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The p.q.-Baer Property of Fixed Rings under Finite Group Action

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ABSTRACT

A ring *R* is called right principally quasi-Baer (simply, right p.q.-Baer) if the right annihilator of every principal right ideal of *R* is generated by an idempotent. For a ring *R*, let *G* be a finite group of ring automorphisms of *R*. We denote the fixed ring of *R* under *G* by R^G. In this work, we investigated the right p.q.-Baer property of fixed rings under finite group action. Assume that *R* is a semiprime ring with a finite group *G* of X-outer ring automorphisms of *R*. Then we show that: 1) If *R* is *G*-p.q.-Baer, then R^G is p.q.-Baer; 2) If *R* is p.q.-Baer, then R^G are p.q.-Baer.

Keywords: p.q.-Baer Property; Fixed Ring; Group Action

1. Introduction

Throughout this paper all rings are associative with identity. Recall from [1] that a ring *R* is called *right principally quasi-Baer* (simply, right p.q.-Baer) if the right annihilator of every principal right ideal of *R* is generated, as a right ideal, by an idempotent of *R*. A left principally quasi-Baer (simply, left p.q.-Baer) ring is defined similarly. Right p.q.-Baer rings have been initially studied in [1]. For more details on (right) p.q.-Baer rings, see [1-6].

Recall from [7] (see also [8]) that a ring *R* is called *quasi-Baer* if the right annihilator of every right ideal is generated, as a right ideal, by an idempotent of *R*. A ring *R* is called *biregular* if for each $x \in R$, $RxR = eR$ for some central idempotent $e \in R$. We note that the class of right p.q.-Baer rings is a generalization of the classes of quasi-Baer rings and biregular rings.

For a ring *R*, we use $Q(R)$ to denote a fixed maximal right ring of quotients of *R*. According to [9] an idempotent e of a ring *R* is called left (resp., right) semicentral if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. Equivalently, an idempotent e is left (resp., right) semicentral if and only if *eR* (resp., Re) is a two-sided ideal of *R*. For a ring *R*, we let $S_l(R)$ (resp., $S_l(R)$) denote the set of all left (resp., right) semicentral idempotents. An idempotent e of a ring *R* is called semicentral reduced if $S_l(eRe) = \{0, e\}$. Recall from [2] that a ring *R* is called semicentral reduced if $S_l(R) = \{0,1\}$, *i.e.*, 1 is a semicentral reduced idempotent of *R*.

For a nonempty subset *X* of a ring *R*, we use $r_R(X)$ and $l_R(X)$ to denote the right annihilator and the left annihilator of *X* in *R*, respectively. If *R* is a semiprime ring and *I* is a two-sided ideal of *R*, then $r_R(I) = l_R(I)$. For a right *R*-module *M* and a submodule *N* of *M*, we use $N_R \leq^{ess} M_R$ and $N_R \leq^{den} M_R$ to denote that N_R is essential in M_R and N_R is dense in M_R, respectively.

For a ring *R*, we let $Aut(R)$ denote the group of ring automorphisms of *R*. Let G be a subgroup of $Aut(R)$. For $r \in R$ and $g \in G$, we let r^g denote the image of *r* under *g*. We use R^G to denote the fixed ring of *R* under *G*, that is $R^G = \{r \in R \mid r^g = r \text{ for every } g \in G\}$.

We begin with the following example.

2. Preliminary

Example 2.1. There exist a ring *R* and a finite group *G* of ring automorphisms of *R* such that *R* is right p.q.-Baer but R^G is not right p.q.-Baer. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ with a field *F* of characteristic 2. Then *R* is right p.q.-Baer. Define $g \in Aut(R)$ by

$$g\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then $g^2 = 1$ since the characteristic of *F* is 2.

Now we show that R^G is not right p.q.-Baer. The fixed ring under *G* is

$$R^G = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \in R \,\middle|\, x, y \in F \right\}$$

By computation we see that the idempotents of R^G are only 0 and 1, thus R^G is semicentral reduced. So if R^G is

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right p.q.-Baer, then R^G is a prime ring by [2, Lemma 4.2], a contradiction. Thus R^G is not right p.q.-Baer.Also we can see that R^G is not left p.q.-Baer.

