On the Infinite Products of Matrices

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ABSTRACT

In different fields in space researches, Scientists are in need to deal with the product of matrices. In this paper, we develop conditions under which a product $\prod_{i=0}^{\infty} P_i$ of matrices chosen from a possibly infinite set of matrices $M = \{P_j, j \in J\}$ converges. There exists a vector norm such that all matrices in M are no expansive with respect to this norm and also a subsequence $\{i_k\}_{k=0}^{\infty}$ of the sequence of nonnegative integers such that the corresponding sequence of operators $\{P_{i_k}\}_{k=0}^{\infty}$ converges to an operator which is paracontracting with respect to this norm. The continuity of the limit of the product of matrices as a function of the sequences $\{i_k\}_{k=0}^{\infty}$ is deduced. The results are applied to the convergence of inner-outer iteration schemes for solving singular consistent linear systems of equations, where the outer splitting is regular and the inner splitting is weak regular.

Keywords: Matrices; Infinite Products; Iteration

1. Introduction

Let the standard iterative method for solving the system of linear equations

$$Ax = b \qquad (1)$$

where $A \in R^{n,n}$ and x, b are n-vectors [1], be induced by the splitting of A into $A = T - Q$, where T is a nonsingular matrix. Starting with an arbitrary vector x_o, the recurrence relation

$$Tx_{k+1} = Qx_k + b \qquad (2)$$

is used to compute a sequence of iterations whose limit should be the solution to Equation (1).

If A is a nonsingular matrix, to obtain a good approximation to the solution of Equation (1), one need not to even solve the system (2) exactly for each x_{k+1}. For each $k \geq 1$, we solve the system (2) by iterations. Then split the matrix T into

$$T = G - H \qquad (3)$$

where the matrix G is invertible. Then, starting with $y_o := z_k, t_k$ inner iterations

$$y_j = G^{-1}Hy_{j-1} + G^{-1}c, \ c = Qz_k + b, j = 1, \cdots, t_k \qquad (4)$$

are computed after which one resets $z_{k+1} = y_{t_k}$. The entire inner-outer iteration process can then be expressed as

follows [2-4]

$$
\begin{aligned}
z_{k+1} &= \left(G^{-1}H\right)^{t_k} z_k + \sum_{i=0}^{t_k-1}\left(G^{-1}H\right)^i G^{-1}b \\
&= \left\{\left(G^{-1}H\right)^{t_k} + \sum_{i=0}^{t_k-1}\left(G^{-1}H\right)^i G^{-1}Q\right\} z_k \\
&\quad + \sum_{i=0}^{t_k-1}\left(G^{-1}H\right)^i G^{-1}b \\
&= P_{t_k} z_k + \sum_{i=0}^{t_k-1}\left(G^{-1}H\right)^i G^{-1}b,
\end{aligned}
\qquad (5)
$$

where

$$P_{t_k} := \left(G^{-1}H\right)^{t_k} + \sum_{i=0}^{t_k-1}\left(G^{-1}H\right)^i G^{-1}Q, \quad k = 1, 2, \cdots \qquad (6)$$

If the spectral radius of both $T^{-1}Q$ and $G^{-1}H$ are smaller than 1 so that the powers of both iteration matrices converge to zero, then for sufficiently large positive integer t we have that if $t_k \geq t \ \forall \ k \geq 1$ [5], the sequence $\{z_k\}$ produced by the inner-outer iterations converges to the solution of Equation (1) from all initial vectors z_o. If A and T have a nonnegative inverse and both iteration matrices $T^{-1}Q$ and $G^{-1}H$ are nonnegative matrices, with the former induced by a regular splitting of A and the latter induced by a weak regular splitting of T, then

the sequence $\{z_k\}$ converges to the solution of Equation (1) whenever $t_k \geq t \;\forall\; k \geq 1$ with no restrictions on t [4]. The process of inner-outer iterations can be represented by means of an iteration matrix at every stage, the spectral radius of such a matrix can no longer be less than 1. Furthermore, even if the spectral radius of the iteration matrix at each stage is 1, this does not ensure the convergence of the inner-outer iteration process even if a fixed number of iterations are used between every two outer iterations [6]. If the number of inner iterations, between every two outer iterations, is allowed to vary, the problem is further compounded [7,8]. Here we shall examine some connections between the work here and problems of convergence of infinite products of matrices such as considered by [6].

