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On Certain Properties of Trigonometrically ρ-Convex Functions

Mohamed Sabri Salem Ali
Department of Mathematics, Faculty of Basic Education, PAAET, Shamiya, Kuwait
Email: mss_ali5@yahoo.com

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ABSTRACT

The aim of this paper is to prove that the average function of a trigonometrically ρ-convex function is trigonometrically ρ-convex. Furthermore, we show the existence of support curves implies the trigonometric ρ-convexity, and prove an extremum property of this function.

Keywords: Generalized Convex Functions; Trigonometrically ρ-Convex Functions; Supporting Functions; Average Functions; Extremum Problems

1. Introduction

In 1908, Phragmén and Lindelöf (See, e.g. [1]) showed that if $F(z)$ is an entire function of order $0<\rho<\infty$, then its indicator which is defined as:

$$h(\theta)=h_F(\theta)=\limsup_{r\to\infty}\frac{\log\left|F\left(re^{i\theta}\right)\right|}{r^{\rho}},\ (\alpha\le\theta\le\beta)$$

has the following property:

If $0<\rho(\beta-\alpha)<\pi$, and $H(\theta)$ is the function of the form

$$H(\theta):=A\cos\rho\theta+B\sin\rho\theta$$

(such functions are called sinusoidal or ρ-trigonometric) which coincides with $h(\theta)$ at α and at β, then for $\alpha\le\theta\le\beta$ we have

$$h(\theta)\le H(\theta).$$

This property is called a trigonometric ρ-convexity ([1,2]).

In this article we shall be concerned with real finite functions defined on a finite or infinite interval $(a,b)\subset\mathbb{R}$.

A well known theorem [3] in the theory of ordinary convex functions states that: A necessary and sufficient condition in order that the function $f:(a,b)\to\mathbb{R}$, be convex is that there is at least one line of support for f at each point x in (a,b).

In Theorem 3.1, we prove this result in case of trigonometrically ρ-convex functions. In Theorem 3.2, we prove the extremum property [4] of convex functions in case of trigonometrically ρ-convex functions. And finally in Theorem 3.3, we show that the average function [5] of a trigonometrically ρ-convex function is also trigonometrically ρ-convex.

2. Definitions and Preliminary Results

In this section we present the basic definitions and results which will be used later , see for example ([1,2], and [6-9]).

Definition 2.1. A function $f:(a,b)\to\mathbb{R}$ is said to be trigonometrically ρ-convex if for any arbitrary closed subinterval $[u,v]$ of (a,b) such that $0<\rho(v-u)<\pi$, the graph of $f(x)$ for $x\in[u,v]$ lies nowhere above the ρ-trigonometric function, determined by the equation

$$H(x)=H(x;u,v,f)=A\cos\rho x+B\sin\rho x$$

where A and B are chosen such that $H(u)=f(u)$, and $H(v)=f(v)$.

Equivalently, if for all $x\in[u,v]$

$$f(x)\le H(x)=\frac{f(u)\sin\rho(v-x)+f(v)\sin\rho(x-u)}{\sin\rho(v-u)}. \tag{1}$$

The trigonometrically ρ-convex functions possess a number of properties analogous to those of convex functions.

For example: If $f:(a,b)\to\mathbb{R}$ is trigonometrically ρ-convex function, then for any $u,v\in(a,b)$ such that $0<\rho(v-u)<\pi$, the inequality $f(x)\ge H(x;u,v,f)$ holds outside the interval (u,v).

Definition 2.2. A function

$$T_u(x) = A\cos\rho x + B\sin\rho x$$

is said to be **supporting function** for $f(x)$ at the point $u\in(a,b)$, if

$$T_u(u) = f(u), \text{ and } T_u(x)\le f(x) \ \forall x\in(a,b). \quad (2)$$

That is, if $f(x)$ and $T_u(x)$ agree at $x=u$ and the graph of $f(x)$ does not lie under the support curve.

Remark 2.1. If $f:(a,b)\to\mathbb{R}$ is differentiable trigonometrically ρ-convex function, then the supporting function for $f(x)$ at the point $u\in(a,b)$ has the form

$$T_u(x) = f(u)\cos\rho(x-u) + f'(u)\sin\rho(x-u).$$

Proof. The supporting function $T_u(x)$ for $f(x)$ at the point $u\in(a,b)$ can be described as follows:

$$T_u(x) = \lim_{v\to u} H(x;u,v,f),$$

where $v\in(a,b)$ such that $0<\rho(v-u)<\pi$, and as

$$f(x)\ge H(x;u,v,f), \ \forall x\in(a,b)\setminus(u,v).$$

Then taking the limit of both sides as $v\to u$, and from (1), one obtains

$$\begin{aligned} f(x) &\ge T_u(x) \\ &= \lim_{v\to u} H(x;u,v,f) \\ &= \lim_{v\to u}\frac{f(u)\sin\rho(v-x)+f(v)\sin\rho(x-u)}{\sin\rho(v-u)} \\ &= f(u)\cos\rho(x-u)+f'(u)\sin\rho(x-u). \end{aligned}$$

Thus, the claim follows.

