjugate to each other. One may take this into account if one knows the involved roots of the two cycles which lead to this factorization. From the coefficients in front of z^{n-2} and z^{n-3} of the cyclotomic polynomial $\frac{p_n(z)}{z-1}$ which are equal to 1 as for all its coefficients one can determine both coefficients a_{m-1} and a_{m-2} in the two factors of their principal form $z^m - a_{m-1} z^{m-1} + a_{m-2} z^{m-2} + \cdots + (-1)^m$. One may begin to determine the coefficients in the factorized form also from the low-order powers of z or better at once from both sides, from the high-order and from the low-order powers of z. From the consequence (3.6) of the Fermat theorem follows, at least, for prime numbers $n = 2m+1$ that in the first factorization of the cyclotomic polynomial $\frac{p_{2m+1}(z)}{z-1}$ into a product of two polynomials of degree m with quadratic radicals only that the coefficients of the powers to $z^0 = 1$ can also be only ± 1 depending on m as an even or odd number. In next steps one has to look what follows from the symmetries and have to introduce unknown quantities for the coefficient which can be successively determined from the multiplication of the two factorized polynomials which have to give the initial cyclotomic polynomials. It is obvious how in the second-highest and second-lowest coefficients appear the square roots $\sqrt{n}$ of n (for prime n) but for coefficients more to the middle of the polynomials it is difficult to derive general formulae.

14. The Number $\frac{-1+\mathrm{i}\sqrt{7}}{2}$ as the Second of the Basis Vectors of a Klein Lattice in the Complex Plane with Unique Prime Factorization

We make now a small digression. Conway and Smith [21] describe from p. 15 on the lattices in the complex plane with respect to the factorization into products of prime complex lattice vectors. The unique factorization of Gaussian integers and also of Eisenstein integers can be found in number-theoretic works (e.g., [8]) but all possible lattices where such a unique factorization into products of prime factors is possible was up to now seldom to find. According to Conway and Smith [21] there are only 9 such lattices (up to rotations and scalings) and the proof for the completeness of these lattices with such a property goes back to Heegner and independently to Baker and Stark. What is very interesting for us although not directly connected with the unique prime factorization is that next to the Gaussian integers (possible basis $(1,\mathrm{i})$ and the Eisenstein integers (possible basis $\left(1,\frac{-1+\mathrm{i}\sqrt{3}}{2}\right)$ the Klein integers ('Kleinian ring') with possible basis $\left(1,\frac{-1+\mathrm{i}\sqrt{7}}{2}\right)$ possesses this property [21]. The complex number $\frac{-1+\mathrm{i}\sqrt{7}}{2}$ plays a main role as second of the fixed points besides the coordinate origin in our construction of the regular heptagon by rhombic bicompasses and ruler (see Section 6). The Klein lattice is formed by the set $z=z_{m,n}$ of all lattice points $z\in\mathbb{C}$ [21]

$$z_{m,n}=m+n\left(\frac{-1+\mathrm{i}\sqrt{7}}{2}\right),\quad \left(m,n=0,\pm1,\pm2,\cdots\right). \tag{14.1}$$

The norm of the lattice points $z_{m,n}$ that is the squared modulus $N(z)=|z|^2$ of an arbitrary complex number z is

$$N\left(z_{m,n}\right)=\left|m-\frac{n}{2}+\mathrm{i}n\frac{\sqrt{7}}{2}\right|^2=m^2-mn+2n^2\geq 0, \tag{14.2}$$

which is a non-negative integer.

Similar pictures as they are known for the distribution of the Gauss primes and for the Eisenstein primes in the complex plane (see, e.g., Conway and Smith [9], p. 224 (in English Ed.) and [21], p. 18 and particularly beautiful (larger) in the book of Guy [22], pp. 56, 57) would be also very interesting for the Klein primes (with 7-fold symmetry C_{7v} as subgroup of permutation group S_7).

15. Conclusions

Present article results mainly from investigations to the problem for which n a regular n-gon besides the regular heptagon ($n=7$) may be constructed by rhombic bicompasses and ruler. We found that the next possible case is the regular tridecagon ($n=13$) although some problems of the realization of the construction remained open. The technical design of rhombic bicompasses, in particular, if the arm length should be variable within certain limits, we do not consider as our problem to which we may contribute. We posed our mathematical problem into the general frame of the solution of the cyclotomic equation and of the factorization of the cyclotomic polynomials for arbitrary n where only concerning the factorization with integer or rational coefficients (in $\mathbb{Z}$ or $\mathbb{Q}$) of the (irreducible) factor polynomials exists a well-known theory. It is a special case of the Galois theory of the solvability of polynomial equations in radicals. Our approach to the factorization in case of $n=13$ and in the other cases with odd n to get factorization with coefficients which do not contain higher than quadratic radicals or cannot be resolved at all by radicals is similar to the usual approach in case of the 17-gon. The first task was to find the different cycles that is illustrated in detail for the cases $n=7$ and $n=13$. The factorization with only quadratic radicals in the coefficients is then straightforward using known details for the coefficients and determining the unknown from the general restrictions for the coefficients in systematic way.

Our bicompasses underlie hard restriction since they possess equal arm lengths and bundles of (maximum) three arms are connected with the two fixed points to guarantee to draw at once two circles of equal radius with correlation of points on the circles with one degree of freedom. The question is whether or not it is possible to solve other geometric construction problems with them as the described ones. However, the trisection of an arbitrary angle which is not possible with compass and ruler seems to be also not possible by bicompasses and ruler. It is due to the form of the equation for the Cosines of the trisected angle which is of third degree with vanishing coefficient in front of the 2-nd degree power which determines the two fixed points. The fixed points coincide in this case and the motion with one degree of freedom of the bicompasses is the rotation of a regular hexagon around the only fixed point. One position of this hexagon determines with 3 of its corner points the right position for the trisection of the angle but it cannot be determined which of the positions is the right. The restrictions to the bicompasses can be weakened. For example, the arm lengths can be varied and the number of arms between the fixed points may be increased to increase the degree of freedom. All this is a wide field and we are maybe only at the beginning of it.

In my previous article [1] I made some remarks about rare biological objects with 7-fold symmetry in heavy contrast to ubiquitous 5-fold symmetry in many plant families (please, amend there the scientific name for oleander into *Nerium oleander*). In the meanwhile I looked through the book of Ernst Haeckel *Kunstformen der Natur* from (1899-1904)[5] with the wonderful drawings and

[5]Edited by O. Breidbach with inclusion of some earlier publications of Ernst Haeckel and biographical notes under the title *Kunstformen der Natur, Kunstformen aus dem Meer*, Prestel-Verlag, 3. Auflage, München 2016.

with the aim to see whether or not are there biological objects with 7-fold symmetry and found two such objects: 1. table 34 (object 2) a colony forming alga from genus *Pediastrum* of the green algae *Chlorophyta* of kingdom *Protista* (or *Plantae*? due to other authors), 2. table 85 (fourth object in last column) a colonial ascidian (sea squirt) *Cynthia* of subphylum *Tunicata* to phylum *Chordata* of kingdom *Animalia*.

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Appendix A. Two Linear Combinations of the Chebyshev Polynomials of First and Second Kind

In Section 2 we came accross a class of polynomials for which sometimes the new notation $\mathrm{V}_n(x)$ is introduced. In this connection there are defined the following two relatives of the Chebyshev polynomials (see [16], p. 60 without notations $\mathrm{V}_n(x)$ and $\mathrm{W}_n(x)$ and, e.g., [18], p. 442)

$$\mathrm{V}_n(\cos(\theta)) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)},$$

$$\mathrm{W}_n(\cos(\theta)) = \frac{\cos\left(\left(n+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}, \tag{A.1}$$

called also Chebyshev polynomials of third and fourth kind. By multiplication of numerator and denominator of the first Equation in (A.1) with $\cos\left(\frac{1}{2}\theta\right)$ and of the second equation with $\sin\left(\frac{1}{2}\theta\right)$ and applying then the addition theorems for trigonometric functions one obtains

$$\mathrm{V}_n(\cos(\theta)) = \frac{\sin((n+1)\theta)+\sin(n\theta)}{\sin(\theta)} = \mathrm{U}_n(\cos(\theta)) + \mathrm{U}_{n-1}(\cos(\theta)),$$

$$\mathrm{W}_n(\cos(\theta)) = \frac{\sin((n+1)\theta)-\sin(n\theta)}{\sin(\theta)} = \mathrm{U}_n(\cos(\theta)) - \mathrm{U}_{n-1}(\cos(\theta)). \tag{A.2}$$

This can be also written in different representations (we use $\mathrm{T}_{-k}(z) = \mathrm{T}_k(z)$ and may use $\mathrm{U}_{n-1}(z) = -\mathrm{U}_{-n-1}(z)$ with special case $\mathrm{U}_{-1}(z) = 0$)

$$\begin{aligned}\mathrm{V}_n(z) &= \frac{2^{2n} n!^2}{(2n)!}\mathrm{P}_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(z) = \frac{2^{2n} n!^2}{(2n)!}\left(\mathrm{P}_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z) + \frac{1+z}{2}\mathrm{P}_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)\right)\\ &= \mathrm{T}_n(z) + (1+z)\mathrm{U}_{n-1}(z) = \sum_{k=0}^{2n}\mathrm{T}_{n-k}(z) = \mathrm{U}_n(z) + \mathrm{U}_{n-1}(z),\end{aligned}$$

$$\begin{aligned}\mathrm{W}_n(z) &= \frac{2^{2n} n!^2}{(2n)!}\mathrm{P}_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(z) = \frac{2^{2n} n!^2}{(2n)!}\left(\mathrm{P}_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z) - \frac{1-z}{2}\mathrm{P}_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)\right)\\ &= \mathrm{T}_n(z) - (1-z)\mathrm{U}_{n-1}(z) = \sum_{k=0}^{2n}(-1)^k\mathrm{T}_{n-k}(z) = \mathrm{U}_n(z) - \mathrm{U}_{n-1}(z),\end{aligned} \tag{A.3}$$

where $\mathrm{P}_n^{(\alpha,\beta)}(z)$ denotes the Jacobi polynomials.