If one is going to employ the inner-outer iteration scheme, then it is very reasonable that often between any two outer iterations only a relatively small number of inner iterations will be computed and only in rare cases much more inner iteration will be allowed. This effectively means that there is a number $n \geq 1$, such that infinitely often at most n inner iterations will be carried out between any two outer ones. This implies that there exists an index $1 \leq n_o \leq n$ such that for an infinite subsequence i_k of the positive integers, $t_{i_k} = n_o$, infinitely often, $P_{t_{i_k}} = P_{m_o}$. We shall prove that under certain convergence properties of P_{m_o} such as P_{m_o} is paracontracting with respect to a vector norm in respect of which all the $P_{i,s}$ are no expansive, the inner-outer iteration (5) for any initial vector z_o. This implies that the inner-outer iteration scheme is convergent when the system (1) is consistent.

Now let we have an infinite set of matrices $M = \{P_j, j \in J\}$, and there exists a vector norm $\|\cdot\|$ on C^n such that each matrix in M is no expansive with respect to $\|\cdot\|$. From M select an infinite sequence of matrices $\{P_i\}_{i=0}^{\infty}$. Then if $\{P_i\}_{i=0}^{\infty}$ contains a subsequence $\{P_{i_k}\}_{k=0}^{\infty}$ which converges to a matrix H which is paracontracting with respect to $\|\cdot\|$ and such that the null space $N(I-H)$ is contained in the intersection of the null spaces $N(I-H)N(I-P_j), j \in J$, then $\exists \lim\limits_{i\to\infty} P_i P_{i-1} \cdots P_o$. Finally, let D be the set of all sequences $(c) = \{c_i\}_{i=0}^{\infty}$ of integers such that each sequence (c) contains an integer $k = k^{(c)}$ such that $c_i = k$ for infinitely many $i's$. Then, according to Th. 3.1 if corresponding to the sequence (c), the matrix P_k is para- contracting, then

$$\exists \lim\limits_{i\to\infty} P_{c_i} \cdots P_{i_o} =: P^{(c)}$$

We shall show that the function $f : (c) \to P^{(c)}$ is continuous.

2. Preliminaries

Let $E \in C^{n,n}$. We shall denote both of the null space and the range of E by $N(E)$ and $R(E)$ respectively. Recall that the Jordan blocks of E corresponding to 0 are 1 × 1 if and only if

$$N(E) \cap R(E) = \{0\}$$

and

$$N(E) + R(E) = C^n,$$

a situation which we shall write as

$$N(E) \oplus R(E) = C^n.$$

Recall further that according to [9] the powers of a matrix $E \in C^{n,n}$ converges if and only if

$$N(I-E) \oplus R(I-E) = C^n$$

and

$$\gamma(E) := \max\{|\lambda|, \lambda \in \sigma(E), \lambda \neq 1\} < 1,$$

where $\sigma(\cdot)$ denotes the spectrum of a matrix.

For a vector $x \in R^n$ we shall write that $x \gg 0 (x > 0)(x \geq 0)$ if all the entries of x are positive numbers. Also, let $\|\cdot\|$ enote a vector norm in. An n × n matrix E is no expansive with respect to $\|\cdot\|$ if for all $x \in C^n$,

$$\|Ex\| \leq \|x\|$$

E is called paracontracting with respect to $\|\cdot\|$ if for all $x \in C^n$

$$Ex \neq x \Leftrightarrow \|Ex\| < \|x\|$$

We denote by $N(\|\cdot\|)$ the set of all matrices in $C^{n,n}$ which are paracontracting with respect to $\|\cdot\|$. Two examples of paracontracting matrices are as follows. For the Euclidian norm it is known that any Hermitian matrix whose eigenvalues lie in (−1, 1] is paracontracting. Suppose now that E is an n × n positive matrix whose spectral radius is 1 and with a Perron vector $x \gg 0$. We claim that such a matrix is paracontracting with respect to $\|\cdot\|_x$, the monotonic vector norm induced by x. Let $y \in R^n$ be any vector satisfying $y \neq Ey$ or, equivalently, not being a multiple of x. We know that

$$\|y\|_x = \min\{\delta > 0, -\delta x \leq y \leq \delta x\}$$

By the positively of E and because $Ex = x$, it follows that for any δ such that $-\delta x \leq y \leq \delta x$, $-\delta x \ll Ey \ll \delta x$, so that $\|Ey\|_x < \|y\|_x$.