Definition 2.2. Let R be a semiprime ring. For $g \in Aut(R)$, let

$$\phi_g = \left\{ x \in Q_m(R) \middle| xr^g = rx \text{ for each } r \in R \right\},$$

where $Q_m(R)$ is the Martindale right ring of quotients of R (see [10] for more on $Q_m(R)$). We say that g is X-outer if $\phi_g = 0$. A subgroup G of $Aut(R)$ is called X-outer on R if every $1 \neq g \in G$ is X-outer. Assume that R is a semiprime ring, then for $g \in Aut(R)$, let

$$\Phi_g = \left\{ x \in Q_m(R) \middle| xr^g = rx \text{ for each } r \in R \right\}.$$

For $g \in Aut(R)$, we claim that $\Phi_g = \phi_g$. Obviously $\phi_g \subseteq \Phi_g$. Conversely, if $x \in \Phi_g$ then $xR = Rx$. There exists $I_R \leq^{den} R_R$ such that $xI \subseteq R$. Therefore $RI \lhd R$, $(RI)_R \leq^{den} R_R$, and $xRI = RxI \subseteq R$. Thus $x \in Q_m(R)$, hence $x \in \phi_g$. Therefore $\Phi_g = \phi_g$. So if G is X-outer on R, then G can be considered as a group of ring automorph-ismms of $Q(R)$ and G is X-outer on $Q(R)$. For more details for X-outer ring automorphisms of a ring, etc., see [10, p. 396] and [11].

We say that a ring R has no nonzero n-torsion (n is a positive integer) if $na = 0$ with $a \in R$ implies $a = 0$.

Lemma 2.3. [12,13]

Let R be a semiprime ring and G a group of ring automorphisms of R. If $R*G$ is semiprime, then R^G is semiprime.

For a ring R, we use $Cen(R)$ to denote the center of R.

Lemma 2.4. For a semiprime ring R, let G be a group of X-outer ring automorphisms of R.

Then $Cen(R*G) = Cen(R^G)$.

Proof.

Let $\alpha = a_1 1 + a_2 g_2 + \cdots + a_n g_n \in Cen(R)$ with $a_i \in R$, 1 the identity of G, and $g_i \in G$.

The $(a_1 1 + a_2 g_2 + \cdots + a_n g_n) b = b(a_1 1 + a_2 g_2 + \cdots + a_n g_n)$ for all $b \in R$. So $a_1 b = ba_1,\ a_2 b^{g_2} = ba_2,\ \cdots,\ a_n b^{g_n} = ba_n$ for all $b \in R$. Since G is X-outer, it follows that $a_2 = \cdots = a_n = 0$. Hence $\alpha = a_1 1 = a_1 \in R$. Also since $\alpha b = b\alpha$ for all $b \in R$, we have that $a_1 \in Cen(R)$. Note that for all $g \in G$, $a_1 g = g a_1 = a_1^{g^{-1}} g$ implies $a_1 = a_1^{g^{-1}}$. So $\alpha = a_1 \in Cen(R)^G$. Thus $Cen(R*G) = Cen(R)^G$.

Conversely, $Cen(R)^G \subseteq Cen(R*G)$ is clear.

Therefore $Cen(R*G) = Cen(R^G)$.

Lemma 2.5. [14] Assume that R is a semiprime ring and G is a finite group of X-outer ring automorphisms of R. Then $Cen\left(Q(R)^G\right) = \left[Cen(Q(R))\right]^G$.

Lemma 2.6. Assume that R is a semiprime ring and $e \in B(Q(R))$. Let I be a two-sided ideal of R such that $I_R \leq^{ess} eR_R$ and $r_R(I) = fR$ with $f \in B(R)$. Then $e = 1 - f$.

Proof. Since R is semiprime, $I_R \leq^{ess} l_R(r_R(I))_R = (1-f)R$. Thus $I_R \leq^{ess} (1-f)Q(R)_R$. As $I_R \leq^{ess} eR_R$, $I_R \leq^{ess} eQ(R)_R$. We note that e and $1-f$ are in $B(Q(R))$. So we have that $e = 1-f$.

Lemma 2.7. [15] Let R be a semiprime ring with a finite group G of X-outer ring automorphisms of R.

1) For $q \in Q(R)^G$, let I be a dense right ideal I of R^G such that $qI \subseteq R^G$. Then IR is a dense right ideal of R and the map $\tilde{q} : IR \to R$ defined by $\tilde{q}\left(\sum a_i r_i\right) = \sum (a_i) r_i$, with $a_i \in I$ and $r_i \in R$, is a right R-homomorphism. Moreover $\tilde{q} \in Q(R)^G$.

2) The map $\sigma : Q(R^G) \to Q(R)^G$ defined by $\sigma(q) = \tilde{q}$ is a ring isomorphism.

3) Let $\tilde{q} \in Q(R)$ and K a dense right ideal of R such that $\tilde{q}K \subseteq R$. Then $K \cap R^G$ is a dense right ideal of R^G and $\tilde{q}\big|_{R^G}(K \cap R^G) \subseteq R^G$, where $\tilde{q}\big|_{R^G}$ is the restriction of $\tilde{q}$ to R^G. Thus $\tilde{q}\big|_{R^G} \in Q(R^G)$.