The orthogonality relations and weight functions for the polynomials $\mathrm{V}_n(x)$ and $\mathrm{W}_n(x)$ in the real interval $-1 \le x \le +1$ are different from that for the Chebyshev polynomials $\mathrm{T}_n(x)$ and $\mathrm{U}_n(x)$ and follow from the general orthogonality relations for the Jacobi polynomials. The weight functions are

$\sqrt{\frac{1-x}{1+x}}$ and $\sqrt{\frac{1+x}{1-x}}$ for $\mathrm{V}_n(x)$ and $\mathrm{W}_n(x)$, respectively, instead of $\frac{1}{\sqrt{1-x^2}}$ and $\sqrt{1-x^2}$ for $\mathrm{T}_n(x)$ and $\mathrm{U}_n(x)$, respectively. This circumstance may justify the separate introduction of $\mathrm{V}_n(x)$ and $\mathrm{W}_n(x)$ but most relations for these polynomials can be directly obtained from that for the Chebyshev polynomials of first and second kind.

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Functions of Bounded $(p(\cdot),2)$-Variation in De la Vallée Poussin-Wiener's Sense with Variable Exponent

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Abstract

In this paper we establish the notion of the space of bounded $(p(\cdot),2)$- variation in De la Vallée Poussin-Wiener's sense with variable exponent. We show some properties of this space $BV^{W}_{(p(\cdot),2)}[a,b]$ and we show that any uniformly bounded composition operator that maps this space into itself necessarily satisfies the so-called Matkowski's conditions.

Keywords

Generalized Variation, De la Vallée Poussin, $(p(\cdot),2)$-Variation in Wiener's Sense, Variable Exponent, Composition Operator, Matkowski's Condition

1. Introduction

In 1881, C. Jordan gave the notion of variation of a function in [1], and from this moment, many generalizations and extensions have been given. Consequently, the study of notions of generalized bounded variation forms an important direction in the field of mathematical analysis. Another important generalization of the space of bounded variation in the Jordan's sense is the notion of the space of functions of second bounded variation studied by Ch. J. de la Vallée Poussin in 1908 in [2]. It is defined as follows:

Definition 1 *Let* π *be a partition of the interval* $[a,b]$ *of the form* $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$, *and* f *be a function* $f:[a,b]\to\mathbb{R}$. *The nonnegative real number*

$$V^{(2)}(f) = V^{(2)}(f;[a,b]) := \sup_{\pi} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|,$$

is called the second variation of f on $[a,b]$, where the supremum is taken over all partitions π of $[a,b]$. In the case that $V^{(2)}(f) < \infty$, we say that f has bounded second variation on $[a,b]$ and we denote it by $f \in BV^{(2)}[a,b]$.

A well-known generalization of the functions of bounded variation was done by N. Wiener in 1924 in [3]. The p-variation of a function f is the supremum of the sums of the p^{th} powers of absolute increments of f over non overlapping intervals. Wiener mainly focused on the case $p = 2$, the 2-variation.

Definition 2 *Let* π *be a partition of the interval* $[a,b]$ *of the form* $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$, f *be a function* $f : [a,b] \to \mathbb{R}$ *and* $1 < p < \infty$. *The nonnegative real number*

$$V_p(f) = V_p(f;[a,b]) := \sup_{\pi} \sum_{j=1}^{n-1} \left| f(t_j) - f(t_{j-1}) \right|^p,$$

is called the Wiener p-variation of f on $[a,b]$ where the supremum is taken over all partitions π of $[a,b]$. In the case that $V_p(f) < \infty$, we say that f has bounded Wiener p-variation on $[a,b]$ and we denote it by $f \in BV_p^W[a,b]$.

The p^{th}-variations were reconsidered in a probabilistic context by R. Dudley in [4] and [5], in 1994 and 1997, respectively. Many basic properties of the variation in the sense of Wiener and a number of important applications of the concept can be found in [6] [7]. The paper by V. V. Chistyakov and O. E. Galkin in [8] in 1998 is very important in the context of p-variation.

The class of nonlinear problems with exponent growth is a new research field and it reflects a new kind of physical phenomena. In 2000 the field began to expand even further. Motivated by problems in the study of electrorheological fluids, Diening [9] raised the question of when the Hardy-Littlewood maximal operator and other classical operators in harmonic analysis are bounded on variable Lebesgue spaces. These and related problems are the subject of active research to this day. These problems are interesting in applications (see [10] [11] [12] [13]) and gave rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent, the origins of which can be traced back to the work of Orlicz [14] in the 1930's. In the 1950's, this study was carried on by Nakano [15] [16] who made the first systematic study of spaces with variable exponent. Later, Polish and Czechoslovak mathematicians investigated the modular function spaces (see for example Musielak [17] [18], Kovacik and Rakosnik [19] and Kozlowski [20]). We refer to the book [13] for detailed information on the theoretical approach for the Lebesgue and Sobolev spaces with variable exponents. Recently, in [21] Castillo, Merentes and Rafeiro studied a new space of functions of generalized bounded variation. They introduced the notion of bounded variation in the Wiener sense with variable exponent $p(\cdot)$ on $[a,b]$ and study some of its properties.

Definition 3 *Given a function* $p : [a,b] \to (1,\infty)$, *a partition*

$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ *of the interval* $[a,b]$, *and a function* $f:[a,b] \to \mathbb{R}$, *the nonnegative real number*

$$V_{p(\cdot)}^{W}(f) = V_{p(\cdot)}^{W}(f,[a,b]) := \sup_{\pi^*} \sum_{=1}^{n} \left| f(t_j) - f(t_{j-1}) \right|^{p(x_{j-1})}, \tag{1.1}$$

is called the Wiener variation with variable exponent (or $p(\cdot)$-variation in Wiener's sense) of f on $[a,b]$ where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0, \cdots, x_{n-1}$ subject to the conditions that for each *j*, $t_j \le x_j \le t_{j+1}$.

In case that $V_{p(\cdot)}^{W}(f;[a,b]) < \infty$, we say that f has bounded Wiener variation with variable exponent (or bounded $p(\cdot)$-variation in Wiener's sense) on $[a,b]$. The symbol $WBV_{p(\cdot)}[a,b] = BV_{p(\cdot)}^{W}[a,b]$ will denote the space of functions of bounded $p(\cdot)$-variation in Wiener's sense with variable exponent on $[a,b]$.

The aim of this paper is to provide a description of the new class formed by the functions of bounded $(p(\cdot),2)$-variation in the sense of Wiener as an extension to the double case of the previous concept. Also, we prove structural properties for mappings of bounded $(p(\cdot),2)$-variation in the Wiener's sense. Finally, we show that any uniformly bounded composition operator that maps the space $BV_{(p(\cdot),2)}^{W}[a,b]$ into itself necessarily satisfies the so-called Matkowski's conditions.

2. Preliminaries

In this section we present some definitions and propositions that will be used through out this paper.

Definition 4 *Let* $1 < p < \infty$, π *be a partition* $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ *of the interval* $[a,b]$, *and* $f:[a,b] \to \mathbb{R}$ *be a function. The nonnegative real number*

$$V_{(p,2)}^{W}(f) = V_{(p,2)}^{W}(f;[a,b]) := \sup_{\pi} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p},$$

is called the De La Vallée Poussin-Wiener variation (or $(p,2)$-variation in Wiener's sense) of f on $[a,b]$ where the supremum is taken over all partitions π of $[a,b]$. In the case that $V_{(p,2)}^{W}(f) < \infty$, we say that f has bounded $(p,2)$-variation on $[a,b]$ and we denote by $f \in BV_{(p,2)}^{W}[a,b]$.

For the interested readers can see some of the properties in [2] [7] and other related problems in [22].

Proposition 1 *Let* $f:[a,b] \to \mathbb{R}$ *be a function with* $a,b > 0$ *and consider* $1 < p < \infty$. *Then*

1) $V_{(p,2)}^{W}(f;[a,b]) = 0$ if and only if f is a liner function.

2) If $V_{(p,2)}^{W}(f;[a,b]) < \infty$, then f is bounded in $[a,b]$.

3) $V_{(p,2)}^{W}(\cdot;[a,b])$ is a convex function.

Proof. 1) Suppose first that f is a linear function. If $f(t)=\alpha t+\beta$ for all $t\in[a,b]$, with $\alpha,\beta\in\mathbb{R}$, then by Definition 4, it follows easily that $V_{(p,2)}^{W}(f;[a,b])=0$.

Now, if $V_{(p,2)}^{W}(f;[a,b])=0$, then by Definition 4 we have

$$\begin{aligned}0&=V_{(p,2)}^{W}(f;[a,b])\\&=\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^p.\end{aligned}$$

Hence, for any partition $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$ of the interval $[a,b]$, we should have that

$$\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^p=0.$$

Then, any term in the sum should be zero. Since the function $t\to t^p$ vanishes only at zero, it follows that

$$\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}=\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\quad\text{for all } j=1,2,\cdots,n-1.$$

Therefore, f is equal to a linear function.

2) Suppose that $f\in BV_{(p,2)}^{W}[a,b]$ and f is not bounded, then there exists a sequence $\{t_n\}_{n\geq1}$, $t_n\in(a,b)$, $n\geq1$ such that $|f(t_n)|\to\infty$ when $n\to\infty$. Let $\{t_m\}_{m\geq1}$ be a subsequence of $\{t_n\}_{n\geq1}$ such that $\{t_m\}_{m\geq1}$ converge to $x\in[a,b]$. Then, as $\{f(t_m)\}_{m\geq1}$ is a subsequence of $\{f(t_n)\}_{n\geq1}$, so

$$|f(t_m)|\to\infty\quad\text{when } n\to\infty.$$

We have that

$$\left|\frac{f(t_{n+1})-f(t_n)}{t_{n+1}-t_n}-\frac{f(t_n)-f(t_{n-1})}{t_n-t_{n-1}}\right|^p\leq V_{(p,2)}^{W}(f;[a,b]),\quad n\geq1.$$

Moreover for $\pi=\{a\leq t\leq t_m\leq\cdots\leq b\}$ we get

$$\left|\frac{f(t_m)-f(t)}{t_m-t}-\frac{f(t)-f(a)}{t-a}\right|^p\leq V_{(p,2)}^{W}(f,[a,t_m])\leq V_{(p,2)}^{W}(f,[a,b]).$$

In consequence, $V_{(p,2)}^{W}(f;[a,b])=\infty$, since

$$\left|\frac{f(t_m)-f(t)}{t_m-t}-\frac{f(t)-f(a)}{t-a}\right|^p\to\infty,$$

as $m\to\infty$, which is a contradiction with $f\in BV_{(p,2)}^{W}[a,b]$. Therefore f is bounded.