The concept of paracontraction was introduced by [4] who showed that the product of any number of matrices in $N(\|\cdot\|)$ is again an element of $N(\|\cdot\|)$. Moreover, they used a result of [3] to show that the powers of any matrix $E \in N(\|\cdot\|)$ converge. Thus, in particular such

matrix has the property that

$$N(I - E) \oplus R(I - E) = C^{n,n}.$$

Finally, recall that a splitting of A into $A = T - Q$ is called regular if T is nonsingular, $T^{-1} \geq 0$ and $Q \geq 0$. Regular splitting where introduced by [10], who showed that for a regular splitting, $\rho(T^{-1}Q) < 1$ if and only if A is nonsingular and $A^{-1} \geq 0$. A splitting $A = T - Q$ is called weak regular if T is nonsingular, $T^{-1} \geq 0$ and $T^{-1}Q \geq 0$. This concept was introduced by [11] who showed that, even allowing for this weakening of the assumption on regular splitting, $\rho(T^{-1}Q) < 1$ if and only if A is nonsingular and $A^{-1} \geq 0$. If $A = T - Q$ is a regular splitting of A, then

$$\rho(T^{-1}Q) \leq 1$$

and

$$R(I - T^{-1}Q) \oplus N(I - T^{-1}Q) = R^n$$

if and only if A is range monotone [12], that is,

$$\left[Ax \geq 0 \text{ and } x \in R(A) \right] \Rightarrow x \geq 0.$$

Moreover, they showed that if there exists a vector $x \gg 0$ such that $T^{-1}Qx \leq x$, then $\rho(T^{-1}Q) \leq 1$ and $R(I - T^{-1}Q) \oplus N(I - T^{-1}Q) = R^n$, and such a positive vector always exists if A is a singular and irreducible M-matrix.

3. Applications to Singular Systems

As we mentioned before, if $A = T - Q$ is a regular splitting for $A \in R^{n,n}$ and A is range monotone, then $\rho(T^{-1}Q) \leq 1$ and

$$R(I - T^{-1}Q) \oplus N(I - T^{-1}Q) = R^n.$$

Now, let $T = G - H$ is a weak regular splitting for T and consider the inner-outer iteration process

$$
\begin{aligned}
z_{k+1} &= \left(G^{-1}H\right)^{Tk} z_k + \sum_{j=0}^{Tk-1}\left(G^{-1}H\right)^j G^{-1}c \\
&= \left\{\left(G^{-1}H\right)^{Tk} + \sum_{j=0}^{Tk-1}\left(G^{-1}H\right)G^{-1}Q\right\} z_k \\
&\quad + \sum_{j=0}^{Tk-1}\left(G^{-1}H\right)^j G^{-1}c \\
&= P_{Tk} z_k + \sum_{j=0}^{Tk-1}\left(G^{-1}H\right)^j G^{-1}c
\end{aligned}
\tag{7}
$$

where

$$P_i = \left(G^{-1}H\right)^i + \sum_{j=0}^{i-1}\left(G^{-1}H\right)^j G^{-1}Q, \quad i = 1, 2, \cdots \tag{8}$$

We observe at once that since $A = T - Q$ is a regular splitting for A and $T = G - H$ is a weak regular split-

ting for T, any of the inner-outer iteration operators $P_i, i \geq 1$, is a nonnegative matrix. Already Nichols in [3] essentially showed the following relation holds:

$$I - P_i = \left(I - R^i\right)\left(I - T^{-1}Q\right), \tag{9}$$

Suppose that the n x n coefficient matrix A (assumed to be nonsingular) in the system (1) is monotone. For each $i \geq 1$, let $A = T_i - Q_i$ be a regular splitting of A and $T_i = G_i - H_i$ be a weak regular splitting. Consider the inner-outer iteration process:

$$z_{i+1} = P_{i,Ti} z_i + \sum_{i=0}^{Ti-1}\left(G_i^{-1}H_i\right)^i G_i^{-1}Q_i b,$$

where as, in the introduction, $T_i \geq 1$ and

$$P_{i,Ti} = \left(G_i^{-1}H_i\right)^{Ti} + \sum_{j=0}^{Ti-1}\left(G_i^{-1}H_i\right)^j G_i^{-1}Q_i$$

If there are splitting $A = T - Q$ and $T = G - H$ such that for infinitely many is $T_i = T$ and $G_i = G$ simultaneously, then for any $z_o \in R^n$,

$$\lim_{i \to \infty} z_i = A^{-1}b$$

Now, suppose $A \in R^{n,n}$ is a range monotone and that $A = T - Q$ and $T = G - H$ are regular and weak regular splitting for A and T respectively, then $\rho(P_i) \leq 1$ and

$$N(I - P_i) \oplus R(I - P_i) = R^n$$

for all $i \geq 1$ [2,10,13].

