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A Characterization of Jacobson Radical in Γ-Banach Algebras

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ABSTRACT

Let V_1 and V_2 be two Γ-Banach algebras and R_i be the right operator Banach algebra and L_i be the left operator Banach algebra of $V_i\left(i=1,2\right)$. We give a characterization of the Jacobson radical for the projective tensor product $V_1 \otimes_\gamma V_2$ in terms of the Jacobson radical for $R_1 \otimes_\gamma L_2$. If V_1 and V_2 are isomorphic, then we show that this characterization can also be given in terms of the Jacobson radical for $R_2 \otimes_\gamma L_1$.

Keywords: Γ-Algebra; Right Quasi Regularity; Tensor Product; Operator Banach Algebra

1. Introduction

In [1,2], using the right quasi regularity property, Kyuno and Coppage and Luh gave a characterization of Jacobson radical in Γ-rings. Many interesting results on the internal properties of Jacobson radical for Γ-rings were developed in [2-5] by different research workers. In [6], some of these results are extended to Γ-algebras. In this paper, we consider two Γ-Banach algebras V_1 and V_2 and consider their projective tensor product $V_1 \otimes_\gamma V_2$. Let R_i be the right operator Banach algebra and L_i be the left operator Banach algebra of $V_i\left(i=1,2\right)$. We give a characterization of Jacobson radical $J\left(V_1 \otimes_\gamma V_2\right)$ in terms of $J\left(R_1 \otimes_\gamma L_2\right)$

Before going to present our main results, we first give some basic terminologies (refer to [5-12]) which are needed in our discussion.

Definition 1.1

Let X be a ring having the unit element e. A new multiplication called the circle composition (refer to [5]) on X is defined by: $x \cdot x' = x + x' - xx'$ .This composition makes sense even when X does not have the unit element. An element x of X is said to be right quasi regular if it has a right quasi inverse w.r.t. this composition, *i.e.*, there exists $x' \in X$ such that $x \cdot x' = x + x' - xx' = 0$.

Definition 1.2

Let V and Γ be two linear spaces over a field F. V is said to be a Γ-algebra over F if, for x, y, $z \in V$; α, $\beta \in \Gamma$; $a \in F$, the following conditions are satisfied:

1) $x\alpha y \in V$;
2) $\left(x\alpha y\right)\beta z = x\alpha\left(y\beta z\right)$;
3) $a\left(x\alpha y\right) = \left(ax\right)\alpha y = x\left(a\alpha\right)y = x\alpha\left(ay\right)$;
4) $x\alpha\left(y+z\right) = x\alpha y + x\alpha z$,
$x\left(\alpha+\beta\right)y = x\alpha y + x\beta y$,
$\left(x+y\right)\alpha z = x\alpha z + y\alpha z$.

The Γ-algebra is denoted by $\left(V,\Gamma\right)$. If V and Γ are normed linear spaces over F, then Γ-algebra $\left(V,\Gamma\right)$ is called a Γ-normed algebra if conditions 1) to 4) hold and further

5) $\|x\alpha y\| \le \|x\| \cdot \|\alpha\| \cdot \|y\|$ holds.

A Γ-normed algebra $\left(V,\Gamma\right)$ is called a Γ-Banach algebra if V is a Banach space. Any Banach algebra can be regarded as a Γ-Banach algebra by suitably choosing Γ.

Definition 1.3

A subset I of a Γ-Banach algebra V is said to be a right (left) Γ-ideal of V if

1) I is a subspace of V (in the vector space sense);
2) $x\alpha y \in I\left(y\alpha x \in I\right) \forall x \in I, \alpha \in \Gamma; y \in V$

i.e., $I\Gamma V \subseteq I\left(V\Gamma I \subseteq I\right)$.

A right Γ-ideal, which is a left Γ-ideal as well, is called a two-sided Γ-ideal or simply a Γ-ideal.