3) Let $f,g:[a,b]\to\mathbb{R}$ be two functions, $\alpha,\beta\in[0,1]$ such that $\alpha+\beta=1$ and $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$ be a partition of $[a,b]$. Since t^p is convex and nondecreasing, we have that

$$\begin{aligned}
&\alpha V_{(p,2)}^{W}\left(f;[a,b]\right)+\beta V_{(p,2)}^{W}\left(g;[a,b]\right)\\
&=\alpha \sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^p\\
&\quad+\beta \sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{g\left(t_{j+1}\right)-g\left(t_j\right)}{t_{j+1}-t_j}-\frac{g\left(t_j\right)-g\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^p\\
&\geq \sup_{\pi}\sum_{j=1}^{n-1}\left|\alpha\left[\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right]\right.\\
&\quad\left.+\beta\left[\frac{g\left(t_{j+1}\right)-g\left(t_j\right)}{t_{j+1}-t_j}-\frac{g\left(t_j\right)-g\left(t_{j-1}\right)}{t_j-t_{j-1}}\right]\right|^p\\
&=\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{(\alpha f+\beta g)\left(t_{j+1}\right)-(\alpha f+\beta g)\left(t_j\right)}{t_{j+1}-t_j}\right.\\
&\quad\left.-\frac{(\alpha f+\beta g)\left(t_j\right)-(\alpha f+\beta g)\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^p\\
&=V_{(p,2)}^{W}\left(\alpha f+\beta g;[a,b]\right).
\end{aligned}$$

Then, $V_{(p,2)}^{W}\left(\cdot;[a,b]\right)$ is a convex function.

Definition 5 *(Norm in* $BV_{(p,2)}^{W}[a,b]$ *) The functional* $\|\cdot\|_{(p,2)}^{W}: BV_{(p,2)}^{W}[a,b]\to\mathbb{R}$ *defined by*

$$\|f\|_{(p,2)}^{W}:=\left|f(a)\right|+\left|f'(a)\right|+V_{(p,2)}^{W}\left(f;[a,b]\right)^{\frac{1}{p}} \tag{2.1}$$

is a norm.

In [7], the authors have shown that the linear space $BV_{(p,2)}^{W}[a,b]$ with the norm (2.1) is a Banach space and $BV_{(p,2)}^{W}[a,b]\subset BV_{p}^{W}[a,b]$.

3. Main Results

In [23] the authors present and study the space of functions of bounded $p(\cdot)$-variation as an extension of the space $BV_{p}^{W}[a,b]$. In this section, our goal is to study the corresponding space of functions of bounded second $p(\cdot)$-variation, with $p(\cdot)$ be a variable exponent, as an extension of $BV_{(p,2)}^{W}[a,b]$.

Definition 6 *Let* p *be a function* $p:[a,b]\to(1,\infty)$, π *be a partition* $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$ *of the interval* $[a,b]$ *and* $f:[a,b]\to\mathbb{R}$ *be a function. The nonnegative real number*

$$V_{(p(\cdot),2)}^{W}(f)=V_{(p(\cdot),2)}^{W}\left(f;[a,b]\right):=\sup_{\pi^*}\sum_{j=1}^{n-1}\left|\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^{p\left(x_{j-1}\right)},$$

is called the De La Vallée Poussin-Wiener variation with variable exponent (or $\left(p(\cdot),2\right)$-variation in De La Vallée Poussin-Wiener's sense) of f on $[a,b]$, where π^* is a tagged partition of the interval $[a,b]$, *i.e.*, a partition of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-2}$ subject to the conditions $t_j \le x_j \le t_{j+1}$ for each j. It is worth to note that by definition (we take supremum over all partitions), the number $V^W_{(p(\cdot),2)}(f)$ does not depend on the election of the argument of the exponent. In the case that $V^W_{(p(\cdot),2)}(f)<\infty$, we say that f has bounded $(p(\cdot),2)$-variation on $[a,b]$.

We will denote by $BV^W_{(p(\cdot),2)}[a,b]=V^W_{(p(\cdot),2)}[a,b]$ the space of functions of bounded $(p(\cdot),2)$-variation in Wiener's sense with variable exponent in $[a,b]$. It is endowed with the functional:

$$\|f\|_{BV^W_{(p(\cdot),2)}[a,b]} = |f(a)| + |f'_+(a)| + \inf\left\{\lambda>0; V^W_{(p(\cdot),2)}\left(\frac{f}{\lambda};[a,b]\right)\le 1\right\}. \tag{3.1}$$

Then,

$$\left(BV^W_{(p(\cdot),2)}[a,b], \|\cdot\|_{BV^W_{(p(\cdot),2)}[a,b]}\right) := \left\{f:[a,b]\to\mathbb{R}; \|f\|_{BV^W_{(p(\cdot),2)}[a,b]}<\infty\right\}.$$

Remark 3.1 *Given a function* $p:[a,b]\to[1,\infty)$.

1) If $p(x)=1$ for all $x\in[a,b]$, then $BV^W_{(p(\cdot),2)}[a,b]=BV^2[a,b]$.

2) If $p(x)=p$ for all $x\in[a,b]$ and $1<p<\infty$ then $BV^W_{(p(\cdot),2)}[a,b]=BV^W_{(p,2)}[a,b]$, *i.e.*, the space of bounded $(p(\cdot),2)$-variation in De la Vallée Poisson-Wiener's sense with variable exponent is exactly the space of bounded $(p,2)$-variation in De la Vallée Poisson-Wiener's sense.

Given a function $p:[a,b]\to(1,\infty)$, that is, a variable exponent function, let us define as in the literature,

$$p^- := \operatorname{essinf}_{x\in[a,b]} p(x) = \sup\left\{\beta\in\mathbb{R}: \left|\left\{x\in[a,b]; p(x)<\beta\right\}\right|=0\right\},$$

and

$$p^+ := \operatorname{esssup}_{x\in[a,b]} p(x) = \inf\left\{\alpha\in\mathbb{R}: \left|\left\{x\in[a,b]; p(x)>\alpha\right\}\right|=0\right\}.$$

It is said that the exponent p is admissible if the range of p is in $(1,\infty)$ and p^+ is finite.

Let us recall a classical concept in the theory of function spaces. Let X be a vector space over $\mathbb{R}$. A convex and left-continuous function $\rho: X\to[0,\infty]$ is called a convex pseudo-modular on X if for arbitrary x and y, there holds:

1) $\rho(0x)=0$,

2) $\rho(\alpha x)=\rho(x)$ for every $\alpha\in\mathbb{R}$ such that $|\alpha|=1$,

3) $\rho(\alpha x+(1-\alpha)y)\le\alpha\rho(x)+(1-\alpha)\rho(y)$ for every $\alpha\in[0,1]$.

It is possible to see that for p be an admissible function, the functional $V^W_{(p(\cdot),2)}(\cdot;[a,b])$ is a convex pseudo-modular.

Proposition 2 *Let* p *be an admissible function. Then* $V^W_{(p(\cdot),2)}(\cdot;[a,b])$ *is a convex pseudo-modular.*

Proof. We have that for any $f\in BV^W_{(p(\cdot),2)}[a,b]$, $V^W_{(p(\cdot),2)}(0f;[a,b])=V^W_{(p(\cdot),2)}(0;[a,b])=0$. Moreover, the fact that for any

$f \in BV^W_{(p(\cdot),2)}[a,b]$, $V^W_{(p(\cdot),2)}(\alpha f;[a,b]) = V^W_{(p(\cdot),2)}(f;[a,b])$ whenever $|\alpha| = 1$ follows immediately from the definition.

Finally, with the same kind of argument than in Proposition 1(c) it follows that for $\alpha \in [0,1]$ and $f, g \in BV^W_{(p(\cdot),2)}[a,b]$ we have that

$$V^W_{(p(\cdot),2)}(\alpha f + (1-\alpha)g;[a,b]) \le \alpha V^W_{(p(\cdot),2)}(f;[a,b]) + (1-\alpha)V^W_{(p(\cdot),2)}(g;[a,b]).$$

Definition 7 *A convex and left-continuous function* $\rho : X \to [0,\infty]$ *is called semimodular on* X *if*

1) $\rho(0) = 0$,

2) $\rho(-x) = \rho(x)$ for every $x \in X$, and

3) if $\rho(\lambda x) = 0$ for every $\lambda \in \mathbb{R}$, then $x = 0$.

For p be an admissible function, the functional $V^W_{(p(\cdot),2)}(\cdot,[a,b])$ is a semimodular on X.

Proposition 3 *Let* p *be an admissible function. Then* $V^W_{(p(\cdot),2)}(\cdot,[a,b])$ *is a semimodular.*

Proof. Let $f \in BV^W_{(p(\cdot),2)}[a,b]$ and π^* be a tagged partition of $[a,b]$, then

$$\begin{aligned} V^W_{(p(\cdot),2)}(-f) &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{-(f(t_{j+1}) - f(t_j))}{t_{j+1} - t_j} - \frac{-(f(t_j) - f(t_{j-1}))}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left(|-1| \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right| \right)^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\ &= V^W_{(p(\cdot),2)}(f). \end{aligned}$$

On the other hand, if

$$\begin{aligned} V^W_{(p(\cdot),2)}(\lambda f) &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{\lambda f(t_{j+1}) - \lambda f(t_j)}{t_{j+1} - t_j} - \frac{\lambda f(t_j) - \lambda f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} \left(|\lambda| \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right| \right)^{p(x_{j-1})} \\ &= \sup_{\pi^*} \sum_{j=1}^{n-1} |\lambda|^{p(x_{j-1})} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = 0, \end{aligned}$$

for every λ, necessarily it follows that $f = 0$.