Theorem 2.1. A trigonometrically ρ-convex function $f:(a,b)\to\mathbb{R}$ has finite right and left derivatives $f_+''(x), f_-'(x)$ at every point $x\in(a,b)$, and $f_-'(x)\le f_+''(x)$ for all $x\in(a,b)$.

Theorem 2.2. Let $f:(a,b)\to\mathbb{R}$ be a two times continuously differentiable function. Then f is trigono-me-trically ρ-convex on (a,b) if and only if $f''(x)+\rho^2 f(x)\ge 0$ for all $x\in(a,b)$.

Property 2.1. Under the assumptions of Theorem 2.1, the function f is continuously differentiable on (a,b) with the exception of an at most countable set.

Property 2.2. A necessary and sufficient condition for the function $f(x)$ to be a trigonometrically ρ-convex in (a,b) is that the function

$$\varphi(x) = f'(x)+\rho^2\int_w^x f(t)\,\mathrm{d}t, \ \ w\in(a,b)$$

is non-decreasing in (a,b).

Lemma 2.1. Let $f:(a,b)\to\mathbb{R}$ be a continuous, 2π-periodic function, and the derivative $f'(x)$ exists and piecewise continuous function and let M be a set of discontinuity points for $f'(x)$ If

$$f'(x_k-0)\le f'(x_k+0), \ \ x_k\in M, \quad (3)$$

and $f\in C^2((a,b)\setminus M)$, where

$$f''(x)+\rho^2 f(x)\ge 0, \ \ x\in(a,b)\setminus M. \quad (4)$$

Then $f(x)$ is trigonometrically ρ-convex on (a,b).

Proof. Consider

$$\varphi(x) = f'(x)+\rho^2\int_w^x f(t)\,\mathrm{d}t, \ \ w\in(a,b). \quad (5)$$

Two cases arise, as follows.

Case 1. Suppose $x=x_k\in M$. Using (5), we observe

$$\varphi(x_k+0)-\varphi(x_k-0) = f'(x_k+0)-f'(x_k-0).$$

From (3), we get $\varphi(x_k+0)\ge\varphi(x_k-0)$.

So, the function $\varphi(x)$ is non-decreasing in M. *Case* 2. Let $x\in(x_{k-1},x_k), x_{k-1}, x_k\in M$, and $(x_{k-1},x_k)\cap M=\varnothing$.

Differentiating both sides of (5) with respect to x, one has

$$\varphi'(x) = f''(x)+\rho^2 f(x).$$

Using (4), one obtains

$$\varphi'(x)\ge 0, \ \ x\in(x_{k-1},x_k).$$

Thus, $\varphi(x)$ is non-decreasing function in (x_{k-1},x_k).

Therefore, from Property 2.2, we conclude that the function $f(x)$ is trigonometrically ρ-convex on (a,b).

3. Main Results

Theorem 3.1. A function $f:(a,b)\to\mathbb{R}$ is trigonometrically ρ-convex on (a,b) if and only if there exists a supporting function for $f(x)$ at each point x in (a,b).

Proof. The necessity is an immediate consequence of F. F. Bonsall [10].

To prove the sufficiency, let x be an arbitrary point in (a,b) and f has a supporting function at this point. For convenience, we shall write the supporting function in the follwoing form:

$$T_x(z) = f(x)\cos\rho(z-x)+K_{x,f}\sin\rho(z-x),$$

where $K_{x,f}$ is a fixed real number depends on x and f.