Definition 1.4

Let V be a Γ-Banach algebra and let $x \in V$, $\alpha \in \Gamma$. Then the mapping $\left[\alpha, x\right]$ defined by $y\left[\alpha, x\right] = y\alpha x \forall y \in V$ is a right Banach space endomorphism of V. The collection R of all endomorphisms generated by $\left[\alpha, x\right]$; $\alpha \in \Gamma$, $x \in V$, is a Banach algebra under the operations:

$$\left[\alpha, x\right] + \left[\alpha, y\right] = \left[\alpha, x+y\right],$$

$$\left[\alpha, x\right] + \left[\beta, x\right] = \left[\alpha+\beta, x\right]$$

$$a[\alpha,x]=[\alpha,ax]=[a\alpha,x]$$

where $a\in F$,

$$[\alpha,x][\beta,y]=[\alpha,x\beta y],\alpha,\beta\in\Gamma,$$

and the norm:

$$\|[\alpha,x]\|=\|\alpha\|_\Gamma\cdot\|x\|_V.$$

This Banach algebra is termed as the right operator Banach algebra of Γ-Banach algebra *V*. We can similarly define the left operator Banach algebra *L* of *V* as the Banach algebra generated by the set of all left endomorphisms of *V* in the form $[x,\alpha]$ where

$$[x,\alpha]y=x\alpha y\forall y\in V.$$

Definition 1.5

Let *V* and V' be Γ-Banach algebras over *F* and ϕ: $V\to V'$ be a mapping. Then ϕ is called a Γ-Banach algebra homomorphism if

1) $\phi(ax+by)=a\phi(x)+b\phi(y)$ and
2) $\phi(x\alpha y)=\phi(x)\alpha\phi(y)$ for all $x,y\in V$; $\alpha\in\Gamma$ and $a,b\in F$

Definition 1.6

Let *X* and *Y* be two normed spaces. The *projective tensor norm* $\|.\|_\gamma$ on $X\otimes Y$ is defined as:

$$\|u\|_\gamma=\inf\left\{\sum_i\|x_i\|\cdot\|y_i\|:u=\sum_i x_i\otimes y_i\right\}$$

where the infimum is taken over all (finite) representations of *u*. The completion of $\left(X\otimes Y,\|.\|_\gamma\right)$ is called the projective tensor product of *X* and *Y*, and is denoted by $X\otimes_\gamma Y$.

Let (V,Γ) and (V',Γ') be Γ-Banach algebras over F_1 and F_2 isomorphic to *F*. The projective tensor product $(V,\Gamma)\otimes_\gamma(V',\Gamma')$ with the projective tensor norm is a $\Gamma\otimes\Gamma'$-Banach algebra over *F*, where a multiplication is defined by the formula*:*

$$(x\otimes y)(\alpha\otimes\beta)(x'\otimes y')=(x\alpha x')\otimes(y'\beta y).$$

where x, $y\in V$; x', $y'\in V'$; $\alpha\in\Gamma$, $\beta\in\Gamma'$.

Definition 1.7

Let *V* be a Γ-Banach algebra. Let $\alpha\in\Gamma$. An element *x* in *V* is said to be α-right quasi regular with *α*-right quasi inverse *y* if $x+y-x\alpha y=0$. *x* is said to be a right quasi regular element of *V* if it is *α*-right quasi regular for each $\alpha\in\Gamma$.

Equivalently, an element $x\in V$ is called right quasi regular if for any $\alpha\in\Gamma$, there exist $\gamma_i\in\Gamma$, $v_i\in V$, $i=1,2,\cdots,n$ such that

$$v\alpha x+\sum_{i=1}^n v\gamma_i v_i-\sum_{i=1}^n v\alpha x\gamma_i v_i=0\forall v\in V$$

An ideal *I* of *V* is said to be right quasi regular if each of its elements is right quasi regular.

We have, right quasi regularity is a radical property in an algebra. The maximal right quasi regular ideal is called the Jacobson radical of *V* and it is denoted by *J*(*V*).

2. Main Results

In [6], we have the following Lemma regarding right quasi regularity of a Γ-Banach algebra and its operator algebra.