Proposition 4 *Let* X *be a vector space,* ρ *be a semimodular on* X *and* $f \in X$. *Then*

1) $\rho(f) \le 1$ if and only if $\|f\|_\rho \le 1$,

2) if $\|f\|_\rho \le 1$, then $\rho(f) \le \|f\|_\rho$,

3) if $\|f\|_\rho > 1$, then $\rho(f) \ge \|f\|_\rho$,

4) for every $f \in X$, $\|f\|_\rho \le \rho(f) + 1$.

Theorem 1 *Let* $f:[a,b] \to \mathbb{R}$ *be a function and* p *be an admissible function, then* $BV^2[a,b] \subset BV^W_{(p(\cdot),2)}[a,b]$.

Proof. Let p be an admissible function, π^* be a tagged partition of the interval $[a,b]$, $f \in BV^2[a,b]$ and

$$\sigma = \left\{ j \in \pi^* : \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right| \le 1 \right\}.$$

$$\begin{aligned}
&\sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\
&= \sum_{j\in\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\
&\le \sum_{j\in\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right| + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\
&\le \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right| + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\
&\le V^{(2)}(f;[a,b]) + \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}.
\end{aligned}$$

Then,

$$\begin{aligned}
V^W_{(p(\cdot),2)}(f) &:= \sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} \\
&\le V^{(2)}(f;[a,b]) + \sup_{\pi^*} \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})}.
\end{aligned}$$

The proof of the fact that $\sup_{\pi^*} \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} < \infty$ will be by contradiction. That is, we assume that $\sup_{\pi^*} \sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = \infty$. Therefore, there exists a tagged partition π^* such that

$$\sum_{j\notin\sigma} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = \infty.$$

Since $j \notin \sigma$ and $p(\cdot) > 1$ we get

$$\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|>1.$$

But this is satisfied only for a finite number of terms, because in opposite case we would get

$$V^{(2)}(f;[a,b])\geq\sum_{j\notin\sigma}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|>\sum_{j\notin\sigma}1\to\infty,$$

which is a contradiction as $f\in BV^2[a,b]$. Then, taking supremum we get

$$V^W_{(p(\cdot),2)}(f;[a,b])=\sup_{\pi^*}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}<\infty.$$

Theorem 2 *Let* p *be an admissible function. If* $f\in BV^W_{(p(\cdot),2)}[a,b]$*, then it follows that for any* $c\in(a,b)$

$$V^W_{(p(\cdot),2)}(f;[a,c])+V^W_{(p(\cdot),2)}(f;[c,b])\leq V^W_{(p(\cdot),2)}(f;[a,b]). \tag{3.2}$$

Proof. By the definition of $V^W_{(p(\cdot),2)}(f;[a,c])$ and $V^W_{(p(\cdot),2)}(f;[c,b])$ we have that, for each $\epsilon>0$, there are partitions $\pi_{(a,c)}$ and $\pi_{(c,b)}$ with $\pi_{(a,c)}:=\{a=\overline{t}_0,\cdots,\overline{t}_m=c\}$ and $\pi_{(c,b)}:=\{c=t_0,\cdots,t_r=b\}$, and sequences of points $\{x_j\}_{j=0}^{m-2}$ and $\{y_j\}_{j=0}^{r-2}$ such that $\overline{t}_j\leq x_j\leq\overline{t}_{j+1}$ for $j=0,\cdots,m-2$ and $t_j\leq y_j\leq t_{j+1}$ for $j=0,\cdots,r-2$ that satisfies

$$\sum_{j=1}^{m-1}\left|\frac{f(\overline{t}_{j+1})-\overline{t}(t_j)}{\overline{t}_{j+1}-\overline{t}_j}-\frac{f(\overline{t}_j)-f(\overline{t}_{j-1})}{\overline{t}_j-\overline{t}_{j-1}}\right|^{p(x_{j-1})}>V^W_{(p(\cdot),2)}(f;[a,c])-\frac{\epsilon}{2},$$

and

$$\sum_{j=1}^{r-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(y_{j-1})}>V^W_{(p(\cdot),2)}(f;[c,b])-\frac{\epsilon}{2}.$$

Taking $\pi=\pi_{(a,c)}\cup\pi_{(c,b)}=\{a=u_0,\cdots,u_{r+m-1}=b\}$ and the points $\{z_j\}_j:=\{x_j\}_{j=0}^{m-2}\cup\{y_j\}_{j=0}^{r-2}$, we get a partition of $[a,b]$ such that

$$\begin{aligned}&\sum_{j=1}^{m+r-2}\left|\frac{f(u_{j+1})-f(u_j)}{u_{j+1}-u_j}-\frac{f(u_j)-f(u_{j-1})}{u_j-u_{j-1}}\right|^{p(z_{j-1})}\\&=\sum_{j=1}^{r-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(y_{j-1})}\\&\quad+\sum_{j=1}^{m-1}\left|\frac{f(\overline{t}_{j+1})-\overline{t}(t_j)}{\overline{t}_{j+1}-\overline{t}_j}-\frac{f(\overline{t}_j)-f(\overline{t}_{j-1})}{\overline{t}_j-\overline{t}_{j-1}}\right|^{p(x_{j-1})},\end{aligned}$$

which implies that

$$\sum_{j=1}^{m+r-2}\left|\frac{f(u_{j+1})-f(u_j)}{u_{j+1}-u_j}-\frac{f(u_j)-f(u_{j-1})}{u_j-u_{j-1}}\right|^{p(z_{j-1})}$$
$$>V^W_{(p(\cdot),2)}(f;[c,b])-\frac{\epsilon}{2}+V^W_{(p(\cdot),2)}(f;[a,c])-\frac{\epsilon}{2}. \tag{3.3}$$

Letting $\epsilon\to 0$ first, and then taking the corresponding supremum in the left-hand side of (3.3), it follows (3.2).

Define
$$\omega_{p(x_{ts\sigma})}(f;[a,b]) := \sup_{t,s,\sigma\in[a,b]}\left\{\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})}\right\}.$$

Lemma 1 ***Basic properties of the*** $(p(\cdot),2)$ ***-variation in De La Vallée Poussin-Wiener's sense*** *Let* $f:[a,b]\to\mathbb{R}$ *be an arbitrary map. We have the following properties:*

(P1) For any $t,s,\sigma\in[a,b]$, we have that

$$\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})}\le\omega_{p(x_{ts\sigma})}(f;[a,b])\le V^W_{(p(\cdot),2)}(f;[a,b]).$$

(P2) Monotonicity: If $t,s\in[a,b]$ and $a\le t\le s\le b$, then $V^W_{(p(\cdot),2)}(f;[a,t])\le V^W_{(p(\cdot),2)}(f;[a,s])$, $V^W_{(p(\cdot),2)}(f;[s,b])\le V^W_{(p(\cdot),2)}(f;[t,b])$, and $V^W_{(p(\cdot),2)}(f;[t,s])\le V^W_{(p(\cdot),2)}(f;[a,b])$.

(P3) Semi-additivity: If $t\in(a,b)$, then

$$V^W_{(p(\cdot),2)}(f;[a,t])+V^W_{(p(\cdot),2)}(f;[t,b])\le V^W_{(p(\cdot),2)}(f;[a,b]).$$

(P4) Change of variable: If $\varphi:[c,d]\to[a,b]$ is a monotone function, then

$$V^W_{(p(\cdot),2)}(f;\varphi[c,d])=V^W_{(p(\cdot),2)}(f\circ\varphi;[c,d]). \tag{3.4}$$

(P5) Regularity: $V^W_{(p(\cdot),2)}(f;[a,b])=\sup\left\{V^W_{(p(\cdot),2)}(f;[s,t]);s,t\in[a,b]\right\}$.

Proof. (P1) We have that for any $t,s,\sigma\in[a,b]$,

$$\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})}$$
$$\le\sup\left\{\left|\frac{f(t)-f(s)}{t-s}-\frac{f(s)-f(\sigma)}{s-\sigma}\right|^{p(x_{ts\sigma})};t,s,\sigma\in[a,b]\right\}$$
$$:=\omega_{p(x_{ts\sigma})}(f;[a,b])$$
$$\le\sup_{\pi}\sum_{j=1}^{m-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}$$
$$=V^W_{(p(\cdot),2)}(f;[a,b]).$$

(P2) Let $a\le t\le s\le b$ and the partition $\pi:=\{a=t_0<t_1<\cdots<t_{m1}=t<\cdots<t_{m2}=s<\cdots<t_n=b\}$. Then

$$V^W_{(p(\cdot),2)}\left(f;[a,t]\right)=\sup_{\pi^*}\sum_{j=1}^{m1}\left|\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^{p\left(x_{j-1}\right)}$$

$$\le\sup_{\pi^*}\sum_{j=1}^{m1}\left|\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^{p\left(x_{j-1}\right)}$$

$$+\sup_{\pi^*}\sum_{j=m1+1}^{m2}\left|\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^{p\left(x_{j-1}\right)}$$

$$\le\sup_{\pi^*}\sum_{j=1}^{m2}\left|\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^{p\left(x_{j-1}\right)}$$

$$=V^W_{(p(\cdot),2)}\left(f;[a,s]\right).$$

The other cases follow in a similar way.

(P3) Semi-additivity: It is obtained in Theorem 2.