For a ring R with a group G of ring automorphism of R, we say that a right ideal I of R is G-invariant if $I^g \subseteq I$ for every $g \in G$, where $I^g = \left\{ a^g \middle| a \in I \right\}$.

Proposition 2.8. [1] Let R be a semiprime ring. Then the followings are equivalent.

1) R is right p.q.-Baer;

2) Every principal two-sided ideal of R is right essential in a ring direct summand of R;

3) Every finitely generated two-sided ideal of R is right essential in a ring direct summand of R;

4) Every principal two-sided ideal of R that is closed as a right ideal is a direct summand of R;

5) For every principal two-sided ideal I of R, $r_R(I)$ is right essential in a direct summand of R;

6) R is left p.q.-Baer.

For a ring R with a group G of ring automorphisms of R, we say that a right ideal I of R is G-invariant if $I^g \subseteq I$ for every $g \in G$, where $I^g = \left\{ a^g \middle| a \in I \right\}$. Assume that R is a semiprime ring with a group G of ring automorphisms of R. We say that R is G-p.q.-Baer if the right annihilator of every finitely generated G-invariant two-sided ideal is generated by an idempotent, as a right ideal. By Proposition 8, if a ring R is semiprime p.q.-Baer with a group G of ring automorphisms of R, then R is G-p.q.-Baer.

A ring R is called right Rickart if the right annihilator of each element is generated by an idempotent of R. A left Rickart ring is defined similarly. A ring R is called Rickart if R is both right and left Rickart. A ring R is said to be reduced if R has no nonzero nilpotent element. We note that reduced Rickart rings are p.q.-Baer rings.

We put

$$B_p(Q(R)) = \{e \in B(Q(R)) \mid \text{there exists } x \in R \text{ with } RxR_R \leq^{ess} eR_R\}.$$

Let $\hat{Q}_{pqB}(R)$ be the subring of $Q(R)$ generated by R and $B_p(Q(R))$.

Lemma 2.9. [16] Assume that *R* is a semiprime ring. Then:

1) The ring $\hat{Q}_{pqB}(R)$ is the smallest right ring of quotients of *R* which is p.q.-Baer;

2) *R* is p.q.-Baer if and only if $B_p(Q(R)) \subseteq R$.

With these preparations, in spite of Example 1, we have the following result for p.q.-Baer property of R^G on a semiprime ring *R* for the case when *G* is finite and X-outer.

3. Main Results

Theorem 3.1. Let *R* be a semiprime ring with a finite group *G* of X-outer ring automorphisms of *R*. Then:

1) If *R* is *G*-p.q.-Baer, then R^G is p.q.-Baer.

2) If *R* is reduced *G*-p.q.-Baer, then R^G is Rickart.

Proof. 1) Assume that *R* is *G*-p.q.-Baer. To show that R^G is p.q.-Baer, it is enough to see that $B_p(Q(R^G)) \subseteq R^G$ by Lemma 9 since R^G is semiprime from Lemma 3. Let $e \in B_p(Q(R^G))$. Then $e \in B(Q(R^G))$, so $\tilde{e} \in B(Q(R^G))$ by Lemma 7. From Lemma 9, there exists $a \in R^G$ such that $R^G aR^G_{R^G} \leq^{ess} eR^G_{R^G}$ because $e \in B_p(Q(R^G))$. Note that $R^G aR^G_{R^G} \leq \tilde{e}R^G_{R^G}$.

We show that $R^G aR^G_{R^G} \leq^{ess} \tilde{e}R^G_{R^G}$. To see this, say $0 \neq \tilde{e}r \in \tilde{e}R^G$ with $r \in R^G$. Then $0 \neq er \in eR^G$. So there exists $b \in R^G$ such that $0 \neq erb \in R^G aR^G$. Hence $0 \neq \tilde{e}rb \in R^G aR^G$.

Observe that $\left[R^G aR^G \oplus r_{R^G}\left(R^G aR^G\right)\right]_{R^G} \leq^{ess} R^G_R$, as R^G is semiprime from Lemma 3. So $R^G aR^G \oplus r_{R^G}\left(R^G aR^G\right)$ is a dense right ideal of R^G since R^G is semiprime. By Lemma 7, $\left[R^G aR^G \oplus r_{R^G}\left(R^G aR^G\right)\right]R$ is a dense right ideal of R. So it is essential in R_R. Hence

$$\left[R^G aR^G + r_{R^G}\left(R^G aR^G\right)R\right]_R \leq^{ess} Q(R)_R.$$

We claim that $RaR_R \leq^{ess} \tilde{e}R_R$.