Lemma 2.1

An element *x* of a Γ-Banach algebra *V* is right quasi regular if and only if for all $\alpha\in\Gamma$, $[\alpha,x]$ is right quasi regular in the right operator Banach algebra *R* of *V*.

Extending this result to the projective tensor product of Γ-Banach algebras, we prove,

Lemma 2.2

Let *V* and V' be two Γ and Γ'-Banach algebras respectively. Let *R* be the right operator Banach algebra of *V* and *L* be the left operator Banach algebra of V'. If $\sum_i x_i\otimes x_i'$ is right quasi regular in $V\otimes_\gamma V'$, then $\sum_i[\alpha,x_i]\otimes[x_i',\alpha']$ is right quasi regular in $R\otimes_\gamma L$ for $\alpha\otimes\alpha'\in\Gamma\otimes\Gamma'$, and conversely.

Proof. Since $\sum_i x_i\otimes x_i'$ is right quasi regular in $V\otimes_\gamma V'$, so, for any $\alpha\otimes\alpha'\in\Gamma\otimes\Gamma'$, there exist $\eta_j=\sum_n\gamma_{jn}\otimes\gamma_{jn}'\in\Gamma\otimes\Gamma'$, $p_j=\sum_m x_{jm}\otimes x_{jm}'\in V\otimes_\gamma V'$, $j=1,2,\cdots,n_0$ such that for any $q=\sum_k v_k\otimes v_k'\in V\otimes_\gamma V'$,

$$\begin{aligned}&q(\alpha\otimes\alpha')\left(\sum_i x_i\otimes x_i'\right)+\sum_{j=1}^{n_0}q\eta_j p_j-\sum_{j=1}^{n_0}q(\alpha\otimes\alpha')\left(\sum_i x_i\otimes x_i'\right)\eta_j p_j=0\\&\Rightarrow\left(\sum_k v_k\otimes v_k'\right)(\alpha\otimes\alpha')\left(\sum_i x_i\otimes x_i'\right)+\sum_{j=1}^{n_0}\left(\sum_k v_k\otimes v_k'\right)\left(\sum_n\gamma_{jn}\otimes\gamma_{jn}'\right)\left(\sum_m x_{jm}\otimes x_{jm}'\right)\\&\quad-\sum_{j=1}^{n_0}\left(\sum_k v_k\otimes v_k'\right)(\alpha\otimes\alpha')\left(\sum_i x_i\otimes x_i'\right)\left(\sum_n\gamma_{jn}\otimes\gamma_{jn}'\right)\left(\sum_m x_{jm}\otimes x_{jm}'\right)=0\\&\Rightarrow\sum_{k,i}v_k\alpha x_i\otimes x_i'\alpha' v_k'+\sum_{j=1}^{n_0}\left(\sum_{k,n,m}v_k\gamma_{jn}x_{jm}\otimes x_{jm}'\gamma_{jn}'v_k'\right)-\sum_{j=1}^{n_0}\left(\sum_{k,i,n,m}v_k\alpha x_i\gamma_{jn}x_{jm}\otimes x_{jm}'\gamma_{jn}'x_i'\alpha' v_k'\right)=0\end{aligned}\tag{2.1}$$

Let $x=\sum_i[\alpha,x_i]\otimes[x_i',\alpha']$. We take $y=\sum_{j=1}^{n_0}\sum_{n,m}[\gamma_{jn},x_{jm}]\otimes[x'_{jm},\gamma'_{jn}]$