(P4) It follows as in ([23], Lemma 2 (P4)). Indeed, let $[c,d]\subset\mathbb{R}$, $\varphi:[c,d]\to[a,b]$ be a (not necessarily strictly) monotone function, π_0 be a tagged partition of the interval $[c,d]$, $T_1=\{\tau_j\}_{j=0}^m\in\pi_0$ and $T=\{t_j\}_{j=0}^m$ with $t_j=\varphi(\tau_j)$, then

$$V^W_{(p(\cdot),2)}\left(f\circ\varphi,T_1\right)$$

$$=\sup_{T_1}\sum_{j=1}^{m}\left|\frac{f\left(\varphi\left(\tau_{j+1}\right)\right)-f\left(\varphi\left(\tau_j\right)\right)}{\tau_{j+1}-\tau_j}-\frac{f\left(\varphi\left(\tau_j\right)\right)-f\left(\varphi\left(\tau_{j-1}\right)\right)}{\tau_j-\tau_{j-1}}\right|^{p\left(x_{j-1}\right)}$$

$$=\sup_{T}\sum_{j=1}^{m}\left|\frac{f\left(t_{j+1}\right)-f\left(t_j\right)}{t_{j+1}-t_j}-\frac{f\left(t_j\right)-f\left(t_{j-1}\right)}{t_j-t_{j-1}}\right|^{p\left(x_{j-1}\right)}$$

$$=V^W_{(p(\cdot),2)}\left(f,T\right)\le V^W_{(p(\cdot),2)}\left(f,\varphi\left([c,d]\right)\right).$$

On the other hand, if a partition $T=\{t_j\}_{j=0}^m$ of $\varphi([c,d])$ is such that $t_{j-1}<t_j$ for $j=1,\cdots,m$ then there exists $\tau_j\in[c,d]$ such that $t_j=\varphi(\tau_j)$ and, again by the monotonicity of φ

$$V^W_{(p(\cdot),2)}\left(f,T\right)=V^W_{(p(\cdot),2)}\left(f\circ\varphi,T_1\right)\le V^W_{(p(\cdot),2)}\left(f,\varphi\left([c,d]\right)\right).$$

(P5) By monotonicity $V^W_{(p(\cdot),2)}\left(f;[a,b]\right)\ge\sup\left\{V^W_{(p(\cdot),2)}\left(f;[s,t]\right);s,t\in[a,b]\right\}$. On the other hand, for any $\alpha<V^W_{(p(\cdot),2)}\left(f;[a,b]\right)$ such that there exists a tagged partition $\Pi=\{t_i\}_{i=0}^n$ of $[a,b]$ with $V^W_{(p(\cdot),2)}\left(f;\Pi\right)\ge\alpha$. We define $\overline{\pi}$ a partition of the interval $[t_0,t_m]$ then $\Pi\in\overline{\pi}$ and $V^W_{(p(\cdot),2)}\left(f;\overline{\pi}\right)\ge V^W_{(p(\cdot),2)}\left(f;\Pi\right)\ge\alpha$, *i.e.*,

$$V^W_{(p(\cdot),2)}\left(f;[a,b]\right)\le\sup\left\{V^W_{(p(\cdot),2)}\left(f;[s,t]\right);s,t\in[a,b]\right\}.$$

Lemma 2 *If* $\beta_1 > \beta_2$, *then* $V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right) \le V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_2};[a,b]\right)$ *for all* $f \in BV^W_{(p(\cdot),2)}[a,b]$.

Proof. Let β_1, β_2 such that $\beta_1 > \beta_2$. Then, consider any partition π of $[a,b]$, $\pi = \{a = t_0, \cdots, t_n = b\}$ and any finite sequence of numbers $x_0, \cdots, x_{n-2}$ subject to the conditions $t_j \le x_j \le t_{j+1}$ for each $j \le n-2$. It follows that

$$\left|\frac{\left(\frac{f}{\beta_1}\right)(t_{i+1}) - \left(\frac{f}{\beta_1}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_1}\right)(t_i) - \left(\frac{f}{\beta_1}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

$$= \left|\frac{1}{\beta_1}\left[\frac{f(t_{i+1}) - f(t_i)}{t_{i+1}-t_i} - \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right]\right|^{p(x_{i-1})}$$

$$\le \left|\frac{1}{\beta_2}\left[\frac{f(t_{i+1}) - f(t_i)}{t_{i+1}-t_i} - \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right]\right|^{p(x_{i-1})}$$

$$= \left|\frac{\left(\frac{f}{\beta_2}\right)(t_{i+1}) - \left(\frac{f}{\beta_2}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_2}\right)(t_i) - \left(\frac{f}{\beta_2}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

as $\frac{1}{\beta_2} \ge \frac{1}{\beta_1}$. Then, as this inequality follows for all terms in the sum

$$\sum_{i=1}^{n-1}\left|\frac{\left(\frac{f}{\beta_1}\right)(t_{i+1}) - \left(\frac{f}{\beta_1}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_1}\right)(t_i) - \left(\frac{f}{\beta_1}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

$$\le \sum_{i=1}^{n-1}\left|\frac{\left(\frac{f}{\beta_2}\right)(t_{i+1}) - \left(\frac{f}{\beta_2}\right)(t_i)}{t_{i+1}-t_i} - \frac{\left(\frac{f}{\beta_2}\right)(t_i) - \left(\frac{f}{\beta_2}\right)(t_{i-1})}{t_i - t_{i-1}}\right|^{p(x_{i-1})}$$

Taking supremum in any partition, it follows that

$$V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right) \le V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_2};[a,b]\right).$$

Proposition 5 *Let* p *be an admissible function. The space* $BV^W_{(p(\cdot),2)}[a,b]$ *is a vectorial space.*

Proof. Let $f, g \in BV^W_{(p(\cdot),2)}[a,b]$ and consider any partition $\pi = \{a = t_0, \cdots, t_n = b\}$ and any finite sequence of numbers $x_0, \cdots, x_{n-2}$ subject to the conditions $t_j \le x_j \le t_{j+1}$ and $\alpha, \beta \in \mathbb{R}$. By definition, there exists β_1, β_2 such that

$$V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right)\le 1<\infty \quad \text{and} \quad V^W_{(p(\cdot),2)}\left(\frac{g}{\beta_2};[a,b]\right)\le 1<\infty.$$

Let $\hat{\beta}:=\max\{\beta_1,\beta_2\}>0$. By Lemma 2, it follows that

$$V^W_{(p(\cdot),2)}\left(\frac{f}{\hat{\beta}};[a,b]\right)<V^W_{(p(\cdot),2)}\left(\frac{f}{\beta_1};[a,b]\right)<\infty$$

$$V^W_{(p(\cdot),2)}\left(\frac{g}{\hat{\beta}};[a,b]\right)<V^W_{(p(\cdot),2)}\left(\frac{g}{\beta_2};[a,b]\right)<\infty.$$

The rest of the proof follows analyzing the possible cases.

1) If $\alpha=\beta=0$, then $\alpha f+\beta g\in BV^W_{(p(\cdot),2)}[a,b]$.

2) If $\alpha\ne 0$ and/or $\beta\ne 0$. Let $\mu=(|\alpha|+|\beta|)\hat{\beta}>0$, and consider any tagged partition π^* of $[a,b]$, $\pi^*=\{a=t_0\le\cdots\le t_n=b\}$ which is any partition π of $[a,b]$ and any finite sequence of numbers $x_0,\cdots,x_{n-2}$ subject to the conditions $t_j\le x_j\le t_{j+1}$ for each $j\le n-2$. Then, by convexity of t^p, when $1<p<\infty$, it follows that

$$\sum_{j=1}^{n-1}\left|\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j+1})-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)}{t_{j+1}-t_j}-\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}$$

$$=\sum_{j=1}^{n-1}\left|\frac{1}{\mu}\frac{\left[\alpha\left(f(t_{j+1})-f(t_j)\right)+\beta\left(g(t_{j+1})-g(t_j)\right)\right]}{t_{j+1}-t_j}-\frac{1}{\mu}\frac{\left[\alpha\left(f(t_j)-f(t_{j-1})\right)+\beta\left(g(t_j)-g(t_{j-1})\right)\right]}{t_j-t_{j-1}}\right|^{p(x_{j-1})}$$

$$\le\sum_{j=1}^{n-1}\left(\frac{|\alpha|}{\mu}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|+\frac{|\beta|}{\mu}\left|\frac{g(t_{j+1})-g(t_j)}{t_j-t_{j-1}}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$\le\sum_{j=1}^{n-1}\left(\frac{|\alpha|}{|\alpha|+|\beta|}\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right.$$
$$\left.+\frac{|\beta|}{|\alpha|+|\beta|}\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$\le\sum_{j=1}^{n-1}\left[\frac{|\alpha|}{|\alpha|+|\beta|}\left(\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}\right.$$
$$\left.+\frac{|\beta|}{|\alpha|+|\beta|}\left(\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}\right]$$

$$\le\frac{|\alpha|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$
$$+\frac{|\beta|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}.$$

Therefore,

$$\sum_{j=1}^{n-1}\left|\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j+1})-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)}{t_{j+1}-t_j}-\frac{\left(\frac{\alpha f+\beta g}{\mu}\right)(t_j)-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}$$

$$\leq\frac{|\alpha|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$+\frac{|\beta|}{|\alpha|+|\beta|}\sum_{j=1}^{n-1}\left(\frac{1}{\hat{\beta}}\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|\right)^{p(x_{j-1})}$$

$$\leq\frac{|\alpha|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{f}{\hat{\beta}};[a,b]\right)+\frac{|\beta|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{g}{\hat{\beta}};[a,b]\right)<\infty.$$

Then, taking supremum over all partitions, we get that

$$\begin{aligned}&V_{(p(\cdot),2)}^{W}\left(\frac{\alpha f+\beta g}{\mu};[a,b]\right)\\&\leq\frac{|\alpha|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{f}{\hat{\beta}};[a,b]\right)+\frac{|\beta|}{|\alpha|+|\beta|}V_{(p(\cdot),2)}^{W}\left(\frac{g}{\hat{\beta}};[a,b]\right)\\&\leq\frac{|\alpha|}{|\alpha|+|\beta|}+\frac{|\beta|}{|\alpha|+|\beta|}=1<\infty.\end{aligned}$$

Therefore $\alpha f+\beta g\in BV_{(p(\cdot),2)}^{W}[a,b]$.

The other properties of a vectorial space follow similarly.

Theorem 3 *Let* p *be an admissible function. The space* $BV_{(p(\cdot),2)}^{W}[a,b]$ *is a normed space.*

Proof. Let p be an admissible function. Let us analyze all the properties of a norm.

1) By definition of $\|\cdot\|_{BV_{(p(\cdot),2)}^{W}[a,b]}$, we have that $\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}\geq 0$ for all $f\in BV_{(p(\cdot),2)}^{W}[a,b]$

2) To prove that $\|\alpha f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=|\alpha|\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}$ for any $\alpha\in\mathbb{R}$, we consider the possible cases:

- If $\alpha=0$, then

$$\|\alpha f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=\|0\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=0=0\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}=\alpha\|f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}$$

for any $f\in BV_{(p(\cdot),2)}^{W}[a,b]$.