First note that $R^G aR^G R \subseteq \tilde{e}R_R$. For the claim, it is enough to show that $R^G aR^G R_R \leq^{ess} \tilde{e}R_R$. Take $0 \neq \tilde{e}r \in \tilde{e}R$ with $r \in R$. Then there exists $r_1 \in R$ such that $0 \neq \tilde{e}rr_1 \in R^G aR^G R + r_{R^G}\left(R^G aR^G\right)R$. Say $\tilde{e}rr_1 = x + y$, where $x \in R^G aR^G R$ and $y \in r_{R^G}\left(R^G aR^G\right)R$.

Then $\tilde{e}rr_1 = \tilde{e}\tilde{e}rr_1 = \tilde{e}x + \tilde{e}y$.

Put $y = \sum p_i s_i$ with $p_i \in r_{R^G}\left(R^G aR^G\right)$ and $s_i \in R$. Then $\tilde{e}y = \sum \tilde{e}p_i s_i$. In this case, $\tilde{e}p_i = 0$ for all *i*. To see this, assume on the contrary that there is p_i such that $\tilde{e}p_i \neq 0$. Note that $0 \neq \tilde{e}p_i \in \tilde{e}R^G$.

Thus there exists $r_0 \in R^G$ such that $0 \neq \tilde{e}p_i r_0 \in R^G aR^G$ because $R^G aR^G_{R^G} \leq^{ess} \tilde{e}R^G_{R^G}$.

Also $\tilde{e}p_i r_0 R^G aR^G \subseteq \tilde{e}p_i R^G aR^G = \tilde{e}0 = 0$ because $p_i \in r_{R^G}\left(R^G aR^G\right)$. Therefore we have that $0 \neq \tilde{e}p_i r_0 \in R^G aR^G \cap r_{R^G}\left(R^G aR^G\right) = 0$, a contradiction. Thus $\tilde{e}p_i = 0$ for all *i*, so $\tilde{e}y = 0$. Hence $\tilde{e}rr_1 = \tilde{e}x + \tilde{e}y = \tilde{e}x$. Now since $x \in R^G aR^G R$, write $x \in \sum t_i z_i$ with $t_i \in R^G aR^G$ and $z_i \in R$. Then $\tilde{e}x = \sum \tilde{e}t_i z_i = \sum t_i z_i = x$ because $t_i \in R^G aR^G \subseteq \tilde{e}R^G$. So $0 \neq \tilde{e}rr_1 = \tilde{e}x = x \in R^G aR^G R$. Therefore $RaR_R \leq^{ess} \tilde{e}R_R$. Note that $Cen(Q(R^G)) = \left[Cen(Q(R))\right]^G$ by Lemma 5. Therefore

$$\tilde{e} \in B(Q(R^G)) \subseteq Cen(Q(R^G)) = \left[Cen(Q(R))\right]^G \subseteq Cen(Q(R))$$

Hence $RaR = RR^G aR^G R \subseteq R\tilde{e}R^G R = \tilde{e}R$.

As *RaR* is a *G*-invariant two-sided ideal of *R* and *R* is *R*-p.q.-Baer, there is $f \in S_l(R)$ such that $r_R(RaR) = fR$. From [9], $f \in B(R)$. As $\tilde{e} \in B(Q(R))$ and $RaR_R \leq^{ess} \tilde{e}R_R$, it follows that $\tilde{e} = 1 - f \in R$ by Lemma 6, so $\tilde{e} \in R^G$.

Therefore $eR^G = \tilde{e}R^G \subseteq R^G$, and thus $e \in R^G$.

So $B_p(Q(R^G)) \subseteq R^G$, and hence R^G is p.q.-Baer by Lemma 9.

2) We recall that a reduced p.q.-Baer ring is Rickart. Thus if *R* is reduced *G*-p.q.-Baer, then R^G is Rickart from 1).

4. Conclusion

In [14], the quasi-Baer property of fixed rings under finite group actions on a semiprime ring and their applications to C^*-algebras have been studied (see also [17,18]). Motivated by investigations in [14], in this paper we investigate the right p.q.-Baer property of fixed rings under finite group actions on a given semiprime ring. Assume that *R* is a semiprime ring with a finite group *G* of X-outer ring automorphisms of *R*. Then we show that if *R* is *G*-p.q.-Baer, then R^G is p.q.-Baer. Thus if *R* is a semiprime p.q.-Baer ring with finite group *G* of X-outer ring automorphisms of *R*, then R^G is p.q.-Baer.

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