Now,

$$\begin{aligned}(x+y-xy)\left(\sum_k v_k\otimes v_k'\right)&=\Bigg(\left(\sum_i[\alpha,x_i]\otimes[x_i',\alpha']\right)+\sum_{j=1}^{n_0}\sum_{n,m}[\gamma_{jn},x_{jm}]\otimes[x'_{jm},\gamma'_{jn}]\\&\quad-\left(\sum_i[\alpha,x_i]\otimes[x_i',\alpha']\right)\left(\sum_{j=1}^{n_0}\sum_{n,m}[\gamma_{jn},x_{jm}]\otimes[x'_{jm},\gamma'_{jn}]\right)\Bigg)\left(\sum_k v_k\otimes v_k'\right)\\&=\left(\sum_i[\alpha,x_i]\otimes[x_i',\alpha']\right)\left(\sum_k v_k\otimes v_k'\right)+\sum_{j=1}^{n_0}\sum_{n,m}[\gamma_{jn},x_{jm}]\otimes[x'_{jm},\gamma'_{jn}]\left(\sum_k v_k\otimes v_k'\right)\\&\quad-\left(\sum_i[\alpha,x_i]\otimes[x_i',\alpha']\right)\left(\sum_{j=1}^{n_0}\sum_{n,m}[\gamma_{jn},x_{jm}]\otimes[x'_{jm},\gamma'_{jn}]\right)\left(\sum_k v_k\otimes v_k'\right)\\&=\sum_{k,i}v_k\alpha x_i\otimes x_i'\alpha'v_k'+\sum_{j=1}^{n_0}\left(\sum_{k,n,m}v_k\gamma_{jn}x_{jm}\otimes x'_{jm}\gamma'_{jn}v_k'\right)-\sum_{j=1}^{n_0}\left(\sum_{k,i,n,m}v_k\alpha x_i\gamma_{jn}x_{jm}\otimes x'_{jm}\gamma'_{jn}x_i'\alpha'v_k'\right)=0\end{aligned}$$

(by (2.1)).

But, $\sum_k v_k\otimes v_k'\in V\otimes_\gamma V'$ is arbitrary.

So, $x+y-xy=0$. Thus, x, *i.e.*, $\sum_i[\alpha,x_i]\otimes[x_i',\alpha']$ is right quasi regular in $R\otimes_\gamma L$.

The converse follows in the same way. □

In [13], we have defined the following ideal for the projective tensor product of V and V'.

Lemma 2.3

Let V and V' be two Γ and Γ'-Banach algebras respectively. Let R be the right operator Banach algebra of V and L be the left operator Banach algebra of V'. Let J be an ideal of $R\otimes_\gamma L$. We define:

$$J^0=\left\{\left(\sum_i x_i\otimes x_i'\right)\in V\otimes_\gamma V':\sum_i[\Gamma,x_i]\otimes[x_i',\Gamma']\subseteq J\right\}$$

where $[\Gamma,x_i]=\left\{\sum_j[\alpha_j,x_i]:\alpha_j\in\Gamma\right\}$, and

$$[x_i',\Gamma']=\left\{\sum_j[x_i',\alpha_j']:\alpha_j'\in\Gamma'\right\}$$

Then J^0 is an ideal of $V\otimes_\gamma V'$.

Using the above defined ideal, now, we give the characterization of Jacobson radical for the projective tensor product of two Γ-Banach algebras $V_i\,(i=1,2)$ in terms of the Jacobson radical of the projective tensor product of corresponding right and left operator Banach algebras.

Theorem 2.4

Let V_i be a Γ-Banach algebra (over F) with right operator Banach algebra R_i and left operator Banach algebra $L_i\,(i=1,2)$ respectively. Then the Jacobson radical of $V_1\otimes_\gamma V_2$ is given by: $J(V_1\otimes_\gamma V_2)=[J(R_1\otimes_\gamma L_2)]^0$.

Proof. Let $\sum_i x_i\otimes x_i'\in J(V_1\otimes_\gamma V_2)$.

Then $\sum_i x_i\otimes x_i'$ is a right quasi regular element of $V_1\otimes_\gamma V_2$. By Lemma 2.2, for any $\alpha,\ \alpha'\in\Gamma$, $\sum_i[\alpha,x_i]\otimes[x_i',\alpha']$ is a right quasi regular element of $R_1\otimes_\gamma L_2$, *i.e.*,