- If $\alpha\neq 0$, then

$$\begin{aligned}\|\alpha f\|_{BV_{(p(\cdot),2)}^{W}[a,b]}&=|\alpha f(a)|+|\alpha f_{+}'(a)|+\inf\left\{\lambda>0;V_{(p(\cdot),2)}^{W}\left(\frac{\alpha f}{\lambda};[a,b]\right)\leq 1\right\}\\&=|\alpha||f(a)|+|\alpha||f_{+}'(a)|+\inf\left\{\lambda>0;V_{(p(\cdot),2)}^{W}\left(\frac{f}{\frac{\lambda}{\alpha}};[a,b]\right)\leq 1\right\}\end{aligned}$$

$$= |\alpha| |f(a)| + |\alpha| |f'_+(a)| + \inf \left\{ \alpha \frac{\lambda}{\alpha} > 0; V^W_{(p(\cdot),2)} \left(\frac{f}{\frac{\lambda}{\alpha}}; [a,b] \right) \le 1 \right\}$$

$$= |\alpha| |f(a)| + |\alpha| |f'_+(a)| + \alpha \inf \left\{ \frac{\lambda}{\alpha} > 0; V^W_{(p(\cdot),2)} \left(\frac{f}{\frac{\lambda}{\alpha}}; [a,b] \right) \le 1 \right\}$$

$$= |\alpha| |f(a)| + |\alpha| |f'_+(a)| + \alpha \inf \left\{ \beta > 0; V^W_{(p(\cdot),2)} \left(\frac{f}{\beta}; [a,b] \right) \le 1 \right\}$$

$$= |\alpha| \|f\|_{BV^W_{(p(\cdot),2)}[a,b]},$$

3) Property $\|f+g\|_{BV^W_{(p(\cdot),2)}[a,b]} \le \|f\|_{BV^W_{(p(\cdot),2)}[a,b]} + \|g\|_{BV^W_{(p(\cdot),2)}[a,b]}$ is satisfied by using that $|f+g| \le |f| + |g|$, $\left|(f+g)'_+\right| = |f'_+ + g'_+| \le |f'_+| + |g'_+|$ and the previous proposition.

4) Let us see that $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]} = 0$ if and only if $f = 0$.

- If $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]} = 0$, then by definition of the norm, $f(a) = 0$ and $f'_+(a) = 0$, and

$$\inf \left\{ \lambda > 0; V^W_{(p(\cdot),2)} \left(\frac{f}{\lambda}; [a,b] \right) \le 1 \right\} = 0.$$

Hence, we have by Proposition 3 and Proposition 4 (2) that

$$V^W_{(p(\cdot),2)}(f; [a,b]) \le \|f\|_{BV^W_{(p(\cdot),2)}[a,b]}.$$

Therefore, $V^W_{(p(\cdot),2)}(f; [a,b]) = 0$, and hence,

$$\sup_{\pi^*} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = 0.$$

Therefore, for any tagged partition π^* of the interval $[a,b]$, that is a partition $\pi = \{a = t_0 < \cdots < t_n = b\}$ together with a finite sequence of numbers $x_0, \cdots, x_n$ subject to the conditions $t_j \le x_j \le t_{j+1}$ for each j, we have that

$$\left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|^{p(x_{j-1})} = 0, \quad \forall j \in \{1, \cdots, n-1\}.$$

So that

$$\frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} = \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}}, \quad \forall j \in \{1, \cdots, n-1\}.$$

Consider the partition $\pi = \{a \le t_1 < t_2 = c < t \le b\}$. We get that

$$\lim_{c \to a+} \frac{f(t) - f(c)}{t - c} = \lim_{c \to a+} \frac{f(c) - f(a)}{c - a} = f'_+(a) = 0.$$

Then

$$\frac{f(t)-f(a)}{t-a}=0.$$

As $f(a)=0$ is obtained that $f(t)=0$ for all $t\in[a,b]$.

- In other hand, if $f=0$, then $f(t)=0$ for all $t\in[a,b]$. Hence, $f'_+(a)=0$ and $V^W_{(p(\cdot),2)}(f;[a,b])=V^W_{(p(\cdot),2)}(0;[a,b])=0$. Therefore, by definition, $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}=0$.

Theorem 4 *Let* p *be an admissible function. The space* $BV^W_{(p(\cdot),2)}[a,b]$ *is a Banach space endowed with the norm in* (3.1).

Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $BV^W_{(p(\cdot),2)}[a,b]$. Then, for all $\epsilon>0$, there exists $N(\epsilon)$ such that

$$\|f_m-f_n\|_{BV^W_{(p(\cdot),2)}[a,b]}<\epsilon,\quad \forall m,n>N(\epsilon).$$

Therefore, by definition it follows that

$$\inf\left\{\lambda>0;V^W_{(p(\cdot),2)}\left(\frac{f_m-f_n}{\lambda};[a,b]\right)\le 1\right\}<\epsilon,\quad \forall m,n>N(\epsilon),\tag{3.5}$$

$$\left|(f_m-f_n)(a)\right|<\epsilon,\quad \forall m,n>N(\epsilon),\tag{3.6}$$

and

$$\left|(f_m-f_n)'_+(a)\right|<\epsilon,\quad \forall m,n>N(\epsilon).\tag{3.7}$$

Then, by (3.5) and Proposition 4 (2) we have that

$$V^W_{(p(\cdot),2)}(f_m-f_n;[a,b])<\epsilon.$$

It implies that for fixed t, $\{f_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Indeed,

$$V^W_{(p(\cdot),2)}\left(\frac{f_m-f_n}{\epsilon}\right)\le 1,\quad \forall m,n>N(\epsilon)$$

then for all $x,y,z\in[a,b]$, $f=f_m-f_n$ we get

$$\left|\frac{1}{\epsilon}\left(\frac{f(z)-f(y)}{z-y}-\frac{f(y)-f(x)}{y-x}\right)\right|^{p(y)}\le V^W_{(p(\cdot),2)}\left(\frac{f_m-f_n}{\epsilon}\right)\le 1$$

so

$$\left|\frac{f(z)-f(y)}{z-y}-\frac{f(y)-f(x)}{y-x}\right|^{p(y)}\le\epsilon^{p(y)}.$$

As

$$\begin{aligned}\left|\frac{f(z)-f(y)}{z-y}\right|^{p(y)}&\le\left\|\frac{f(z)-f(y)}{z-y}\right|-\left|\frac{f(y)-f(x)}{y-x}\right\|^{p(y)}\\&\le\left|\frac{f(z)-f(y)}{z-y}-\frac{f(y)-f(x)}{y-x}\right|^{p(y)}\end{aligned}$$

thus

$$\left|\frac{f(z)-f(y)}{z-y}\right|^{p(y)} \le \epsilon^{p(y)}.$$

Therefore

$$|f(z)-f(y)|^{p(y)} \le (\epsilon|z-y|)^{p(y)}$$

and by property of log

$$p(y)\log|f(z)-f(y)| \le p(y)\log(\epsilon|z-y|).$$

Then

$$\log|f(z)-f(y)| \le \log(\epsilon|z-y|)$$

and hence

$$|f(z)-f(y)| \le \epsilon' = \exp^{\log(\epsilon|z-y|)} = \epsilon|z-y|.$$

i.e.

$$|(f_m-f_n)(z)-(f_m-f_n)(y)| \le \epsilon', \quad \forall m,n > N(\epsilon).$$

Let $f(t) := \lim_{n\to\infty} f_n(t)$ for any $t\in[a,b]$ and let π be any partition $\pi := \{a=t_0,\cdots,t_k=b\}$ of $[a,b]$ and a sequence $x_0,\cdots,x_{k-1}$ such that $t_j \le x_j \le t_{j+1}$ for any $1\le j<k-1$. It follows that for all $m,n > N(\epsilon)$

$$\sum_{j=1}^{k}\left|\frac{(f_m-f_n)(t_{j+1})-(f_m-f_n)(t_j)}{t_{j+1}-t_j}-\frac{(f_m-f_n)(t_j)-(f_m-f_n)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})} < \epsilon.$$

Then, letting $n\to\infty$, for any $m > N(\epsilon)$ it follows that

$$\sum_{j=1}^{k}\left|\frac{(f_m-f)(t_{j+1})-(f_m-f)(t_j)}{t_{j+1}-t_j}-\frac{(f_m-f)(t_j)-(f_m-f)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})} < \epsilon. \tag{3.8}$$

Therefore, as (3.8) follows for any tagged partition π^* of $[a,b]$, taking supremum over all tagged partitions it follows that

$$V^W_{(p(\cdot),2)}(f_m-f;[a,b]) < \epsilon, \quad \forall m > N(\epsilon). \tag{3.9}$$

Moreover, by (3.6) and (3.7), we have that

$$|(f_m-f_n)(a)| < \epsilon, \quad \left|(f_m-f_n)'_+(a)\right| < \epsilon, \quad \forall m,n > N(\epsilon).$$

Then, letting $n\to\infty$, we have that

$$|(f_m-f)(a)| < \epsilon, \quad \left|(f_m-f)'_+(a)\right| < \epsilon, \quad \forall m > N(\epsilon). \tag{3.10}$$

Then, (3.9) and (3.10) imply that for m sufficiently large

$$\|f_m-f\|_{BV^W_{(p(\cdot),2)}[a,b]} < 3\epsilon.$$

Hence, as

$$\|f\|_{BV^{W}_{(p(\cdot),2)}[a,b]} \le \|f_m - f\|_{BV^{W}_{(p(\cdot),2)}[a,b]} + \|f_m\|_{BV^{W}_{(p(\cdot),2)}[a,b]} < \infty,$$

we obtain that $f \in BV^{W}_{(p(\cdot),2)}[a,b]$.

Theorem 5 *Let* p *be an admissible function. Then, we have:*

1) If $f \in BV^{W}_{(p(\cdot),2)}[a,b]$, then f is bounded in all the interval $[a,b]$.

2) $BV^{W}_{(p(\cdot),2)}[a,b] \hookrightarrow BV^{W}_{(q(\cdot),2)}[a,b]$ for functions $q(x) \ge p(x)$.