From Definition 2.2, one has

$$T_x(x) = f(x), \text{ and } T_x(z)\le f(z) \ \forall z\in(a,b).$$

It follows that,

$$f(x)\cos\rho(z-x)+K_{x,f}\sin\rho(z-x)\le(z) \ \ \forall z\in(a,b). \quad (6)$$

For all $u,v\in(a,b)$, choose any $u\ne v$ such that $0<\rho(v-u)<\pi$, and $\lambda,\mu\ge 0$ with $\lambda+\mu=1$ and let

$x = \lambda u + \mu v.$

Applying (6) twice at $z = u$ and at $z = v$ yields

$$f(x)\cos\rho(u-x)+K_{x,f}\sin\rho(u-x)\le f(u),$$

$$f(x)\cos\rho(v-x)+K_{x,f}\sin\rho(v-x)\le f(v).$$

Multiplying the first inequality by $\sin\rho\lambda(v-u)$, the second by $\sin\rho\mu(v-u)$, and adding them, we obtain

$$\begin{aligned}&f(x)\left[\sin\rho\lambda(v-u)\cos\rho(u-x)\right.\\&\left.-\cos\rho(v-x)\sin\rho\mu(u-v)\right]\\&+K_{x,f}\sin\rho\lambda(v-u)\sin\rho(u-x)\\&-\sin\rho(v-x)\sin\rho\mu(u-v)\\&\le f(u)\sin\rho\lambda(v-u)+f(v)\sin\rho\mu(v-u).\end{aligned}$$

Consequently

$$f(x)\le\frac{f(u)\sin\rho(v-x)+f(v)\sin\rho(x-u)}{\sin\rho(v-u)}$$

for all $x\in[u,v]$, which proves that the function $f(x)$ is trigonometrically ρ-convex on (a,b).

Hence, the theorem follows.

Remark 3.1. For a trigonometrically ρ-convex function $f:(a,b)\to\mathbb{R}$, the constant $K_{x,f}$ in the above theorem is equal to $f'(x)$ if f is differentiable at the point x in (a,b), otherwise, $K_{x,f}\in\left[f'_-(x),f''_+(x)\right]$.

Theorem 3.2. Let $f:(a,b)\to\mathbb{R}$ be a trigonometrically ρ-convex function such that $0<\rho(b-a)<\pi$ and let $T_u(x)$ be a supporting function for $f(x)$ at the point $u\in(a,b)$. Then the function

$$G(u)=\int_a^b\left[f(x)-T_u(x)\right]\mathrm{d}x$$

has a minimum value at $u=\dfrac{a+b}{2}$.

Proof. From Definition 2.2, we have

$$T_u(u)=f(u), \tag{7}$$

and

$$T_u(x)\le f(x)\quad\forall x\in(a,b), \tag{8}$$

and $T_u(x)$ can be written in the form

$$\begin{aligned}T_u(x)&=f(u)\cos\rho(x-u)+K_{u,f}\sin\rho(x-u)\\&=K\sin\rho(x+\alpha-u),\end{aligned} \tag{9}$$

where $K=\sqrt{f^2(u)+K_{u,f}^2}$, and $\tan\rho\alpha=\dfrac{f(u)}{K_{u,f}}.$

Using (9), one obtains

$$\begin{aligned}&\int_a^b T_u(x)\mathrm{d}x\\&=K\int_a^b\sin\rho(x+\alpha-u)\mathrm{d}x\\&=\frac{2}{\rho}\sin\rho\left(\frac{b-a}{2}\right)K\sin\rho\left[\left(\frac{b+a}{2}\right)+\alpha-u\right]\\&=\frac{2}{\rho}\sin\rho\left(\frac{b-a}{2}\right)T_u\left(\frac{a+b}{2}\right)\end{aligned}$$

Consequently,

$$G(u)=\int_a^b f(x)\mathrm{d}x-\frac{2}{\rho}\sin\rho\left(\frac{b-a}{2}\right)T_u\left(\frac{a+b}{2}\right). \tag{10}$$

Using (7) at $u=\dfrac{a+b}{2}$, the function $G(u)$ becomes

$$G\left(\frac{a+b}{2}\right)=\int_a^b f(x)\mathrm{d}x-\frac{2}{\rho}\sin\rho\left(\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right). \tag{11}$$

But from (8) ,we observe $T_u\left(\dfrac{a+b}{2}\right)\le f\left(\dfrac{a+b}{2}\right)$ for all $u\in(a,b)$.

Now using (10) and (11), it follows that

$G(u)\ge G\left(\dfrac{a+b}{2}\right)$ for all $u\in(a,b)$.

Hence, the minimum value of the function $G(u)$ occurs at $u=\dfrac{a+b}{2}$.

Theorem 3.3. Let $f(x)$ be a non-negative, 2π-periodic, and trigonometrically ρ-convex function with a continuous second derivative on $\mathbb{R}^+$ and let $F(x)$ be a 2π-periodic function defined in $[0,2\pi]$ as follows

$$F(x)=\frac{1}{x}\int_0^x f(t)\mathrm{d}t,\quad x\in[0,2\pi]. \tag{12}$$

If $f'(0)\ge 0$, and

$$f(0)=\frac{1}{2\pi}\int_0^{2\pi}f(t)\mathrm{d}t. \tag{13}$$

Then, $F(x)$ is trigonometrically ρ-convex function.