$$\sum_i[\alpha,x_i]\otimes[x_i',\alpha']\in J(R_1\otimes_\gamma L_2).$$

So,

$$\sum_i[\Gamma,x_i]\otimes[x_i',\Gamma]\subseteq J(R_1\otimes_\gamma L_2).$$

Hence,

$$\sum_i x_i\otimes x_i'\in[J(R_1\otimes_\gamma L_2)]^0.$$

Thus,

$$J(V_1\otimes_\gamma V_2)\subseteq[J(R_1\otimes_\gamma L_2)]^0.$$

Conversely, let

$$\sum_i x_i\otimes x_i'\in[J(R_1\otimes_\gamma L_2)]^0.$$

Then

$$\sum_i[\Gamma,x_i]\otimes[x_i',\Gamma]\subseteq J(R_1\otimes_\gamma L_2).$$

So, for any $\alpha,\ \alpha'\in\Gamma$, $\sum_i[\alpha,x_i]\otimes[x_i',\alpha']$ is a right quasi regular element of $R_1\otimes_\gamma L_2$. By Lemma 2.2, $\sum_i x_i\otimes x_i'$ is a right quasi regular element of $V_1\otimes_\gamma V_2$, *i.e.* $\sum_i x_i\otimes x_i'\in J(V_1\otimes_\gamma V_2)$ So,

$\left[J\left(R_1 \otimes_\gamma L_2\right)\right]^0 \subseteq J\left(V_1 \otimes_\gamma V_2\right)$.

Thus, $J\left(V_1 \otimes_\gamma V_2\right) = \left[J\left(R_1 \otimes_\gamma L_2\right)\right]^0$. $\square$

Let the Γ-Banach algebras V_1 and V_2 are isomorphic. In that case, we have the following result.

Theorem 2.5

Let V_i be a Γ-Banach algebra (over F) with right operator Banach algebra R_i and left operator Banach algebra $L_i\left(i=1,2\right)$ respectively. If there exists a Γ-Banach algebra isomorphism f from V_1 onto V_2, then $R_1 \otimes_\gamma L_2$ is a homomorphic image of $R_2 \otimes_\gamma L_1$.

Proof. Let $\sum_n r_n \otimes l_n \in R_2 \otimes_\gamma L_1$, where $l_n = \left[y_n, \beta_n\right]$, $r_n = \left[\alpha'_n, x'_n\right]$. We define $\phi : R_2 \otimes_\gamma L_1 \to R_1 \otimes_\gamma L_2$ by

$$\phi\left(\sum_n r_n \otimes l_n\right) = \phi\left(\sum_n \left[\alpha'_n, x'_n\right] \otimes \left[y_n, \beta_n\right]\right) = \sum_n \left[\alpha'_n, x_n\right] \otimes \left[f\left(y_n\right), \beta_n\right],$$

where $x'_n = f\left(x_n\right)$, $x_n \in V_1$.

Let $r_1^* \in R_1^*$ (The dual space of R_1).

We define $r_2^* : R_2 \to C$ by $r_2^*\left(\left[\alpha', x'\right]\right) = r_1^*\left(\left[\alpha', x\right]\right)$, where $x' = f\left(x\right)$.

Then $r_2^* \in R_2^*$.

Similarly, for $l_2^* \in L_2^*$, we can define $l_1^* \in L_1^*$ by $l_1^*\left(\left[y, \beta\right]\right) = l_2^*\left(\left[f\left(y\right), \beta\right]\right)$.

Now, let

$$\sum_n r_n \otimes l_n = \sum_m \tilde{r}_m \otimes \tilde{l}_m,$$

where

$$\tilde{r}_m = \left[\tilde{\alpha}'_m, \tilde{x}'_m\right], \tilde{l}_m = \left[\tilde{y}_m, \tilde{\beta}_m\right]$$

$$\Rightarrow \left(\sum_n r_n \otimes l_n\right)\left(h,k\right) = \left(\sum_m \tilde{r}_m \otimes \tilde{l}_m\right)\left(h,k\right)$$

$$\forall h \in R_2^*, k \in L_1^*.$$

In particular, taking $h = r_2^*$, $k = l_1^*$, we get,

$$\left(\sum_n r_n \otimes l_n\right)\left(r_2^*, l_1^*\right) = \left(\sum_m \tilde{r}_m \otimes \tilde{l}_m\right)\left(r_2^*, l_1^*\right)$$