Let us proof (a). Suppose that $f \in BV^{W}_{(p(\cdot),2)}[a,b]$ and f is not bounded. Then, there exists a sequence $\{t_n\}_{n\ge1}$, $t_n \in (a,b)$, $n \ge 1$ such that $|f(t_n)| \to \infty$ when $n \to \infty$. Let $\{t_m\}_{m\ge1}$ be a subsequence of $\{t_n\}_{n\ge1}$ such that $\{t_m\}_{m\ge1}$ converge to $x \in [a,b]$. As $\{f(t_m)\}_{m\ge1}$ is a subsequence of $\{f(t_n)\}_{n\ge1}$, so

$$|f(t_m)| \to \infty \quad \text{when } n \to \infty.$$

Case 1: Suppose that $x = a$ and let t such that $a \le t_m < t < b$ for some $t_m \in \{t_m\}_{m\ge1}$, then

$$\left|\frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(t_m)}{t-t_m}\right|^{p(x_t)} \le V^{W}_{(p(\cdot),2)}(f;[a,b])$$

and since $u \to u^s$ is continuous

$$\left|\frac{f(b)-f(t)}{b-t} - \frac{\lim\limits_{m\to\infty} f(t)-f(t_m)}{t-x}\right|^{p(x_t)}$$

$$= \lim_{m\to\infty}\left|\frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(t_m)}{t-t_m}\right|^{p(x_t)} \le V^{W}_{(p(\cdot),2)}(f;[a,b]).$$

On the other hand $|f(t)-f(t_m)|$ tend to infinity as $m \to \infty$. Then

$$\lim_{m\to\infty}\left|\frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(t_m)}{t-t_m}\right|^{p(x_t)} = \infty,$$

and hence $V^{W}_{(p(\cdot),2)}(f;[a,b]) = \infty$, which is a contradiction.

Case 2: Suppose that $x \ne a$ and let t such that $a < t < t_m < b$ for some $t_m \in \{t_m\}_{m\ge1}$, then

$$\left|\frac{f(t_m)-f(t)}{t_m-t} - \frac{f(t)-f(a)}{t-a}\right|^{p(x_t)} \le V^{W}_{(p(\cdot),2)}(f;[a,b]).$$

Since $u \to u^s$ is continuous

$$\left|\frac{\lim\limits_{m\to\infty} f(t_m)-f(t)}{x-t} - \frac{f(t)-f(a)}{t-a}\right|^{p(x_t)}$$

$$= \lim_{m\to\infty}\left|\frac{f(t_m)-f(t)}{t_m-t} - \frac{f(t)-f(a)}{t-a}\right|^{p(x_t)} \le V^{W}_{(p(\cdot),2)}(f;[a,b]).$$

On the other hand $|f(t_m)-f(t)|$ tend to infinity as $m \to \infty$ then

$$\left|\frac{\lim_{m\to\infty} f(t_m)-f(t)}{x-t}-\frac{f(t)-f(a)}{t-a}\right|^{p(x_t)} \to \infty \quad \text{as } n\to\infty,$$

and then $V^W_{(p(\cdot),2)}(f)=\infty$, which is a contradiction.

Let us proof (b). Taking $\|f\|_{BV^W_{(p(\cdot),2)}[a,b]}=1$, since $V^W_{(p(\cdot),2)}(f;[a,b])\le 1$, it follows that

$$\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})} \le 1,$$

for any tagged partition $\pi:=\{a=t_0<\cdots<t_n=b\}$ and any sequence of points x_j such that $t_j\le x_j\le t_{j+1}$ for $j=0,\cdots,n-2$. Therefore,

$$\begin{aligned}&\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{q(x_{j-1})}\\ &\le \sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})} \le 1,\end{aligned}$$

since in particular $\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})} \le 1$ for any $1\le j\le n-1$. Taking supremum to both sides, we obtain that $V^W_{(q(\cdot),2)}(f;[a,b])\le V^W_{(p(\cdot),2)}(f;[a,b])$. Then, by definition it follows that

$$\|f\|_{BV^W_{(q(\cdot),2)}[a,b]}\le \|f\|_{BV^W_{(p(\cdot),2)}[a,b]},$$

and the general case follows from the homogeneity of the norm. □

4. Functions in $BV^W_{(p(\cdot),2)}[a,b]$ and Hölder Continuous Functions

In this section we prove also that if a function is the composition of a bounded monotone function with a $(\gamma(\cdot)+1)$-Hölder continuous function with $\gamma(\cdot)=1/p(\cdot)$, then the function is in $BV^W_{(p(\cdot),2)}[a,b]$.

Definition 8 *A function* $g:[a,b]\to\mathbb{R}$ *is Hölder continuous of exponent* γ, *where* $\gamma(\cdot)$ *is a positive function such that* $0\le\gamma(x)\le 1$, *if*

$$|g(t_i)-g(t_{i-1})|\le C|t_i-t_{i-1}|^{\gamma(x_{i-1})}$$

for all $x_{i-1}\in[a,b]$. The least number C satisfying the above inequality is called the Hölder constant of g.

Proposition 6 *Let* p *be an admissible function and* $f:[a,b]\to\mathbb{R}$ *such that* $f=g\circ\varphi$, *where* $\varphi:[a,b]\to\mathbb{R}$ *is a bounded monotone function and* $g:\varphi[a,b]\to\mathbb{R}$ *is* $(\gamma(\cdot)+1)$*-Hölder continuous with* $\gamma(\cdot)=\frac{1}{p(\cdot)}$. *Then*

$f \in BV^W_{(p(\cdot),2)}[a,b]$.

Proof. Assume that φ is nondecreasing. Since $\varphi([a,b]) = [\varphi(a),\varphi(b)]$, by virtue of the change of variable

$$V^W_{(p(\cdot),2)}(f;[a,b]) = V^W_{(p(\cdot),2)}(g\circ\varphi;[a,b]) = V^W_{(p(\cdot),2)}(g;[\varphi(a),\varphi(b)]). \tag{4.1}$$

If $T = \{t_i\}_{i=0}^n$ is a partition of $[\varphi(a),\varphi(b)]$ and $\{x_j\}$ is a sequence of points $x_j \in (t_j,t_{j+1})$ for $j = 0,\cdots,n-2$ then

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left|\frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}-\frac{g(t_i)-g(t_{i-1})}{t_i-t_{i-1}}\right|^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}\left(\left|\frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\right|+\left|\frac{g(t_i)-g(t_{i-1})}{t_i-t_{i-1}}\right|\right)^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}\left(\frac{C|t_{i+1}-t_i|^{\gamma(x_{i-1})+1}}{|t_{i+1}-t_i|}+\frac{C|t_i-t_{i-1}|^{\gamma(x_{i-1})+1}}{|t_i-t_{i-1}|}\right)^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}\left(C|t_{i+1}-t_i|^{\gamma(x_{i-1})}+C|t_i-t_{i-1}|^{\gamma(x_{i-1})}\right)^{p(x_{i-1})}\\
&\le \sum_{i=1}^{n-1}2^{p(x_{i-1})}\left(C^{p(x_{i-1})}|t_{i+1}-t_i|^{(\gamma(x_{i-1}))p(x_{i-1})}+C^{p(x_{i-1})}|t_i-t_{i-1}|^{(\gamma(x_{i-1}))p(x_{i-1})}\right)\\
&\le \sum_{i=1}^{n-1}2^{p^+}\left(C^{p^+}|t_{i+1}-t_i|+C^{p^+}|t_i-t_{i-1}|\right)\le 2^{p^++1}C^{p^+}|\varphi(b)-\varphi(a)|.
\end{aligned}$$

Therefore, by taking supremum over any tagged partition, it follows that

$$V^W_{(p(\cdot),2)}(g;[\varphi(a),\varphi(b)]) \le 2^{p^++1}C^{p^+}|\varphi(b)-\varphi(a)| < \infty$$

by the boundedness of φ. Hence, by (4.1)

$$V^W_{(p(\cdot),2)}(f;[a,b]) = V^W_{(p(\cdot),2)}(g;[\varphi(a),\varphi(b)]) < \infty.$$

5. The Matkowski's Condition

Let us show as an application that, any uniformly bounded composition operator that maps the space $BV^W_{(p(\cdot),2)}[a,b]$ into itself satisfies the Matkowski's condition.

Theorem 6 *Suppose that the composition operator* H *generated by* h *maps* $BV^W_{(p(\cdot),2)}[a,b]$ *into itself and satisfies the following inequality*

$$\|Hf_1 - Hf_2\|^W_{(p(\cdot),2)} \le \gamma\left(\|f_1-f_2\|^W_{(p(\cdot),2)}\right),\quad \left(f_1,f_2 \in BV^W_{(p(\cdot),2)}[a,b]\right), \tag{5.1}$$

for any function $\gamma:[0,\infty)\to[0,\infty)$. Then, there exist functions $\alpha,\beta \in BV^W_{(p(\cdot),2)}[a,b]$ such that

$$h(t,x) = \alpha(t)x+\beta(t),\quad t\in[a,b], x\in\mathbb{R}. \tag{5.2}$$

Proof. By hypothesis, for $x\in\mathbb{R}$ fixed, the constant function $f(t) = x$, $t\in[a,b]$ belongs to $BV^W_{(p(\cdot),2)}[a,b]$. Since H maps $BV^W_{(p(\cdot),2)}[a,b]$ into itself, we have that $(Hf)(t)=h(t,f(t))\in BV^{W}_{(p(\cdot),2)}[a,b]$.