Proof. The proof mainly depends on Lemma 2.1. So, we show that the function $F(x)$ satisfies all conditions in this lemma.

Suppose that

$$g(x):=\frac{1}{x}\int_0^x f(t)\mathrm{d}t,\quad x\in\mathbb{R}^+. \tag{14}$$

It is obvious that, $g(0)=f(0)$.

First, we study the behavior of the function $F(x)$ inside the interval $(0,2\pi)$.

It is clear from (12) that $F(x)$s is an absolutely continuous function, has a derivative of third order.

But from the periodicity of $F(x)$ and (13), we get

$$F(0) = g(0) = f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = g(2\pi) = F(2\pi). \quad (15)$$

Using the following substitution $t = x\tau$.

It follows that, $F(x)$ can be written as

$$F(x) = \int_0^1 f(x\tau)\,d\tau \quad \text{and} \quad F''(x) = \int_0^1 \tau^2 f''(x\tau)\,d\tau .$$

Consequently,

$$F''(x) + \rho^2 F(x) = \int_0^1 \tau^2 \left(f''(x\tau) + \rho^2 f(x\tau)\right) + \rho^2\left(1-\tau^2\right) f(x\tau)\,d\tau. \quad (16)$$

Since $f(x)$ is non-negative, trigonometrically ρ-convex function, and $0 \le \tau \le 1$, then from Theorem 2.2 and (16) it follows that

$$F''(x) + \rho^2 F(x) \ge 0, \; x \in (0, 2\pi). \quad (17)$$

Second, we prove that

$$F'(2\pi - 0) \le F'(2\pi + 0). \quad (18)$$

From the definition of $g(x)$ in (14) and the periodicity of $F(x)$, we observe that
$F'(2\pi - 0) = g'(2\pi)$, and $F'(2\pi + 0) = g'(0)$.

Again using (14), we have

$$g'(x) = \frac{f(x) - g(x)}{x}. \quad (19)$$

Thus, from (15) and (19), one has $g'(2\pi) = 0$, and $g'(0) = \frac{1}{2} f'(0)$.

Hence, from (13), we infer that

$$F'(2\pi - 0) = g'(2\pi) = 0 \le \frac{1}{2} f''(0) = g'(0) = F'(2\pi + 0),$$

and the inequality in (18) is proved.

Now using (17), (18), and Lemma 2.1, we conclude that $F(x)$ is trigonometrically ρ-convex function, and the theorem is proved.

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REFERENCES

[1] B. Ya. Levin, "Lectures on Entire Functions," American Mathematical Society, 1996.

[2] L. S. Maergoiz, "Asymptotic Characteristics of Entire Functions and Their Applications in Mathematics and Biophysics," Kluwer Academic Publishers, New York, 2003.

[3] A. W. Roberts and D. E. Varberg, "Convex Functions," Academic Press, New York, 1973.

[4] M. J. Miles, "An Extremum Property of Convex Functions," *American Mathematical Monthly*, Vol. 76, 1969, pp. 921-922. doi:10.2307/2317948

[5] A. M. Bruckner and E. Ostrow, "Some Functions Classes Related to the Class of Convex Functions," *Pacific Journal of Mathematics*, Vol. 12, 1962, pp. 1203-1215.

[6] E. F. Beckenbach, "Convex Functions," *Bulletin of the American Mathematical Society*, Vol. 54, No. 5, 1948, pp. 439-460. doi:10.1090/S0002-9904-1948-08994-7

[7] J. W. Green, "Support, Convergence, and Differentiability Properties of Generalized Convex Functions," *Proceedings of the American Mathematical Society*, Vol. 4, No. 3, 1953, pp. 391-396. doi:10.1090/S0002-9939-1953-0056039-2

[8] M. M. Peixoto, "On the Existence of Derivatives of Generalized Convex Functions," *Summa Brasilian Mathematics*, Vol. 2, No. 3, 1948, pp. 35-42.

[9] M. M. Peixoto, "Generalized Convex Functions and Second Order Differential Inequlities," *Bulletin of the American Mathematical Society*, Vol. 55, No. 6, 1949, pp. 563-572. doi:10.1090/S0002-9904-1949-09246-7

[10] F. F. Bonsall, "The Characterization of Generalized Convex Functions," *The Quarterly Journal of Mathematics Oxford Series*, Vol. 1, 1950, pp. 100-111. doi:10.1093/qmath/1.1.100