$$\Rightarrow \sum_n r_2^*\left(r_n\right) l_1^*\left(l_n\right) = \sum_m r_2^*\left(\tilde{r}_m\right) \otimes l_1^*\left(\tilde{l}_m\right)$$

$$\Rightarrow \sum_n r_2^*\left(\left[\alpha'_n, x'_n\right]\right) l_1^*\left(\left[y_n, \beta_n\right]\right)$$

$$= \sum_m r_2^*\left(\left[\tilde{\alpha}'_m, \tilde{x}'_m\right]\right) l_1^*\left(\left[\tilde{y}_m, \tilde{\beta}_m\right]\right)$$

$$\Rightarrow \sum_n r_1^*\left(\left[\alpha'_n, x_n\right]\right) l_2^*\left(\left[f\left(y_n\right), \beta_n\right]\right)$$

$$= \sum_m r_1^*\left(\left[\tilde{\alpha}'_m, \tilde{x}_m\right]\right) l_2^*\left(\left[f\left(\tilde{y}_m\right), \tilde{\beta}_m\right]\right),$$

where $x'_n = f\left(x_n\right)$, and $\tilde{x}'_m = f\left(\tilde{x}_m\right)$.

$$\Rightarrow \left(\sum_n \left[\alpha'_n, x_n\right] \otimes \left[f\left(y_n\right), \beta_n\right]\right)\left(r_1^*, l_2^*\right)$$

$$= \left(\sum_m \left[\tilde{\alpha}'_m, \tilde{x}_m\right] \otimes \left[f\left(\tilde{y}_m\right), \tilde{\beta}_m\right]\right)\left(r_1^*, l_2^*\right)$$

$$\Rightarrow \left(\phi\left(\sum_n r_n \otimes l_n\right)\right)\left(r_1^*, l_2^*\right) = \left(\phi\left(\sum_m \tilde{r}_m \otimes \tilde{l}_m\right)\right)\left(r_1^*, l_2^*\right).$$

But $r_1^* \in R_1^*$ and $l_2^* \in L_2^*$ are arbitrary. So, $\phi\left(\sum_n r_n \otimes l_n\right) = \phi\left(\sum_m \tilde{r}_m \otimes \tilde{l}_m\right)$ Thus ϕ is well defined.

Now, Let $a, b \in F$. Then

$$\phi\left(a\sum_n r_n \otimes l_n + b\sum_m \tilde{r}_m \otimes \tilde{l}_m\right) = \phi\left(\sum_n a r_n \otimes l_n + \sum_m b\tilde{r}_m \otimes \tilde{l}_m\right)$$

$$= \phi\left(\sum_n a\left[\alpha'_n, x'_n\right] \otimes \left[y_n, \beta_n\right] + \sum_m b\left[\tilde{\alpha}'_m, \tilde{x}'_m\right] \otimes \left[\tilde{y}_m, \tilde{\beta}_m\right]\right)$$

$$= \phi\left(\sum_n \left[a\alpha'_n, x'_n\right] \otimes \left[y_n, \beta_n\right] + \sum_m \left[b\tilde{\alpha}'_m, \tilde{x}'_m\right] \otimes \left[\tilde{y}_m, \tilde{\beta}_m\right]\right)$$

$$= \sum_n \left[a\alpha'_n, x_n\right] \otimes \left[f\left(y_n\right), \beta_n\right] + \sum_m \left[b\tilde{\alpha}'_m, \tilde{x}_m\right] \otimes \left[f\left(\tilde{y}_m\right), \tilde{\beta}_m\right],$$

where $x'_n = f\left(x_n\right)$, and $\tilde{x}'_m = f\left(\tilde{x}_m\right)$.