From inequality (5.1) and definition of the norm $\|\cdot\|^{W}_{(p(\cdot),2)}$, we have for $f_1,f_2\in BV^{W}_{(p(\cdot),2)}[a,b]$,

$$\inf\left\{\lambda>0;V^{W}_{(p(\cdot),2)}\left(\frac{Hf_1-Hf_2}{\lambda};[a,b]\right)\le 1\right\}$$
$$\le\|Hf_1-Hf_2\|^{W}_{(p(\cdot),2)}\le\gamma\left(\|f_1-f_2\|^{W}_{(p(\cdot),2)}\right),$$

and then

$$V^{W}_{(p(\cdot),2)}\left(\frac{Hf_1-Hf_2}{\gamma\left(\|f_1-f_2\|^{W}_{(p(\cdot),2)}\right)};[a,b]\right)\le 1. \tag{5.3}$$

Consider $a\le s<t\le b$ and let $\pi_m:=\{t_0,t_1,\cdots,t_{2m}\}\in\pi$ be the equidistant partition defined by

$$t_0=s,\ t_j-t_{j-1}=\frac{t-s}{2m},\quad (j=1,2,\cdots,2m).$$

Given $u,v\in\mathbb{R}$ with $u\ne v$, define $f_1,f_2:[a,b]\to\mathbb{R}$ by

$$f_1(x):=\begin{cases} v, & \text{if } x=t_j \text{ for some even } j,\\ \dfrac{u+v}{2}, & \text{if } x=t_j \text{ for some odd } j,\\ \text{linear}, & \text{otherwise}\end{cases}$$

and

$$f_2(x):=\begin{cases} \dfrac{u+v}{2}, & \text{if } x=t_j \text{ for some even } j,\\ u, & \text{if } x=t_j \text{ for some odd } j,\\ \text{linear}, & \text{otherwise.}\end{cases}$$

Then, the difference f_1-f_2 satisfies that $|f_1(x)-f_2(x)|=\frac{|u-v|}{2}$ for all $x\in[a,b]$. Therefore, by the inequality (5.1)

$$\|Hf_1-Hf_2\|^{W}_{(p(\cdot),2)}\le\gamma\left(\|f_1-f_2\|^{W}_{(p(\cdot),2)}\right)$$
$$\le\gamma\left(\frac{|u-v|}{2}\right),$$

and hence, by definition

$$V^{W}_{(p(\cdot),2)}\left(\frac{Hf_1-Hf_2}{\gamma\left(\frac{|u-v|}{2}\right)};[a,b]\right)\le 1. \tag{5.4}$$

From the inequality (5.4), and the definition, it follows that for any partition $\{t_0, t_2, t_4, \cdots, t_{2(m-1)}\}$ of $[a,b]$

$$\sum_{j=1}^{m-1}\left|\frac{h(f_1)(t_{2j})-h(f_2)(t_{2j})-h(f_1)(t_{2j-1})+h(f_2)(t_{2j-1})}{|t_{2j}-t_{2j-1}|\gamma\left(\frac{|u-v|}{2}\right)}\right.$$
$$\left.-\frac{h(f_1)(t_{2j-1})-h(f_2)(t_{2j-1})-h(f_1)(t_{2j-2})+h(f_2)(t_{2j-2})}{|t_{2j-1}-t_{2j-2}|\gamma\left(\frac{|u-v|}{2}\right)}\right|^{p(x_{j-1})}\leq 1.$$

However, by the definition of f_1 and f_2, we have that

$$\sum_{j=1}^{m-1}\left|\frac{h(f_1)(t_{2j})-h(f_2)(t_{2j})-h(f_1)(t_{2j-1})+h(f_2)(t_{2j-1})}{|t_{2j}-t_{2j-1}|\gamma\left(\frac{|u-v|}{2}\right)}\right.$$
$$\left.-\frac{h(f_1)(t_{2j-1})-h(f_2)(t_{2j-1})-h(f_1)(t_{2j-2})+h(f_2)(t_{2j-2})}{|t_{2j-1}-t_{2j-2}|\gamma\left(\frac{|u-v|}{2}\right)}\right|^{p(x_{j-1})}$$
$$=\sum_{j=1}^{m-1}\left(\frac{2\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\frac{t-s}{2m}\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})}$$
$$=\sum_{j=1}^{m-1}\left(\frac{4m}{t-s}\frac{\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})}\leq 1.$$

Then, since $1<p(x_{j-1})<\infty$ and $j=1,2,\cdots,2m$, it follows that

$$\sum_{j=1}^{m-1}\left(\frac{4}{t-s}\frac{\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})}$$
$$\leq\sum_{j=1}^{m-1}\left(\frac{4m}{t-s}\frac{\left|h(v)+h(u)-2h\left(\frac{u+v}{2}\right)\right|}{\gamma\left(\frac{|u-v|}{2}\right)}\right)^{p(x_{j-1})}\leq 1.$$

Hence, necessarily

$$h(v)+h(u)-2h\left(\frac{u+v}{2}\right)=0.$$

So that, we conclude that $h(s,\cdot)$ satisfies the Jensen equation in $\mathbb{R}$. The continuity of h with respect to the second variable implies that for every $t\in[a,b]$ there exists $\alpha,\beta:[a,b]\to\mathbb{R}$ such that

$$h(t,x)=\alpha(t)x+\beta(t),\quad (t\in[a,b],x\in\mathbb{R}).$$

Since $\beta(t)=h(t,0)$, $t\in[a,b]$, $\alpha(t)=h(t,1)-\beta(t)$ and $h(\cdot,x)\in BV^W_{(p(\cdot),2)}[a,b]$ for each $x\in\mathbb{R}$ we obtain that $\alpha,\beta\in BV^W_{(p(\cdot),2)}[a,b]$.

Now we will give the definition of uniformly bounded mapping introduced by J. Matkowski in [24].

Definition 9 *Let X and Y be two metric (or normed) spaces. A mapping $H:X\to Y$ is uniformly bounded if, for any $t>0$ there exists a nonnegative real number $\gamma(t)$ such that for any nonempty set $B\subset X$ we have*

$$diam(B)\le t\to diamH(B)\le\gamma(t).$$

With the same kind of argument than in ([23], Theorem 7), we can see that any uniformly bounded composition operator acting between general Lipschitz function normed space must be of the form (5.2):

Theorem 7 *Let $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ and H the composition operator associated to h. Suppose that H maps $BV^W_{(p(\cdot),2)}[a,b]$ into itself and it is uniformly continuous, then there exists functions $\alpha,\beta\in BV^W_{(p(\cdot),2)}[a,b]$, such that*

$$h(t,x)=\alpha(t)x+\beta(t),\quad (t\in[a,b],x\in\mathbb{R}).$$

Proof. It follows as ([23], Theorem 7) by Theorem 6.

6. Absolutely Continuous Functions

We now define the analog of absolute *p*-continuous functions of order two in the framework of variable space.

Definition 10 *Given a function $p:[a,b]\to(1,\infty)$, by modulus of $p(\cdot)$-continuity of order two of a function $f:[a,b]\to\mathbb{R}$, we mean*

$$\omega_\delta^{(p(\cdot),2)}(f):=\sup_{\|\pi^*\|\le\delta}\sup_{\pi^*}\sum_{j=1}^{n-1}\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})},$$

where the supremum is taken over all tagged partitions $\pi^*=\{a=t_0<t_1<\cdots<t_n=b\}$ of the interval $[a,b]$ together with a finite sequence of numbers $x_0,\cdots,x_{n-2}$ subject to the conditions $t_j\le x_j\le t_{j+1}$ for each j such that the norm of π^* is at most δ.

Lemma 3 *Let p be an admissible function. The modulus of $p(\cdot)$-continuity of order two is a sub-additive function.*

Proof. Let $f,g:[a,b]\to\mathbb{R}$.

$$\begin{aligned}
&\omega_\delta^{(p(\cdot),2)}(f+g)\\
&=\sup_{\|\pi\|\le\delta}\sup_{\pi}\sum_{j=1}^{n-1}\left|\frac{(f+g)(t_{j+1})-(f+g)(t_j)}{t_{j+1}-t_j}-\frac{(f+g)(t_j)-(f+g)(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\\
&\le 2^{p^+-1}\sup_{\|\pi\|\le\delta}\sup_{\pi}\sum_{j=1}^{n-1}\left(\left|\frac{f(t_{j+1})-f(t_j)}{t_{j+1}-t_j}-\frac{f(t_j)-f(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\right.\\
&\quad\left.+\left|\frac{g(t_{j+1})-g(t_j)}{t_{j+1}-t_j}-\frac{g(t_j)-g(t_{j-1})}{t_j-t_{j-1}}\right|^{p(x_{j-1})}\right)\\
&=2^{p^+-1}\left(\omega_\delta^{(p(\cdot),2)}(f)+\omega_\delta^{(p(\cdot),2)}(g)\right).
\end{aligned}$$

If $f\in BV^W_{(p(\cdot),2)}[a,b]$ and $\lim_{\delta\to 0}\omega_\delta^{(p(\cdot),2)}(f)=0$, we say that f is absolutely $p(\cdot)$-continuous of order two, that is, $f\in C^{(p(\cdot),2)}[a,b]$.

Theorem 8 *Let* p *be an admissible function. Then* $C^{(p(\cdot),2)}[a,b]$ *is a closed subspace of* $BV^W_{(p(\cdot),2)}[a,b]$.

Proof. We take a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $C^{(p(\cdot),2)}[a,b]$ such that

$$s-\lim_{n\to\infty} f_n = f\in BV^W_{(p(\cdot),2)}[a,b]. \tag{6.1}$$

By the sub-additivity of $\omega_\delta^{(p(\cdot),2)}(f)$ we have that

$$\omega_\delta^{(p(\cdot),2)}(f)\le\omega_\delta^{(p(\cdot),2)}(f-f_n)+\omega_\delta^{(p(\cdot),2)}(f_n).$$

Moreover, since $V^W_{(p(\cdot),2)}(f)\ge\omega_\delta^{(p(\cdot),2)}(f)$ and $V^W_{(p(\cdot),2)}(2f)\precsim V^W_{(p(\cdot),2)}(f)$, using Proposition 2.3. in [21] and the strong limit (6.1) we have that, for each fixed δ, $\omega_\delta^{(p(\cdot),2)}(f-f_n)\to 0$ when $n\to\infty$. Since $\omega_\delta^{(p(\cdot),2)}(f_n)\to 0$ when $\delta\to 0$ by hypothesis, we obtain that $\omega_\delta^{(p(\cdot),2)}(f)\to 0$ when $\delta\to 0$.

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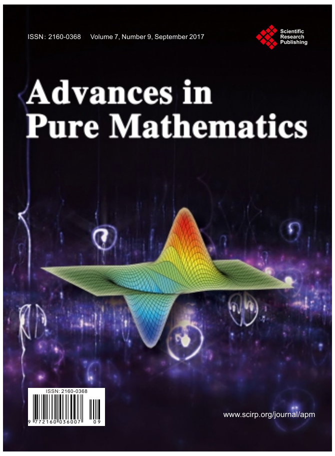

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