Once again, suppose that $A = T - Q$ and $T = G - H$ are regular and weak regular splitting for A and T respectively. Note that the range monotone of A was used only to deduce that $I - T^{-1}Q$ is an M-matrix of index at most 1. Another condition which ensures that $I - T^{-1}Q$ is an M-matrix of index at most 1 is that there exists a positive vector x such that $Ax \geq 0$ for then $T^{-1}Qx \leq x$. Furthermore, such a vector exists when A is a singular and irreducible M-matrix. When A is such an M-matrix, then, in fact, there exists a positive vector x such that $Ax = 0$. But then also

$$T^{-1}Ax = x - T^{-1}Qx = 0$$

so that $x = T^{-1}Qx$, and hence

$$P_i x = T^{-1}Qx + R^i\left(I - T^{-1}Q\right)x = T^{-1}Qx = x$$

We can thus conclude that when A is an irreducible M-matrix, not only the conclusions of the above result hold, but $P_i x = x$ so that $\|P_i\|_x = 1$. Hence for each $i \geq 1$, P_i is no expansive with respect to the norm $\|\cdot\|_x$. We also see that

$$0 = G^{-1}Ax = x - G^{-1}Hx - G^{-1}Qx \leq x - G^{-1}Hx = x - Rx$$

Now we know that $Q \geq 0$. Thus if either $G^{-1}Qx \gg 0$

or $G^{-1}Hx \gg 0$, then it follows that $x \gg Rx$ so that inductively,

$$1 > \|R\|_x \geq \|R\|_x^2 \geq \cdots$$

Let $V := T^{-1}Q$. Then from the relation

$$P_i - V = R^i (I - V)$$

we see that, not only

$$\lim_{i \to \infty} P_i = V, \qquad (10)$$

a fact that already follows from $\rho(R) < 1$, but that the rate of convergence behaves as $\|R\|_x$.

Theorem: Let $M = \{P_j, j \in J\}$ be a set of matrices in $C^{n,n}$, let $\{P_i\}_{i=0}^{\infty}$ be a sequence of matrices chosen from M, and consider the iteration scheme

$$x_{i+1} = P_i x_i, \ i \ = \ 0, \ 1, \ 2, \ \cdots$$

Suppose that all $P_j \in M$ are no expansive with respect to the same vector norm $\|\cdot\|$ and there exists a subsequence $\{P_{i_k}\}_{k=0}^{\infty}$ of the sequence $\{P_i\}_{i=0}^{\infty}$ such that $\lim_{k \to \infty} P_{i_k} = V$ where V is a matrix with the following properties:

- V is paracontracting with respect to $\|\cdot\|$.
- $N(I - V) \subseteq \bigcap_{j \in J} N(I - P_j)$

Then for any $x_o \in C^n$ the sequence $x_{i+1} = P_i x_i$ is convergent and

$$\lim_{i \to \infty} x_i \in N(I - V) \subseteq \bigcap_{j \in J} N(I - P_j).$$

The proof is given in [2].

From the above analysis and previous theorem we can now state the following result concerning the convergence of the inner-outer iteration process:

Theorem: Let $A \in R^{n,n}$ and suppose that $A = T - Q$ and $T = G - H$ are a regular splitting and a weak regular splitting for A and T, respectively, and consider the inner-outer iteration process (7) for solving the consistent linear system $Ax = b$. Suppose there exists a vector $x \gg 0$ such that $Ax \geq 0$ and one of the following conditions is satisfied:

1) For some integer j, P_j is paracontracting and for infinitely many integers k, $Tk = j$.

2) $T^{-1}Q$ is paracontracting with respect to $\|\cdot\|_x$, the sequence $\{Tk\}_{k=0}^{\infty}$ is unbounded, and either $G^{-1}Qx \gg 0$ or $G^{-1}Hx \gg 0$.

The sequence of iterations $\{zk\}_{k=1}^{\infty}$ generated by the scheme given in (7) converges to a solution of the system $Ax = b$.

Proof. We have the identity that

$$I - P_i = \sum_{j=0}^{i-1} (R_i)^j G^{-1} A$$

from which it follows that x is a positive vector for which

$$x \geq P_1 x \geq P_2 x \geq \cdots$$

showing that for each $i \geq 1$, P_i is no expansive with respect to the monotonic vector norm induced by x. Also the proof of (2) is clear because the unboundness of the sequence $\{Tk\}_{k=0}^{\infty}$ together with the existence of the limit in Equation (10) now means that the sequence of matrices $\{P_{Tk}\}_{k=0}^{\infty}$ contains an infinite subsequence of matrices which converges to the paracontracting matrix V.

4. Conclusion

The conditions under which the product $\prod_{i=0}^{\infty} P_i$ of matrices converges are explained and we apply the results for the convergence of inner-outer iteration schemes for solving singular consistent linear system of equations.

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