$$= \sum_n a\left[\alpha'_n, x_n\right] \otimes \left[f\left(y_n\right), \beta_n\right] + \sum_m b\left[\tilde{\alpha}'_m, \tilde{x}_m\right] \otimes \left[f\left(\tilde{y}_m\right), \tilde{\beta}_m\right]$$

$$= a\left(\sum_n \left[\alpha'_n, f(x_n)\right] \otimes \left[f\left(y_n\right), \beta_n\right]\right) + b\left(\sum_m \left[\alpha'_m, f\left(x'_m\right)\right] \otimes \left[f\left(y'_m\right), \beta'_m\right]\right)$$

$$= a\phi\left(\sum_n r_n \otimes l_n\right) + b\phi\left(\sum_m \tilde{r}_m \otimes \tilde{l}_m\right)$$

Again,

$$\phi\left(\left(\sum_n r_n \otimes l_n\right)\left(\sum_m \tilde{r}_m \otimes \tilde{l}_m\right)\right) = \phi\left(\sum_{n,m} r_n \tilde{r}_m \otimes \tilde{l}_m l_n\right)$$

$$= \phi\left(\sum_{n,m} \left[\alpha'_n, x'_n\right]\ \left[\tilde{\alpha}'_m, \tilde{x}'_m\right] \otimes \left[\tilde{y}_m, \tilde{\beta}_m\right]\ \left[y_n, \beta_n\right]\right) \quad (2.2)$$

$$= \phi\left(\sum_{n,m} \left[\alpha'_n, x'_n \tilde{\alpha}'_m \tilde{x}'_m\right] \otimes \left[\tilde{y}_m \tilde{\beta}_m y_n, \beta_n\right]\right)$$

We have, x'_n, $\tilde{x}'_m \in V_2$. So, there exist x_n, $\tilde{x}_m \in V_1$ such that $x'_n = f\left(x_n\right)$, $\tilde{x}'_m = f\left(\tilde{x}_m\right)$.

Now, $x_n \tilde{\alpha}'_m \tilde{x}_m \in V_1$ and

$$f\left(x_n \tilde{\alpha}'_m \tilde{x}_m\right) = f\left(x_n\right) \tilde{\alpha}'_m f\left(\tilde{x}_m\right) = x'_n \tilde{\alpha}'_m \tilde{x}'_m$$

So, the expression (2.2) is equal to

$$\sum_{n,m}[\alpha'_n, x_n\tilde{\alpha}'_m\tilde{x}_m]\otimes\left[f\left(\tilde{y}_m\tilde{\beta}_m y_n\right),\beta_n\right]$$
$$=\sum_{n,m}[\alpha'_n, x_n\tilde{\alpha}'_m\tilde{x}_m]\otimes\left[f\left(\tilde{y}_m\right)\tilde{\beta}_m f\left(y_n\right),\beta_n\right]$$
$$=\sum_{n,m}[\alpha'_n, x_n]\;[\tilde{\alpha}'_m,\tilde{x}_m]\otimes\left[f\left(\tilde{y}_m\right),\tilde{\beta}_m\right]\;\left[f\left(y_n\right),\beta_n\right]$$
$$=\left(\sum_n[\alpha'_n, x_n]\otimes\left[f\left(y_n\right),\beta_n\right]\right)$$
$$\cdot\left(\sum_m[\tilde{\alpha}'_m,\tilde{x}_m]\otimes\left[f\left(\tilde{y}_m\right),\tilde{\beta}_m\right]\right)$$
$$=\phi\left(\sum_n r_n\otimes l_n\right)\phi\left(\sum_m\tilde{r}_m\otimes\tilde{l}_m\right)$$

So, $\phi: R_2\otimes_\gamma L_1\to R_1\otimes_\gamma L_2$ is a homomorphism.

Since f is onto, so, ϕ is also onto. Also, it can be shown that ϕ is one-one.

Thus, $R_1\otimes_\gamma L_2=\phi\left(R_2\otimes_\gamma L_1\right)$.

Corollary 2.6

Let the Γ-Banach algebras V_1 and V_2, as defined in Theorem 2.4 are isomorphic. Then we have,

$$J\left(V_1\otimes_\gamma V_2\right)=\left[J\left(\phi\left(R_2\otimes_\gamma L_1\right)\right)\right]^0.$$

Remark 2.7

If the isomorphism f from V_1 onto V_2 is isometric, then we can show that $\phi: R_2\otimes_\gamma L_1\to R_1\otimes_\gamma L_2$ is also an isometry. So, in that case,

$$J\left(V_1\otimes_\gamma V_2\right)\cong\left[J\left(\phi\left(R_2\otimes_\gamma L_1\right)\right)\right]^0.$$

The notion of direct summand for Γ-rings is discussed in [10] by Booth. For a Γ-Banach algebra V, an ideal P is called direct summand if there exists a Γ-ideal Q of V such that every element v of V is uniquely expressible in the form $v = p + q$, $p\in P$, $q\in Q$, and V is written as $V=P\oplus Q$. Clearly, if $V=P\oplus Q$, then for $p\in P$, $q\in Q$, $p\alpha q=0\,\forall\alpha\in\Gamma$.

Now, we prove:

Deduction 2.8

If P is the direct summand for the Γ-Banach algebra $V_1\otimes_\gamma V_2$, then $J(P)$ is the direct summand for $J\left(V_1\otimes_\gamma V_2\right)$.

Proof. Let $V_1\otimes_\gamma V_2=P\oplus Q$ Clearly, $J(P)\cap J(Q)=\{0\}$.

Let $x\in J\left(V_1\otimes_\gamma V_2\right)$ and $x = p + q$, where $p\in P$, $q\in Q$.

Since x is right quasi regular in $V_1\otimes_\gamma V_2$, so, for any $\alpha\otimes\alpha'\in\Gamma\otimes\Gamma$, we have, there exists $y\in V_1\otimes_\gamma V_2$ such that $x+y-x\left(\alpha\otimes\alpha'\right)y=0$.

Let $y=p_1+q_1$, where $p_1\in P$, $q_1\in Q$.

So,

$$(p+q)+(p_1+q_1)-(p+q)(\alpha\otimes\alpha')(p_1+q_1)=0$$
$$\Rightarrow\left(p+p_1-p\left(\alpha\otimes\alpha'\right)p_1\right)+\left(q+q_1-q\left(\alpha\otimes\alpha'\right)q_1\right)=0$$

[since $p\left(\alpha\otimes\alpha'\right)q_1=0$ and $q\left(\alpha\otimes\alpha'\right)p_1=0$]

But $p+p_1-p\left(\alpha\otimes\alpha'\right)p_1\in P$ and $q+q_1-q\left(\alpha\otimes\alpha'\right)q_1\in Q$, and $P\cap Q=\{0\}$.

So, $p+p_1-p\left(\alpha\otimes\alpha'\right)p_1=0$ and $q+q_1-q\left(\alpha\otimes\alpha'\right)q_1=0$, for any $\alpha\otimes\alpha'\in\Gamma\otimes\Gamma$.

Thus p is right quasi regular in P and q is right quasi regular in Q, *i.e.*, $p\in J(P)$ and $q\in J(Q)$.

Hence $J\left(V_1\otimes V_2\right)=J(P)\oplus J(Q)$. □

In [4], there is a characterization of Jacobson radical for Γ-rings in terms of maximal regular left ideals.

Lemma 2.9

Let X be a Γ-ring. Then $J(X)=\cap M$, where the intersection is over all maximal regular left ideals M of X.

Considering this aspect, we can raise the following problem:

Let the structures of maximal regular left ideals of the operator Banach algebras R_1 and L_2 are given. Using this, can we obtain the structure of the Jacobson radical for $V_1\otimes_\gamma V_2$?

In [6], Behrens radical for Γ-Banach algebras is introduced which contains the Jacobson radical. Let Π denote the class of all subdirectly irreducible Γ-Banach algebras V such that the intersection of all non-zero ideals of V contains a non-zero idempotent element. The upper radical R_B determined by the class Π is called the Behrens radical for V.

Lemma 2.10

For a simple Γ-Banach algebra V, $J(V)\subseteq R_B(V)$.

Now, another problem can be raised:

Can we derive analogous result as in Theorem 2.4 in case of the Behrens radical for $V_1\otimes_\gamma V_2$?

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