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Totally Umbilical Screen Transversal Lightlike Submanifolds of Semi-Riemannian Product Manifolds

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ABSTRACT

We study totally umbilical screen transversal lightlike submanifolds immersed in a semi-Riemannian product manifold and obtain necessary and sufficient conditions for induced connection ∇ on a totally umbilical radical screen transversal lightlike submanifold to be metric connection. We prove a theorem which classifies totally umbilical ST-anti-invariant lightlike submanifold immersed in a semi-Riemannian product manifold.

Keywords: Semi-Riemannian Product Manifolds; Lightlike Submanifolds; Totally Umbilical Radical ST-Lightlike Submanifolds; Totally Umbilical ST-Anti-Invariant Lightlike Submanifolds

1. Introduction

It is well known that the geometry of lightlike submanifolds of semi-Riemannian manifolds is different from the geometry of submanifolds immersed in a Riemannian manifold since the normal vector bundle of lightlike submanifolds intersect with tangent bundle making it more interesting to study. The general theory of lightlike submanifolds of a semi-Riemannian manifold has been developed by Duggal-Bejancu [1] and Kupeli [2]. Totally umbilical CR-submanifolds of a Kaehler manifold with Riemannian metric were studied by Bejancu [3], Deshmukh and Husain [4] and many more whereas, totally umbilical lightlike submanifolds of semi-Riemannian manifolds of constant curvature was investigated by Duggal-Jin [5] and totally umbilical CR-lightlike submanifolds of an indefinite Kaehler manifold were studied by Duggal-Bejancu [1] and Gogna *et al.* [6]. In [7], B. Sahin initiated the study of transversal lightlike submanifolds of an indefinite Kaehler manifold and investigated the existence of such lightlike submanifolds in an indefinite space form. These submanifolds in Sasakian setting were studied by Yildirim and Sahin [8]. As a generalization of real null curves of indefinite Kaehler manifolds, B. Sahin [9] introduced the notion of screen transversal lightlike submanifolds and obtained many interesting results. In this paper, we study totally umbilical screen transversal lightlike submanifolds of semi-Riemannian product manifolds.

This paper is arranged as follows. In Sections 2 and 3, we give the basic concepts on lightlike submanifolds and semi-Riemannian product manifolds needed for this paper. In Section 4, we study the integrability of distributions involved in the definition of totally umbilical radical screen transversal lightlike submanifolds and obtain necessary and sufficient conditions for induced connection ∇ on totally umbilical radical screen transversal lightlike submanifolds to be metric connection. In Section 5, we prove a theorem which shows that the induced connection ∇ on a totally umbilical ST-anti-invariant lightlike submanifold is a metric connection under some conditions. We also prove a theorem which classifies totally umbilical ST-anti-invariant lightlike submanifold immersed in a semi-Riemannian product manifold.

2. Preliminaries

We follow [1] for the notation and fundamental equation for lightlike submanifolds used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $\left(\bar{M}^{m+n},\bar{g}\right)$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\bar{g}$ and radical distribution $RadTM$ is of rank r, where $1 \le r \le m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, *i.e.*,

$$TM = RadTM \perp S(TM)$$

Consider a screen transversal vector bundle $S\left(TM^{\perp}\right)$, which is a semi-Riemannian complementary vector bundle of *RadTM* in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of *RadTM*, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exists a lightlike transversal vector bundle *ltr*(*TM*) locally spanned by $\{N_i\}$ [[1]; pg-144]. Let *tr*(*TM*) be complementary (but not orthogonal) vector bundle to *TM* in $T\bar{M}|_M$. Then

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M}|_M = S(TM) \perp [RadTM \oplus ltr(TM)] \perp S(TM^{\perp}).$$

Following are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$.

Case 1: *r*-lightlike if $r < \min\{m, n\}$.

Case 2: Co-isotropic if $r = n < m$; $S(TM^{\perp}) = 0$.

Case 3: Isotropic if $r = m < n$; $S(TM) = 0$.

Case 4: Totally lightlike if $r = n = m$; $S(TM) = 0 = S(TM^{\perp})$.

The Gauss and Weingarten formulae are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \; \forall X, Y \in \Gamma(TM) \tag{2.1}$$

and

$$\begin{gathered}\bar{\nabla}_X U = -A_U X + \nabla^t_X U,\\ \forall\, X \in \Gamma(TM),\; U \in \Gamma(tr(TM))\end{gathered} \tag{2.2}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively, ∇ and ∇^t are linear connection on *M* and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y) \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N) \tag{2.4}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W) \tag{2.5}$$

$\forall X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Denote the projection of TM on $S(TM)$ by P. Then, by using (2.1), (2.3)-(2.5) and the fact that $\bar{\nabla}$ is a metric connection, we obtain

$$\begin{gathered}\bar{g}(h^s(X,Y), W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y),\\ \bar{g}(D^s(X,N), W) = \bar{g}(N, A_W X).\end{gathered} \tag{2.6}$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\begin{gathered}\nabla_X PY = \nabla^*_X PY + h^*(X, PY),\\ \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi,\end{gathered} \tag{2.7}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.

In general, the induced connection ∇ on *M* is not a metric connection whereas ∇^* is a metric connection on $S(TM)$.

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifolds of $(\bar{M}, \bar{g})$. For any vector field *X* tangent to *M*, we put

$$FX = fX + \omega X \tag{2.8}$$

where fX and ωX are the tangential and transversal parts of FX respectively. For $V \in \Gamma(tr(TM))$

$$FV = BV + CV \tag{2.9}$$

where *BV* and *CV* are the tangential and transversal parts of *FV* respectively.

3. Semi-Riemannian Product Manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2-dimensional semi-Riemannian manifolds with constant indices $q_1 > 0$ and $q_2 > 0$ respectively. Let $\pi : M_1 \times M_2 \to M_1$, and $\sigma : M_1 \times M_2 \to M_2$ be the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$. We denote the product manifold by $\bar{M} = (M_1 \times M_2, \bar{g})$, where

$$\bar{g}(X,Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(T\bar{M})$, where * denotes the differential mapping. Then we have $\pi_*^2 = \pi_*$, $\sigma_*^2 = \sigma_*$, $\pi_* \sigma_* = \sigma_* \pi_* = 0$ and $\pi_* + \sigma_* = I$ where *I* is the identity map of $\Gamma(M_1 \times M_2)$. Thus $(\bar{M}, \bar{g})$ is a $(m_1 + m_2)$-dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The Riemannian product manifold $\bar{M} = (M_1 \times M_2, \bar{g})$ is characterized by M_1 and M_2 which are totally geodesic submanifolds of $\bar{M}$.

Now, if we put $F = \pi_* - \sigma_*$ then we can easily see that $F^2 = I$ and

$$\bar{g}(FX, Y) = \bar{g}(X, FY) \tag{3.1}$$

for any $X, Y \in \Gamma(T\bar{M})$, where *F* is called almost Riemannian product structure on $M_1 \times M_2$. If we denote the Levi-Civita connection on $\bar{M}$ by $\bar{\nabla}$, then

$$(\bar{\nabla}_X F) Y = 0 \tag{3.2}$$

for any $X, Y \in \Gamma(T\bar{M})$, that is, *F* is parallel with respect to $\bar{\nabla}$.

4. Totally Umbilical Radical ST-Lightlike Submanifolds

In this section, we study totally umbilical radical ST-lightlike submanifolds of a semi-Riemannian product manifold. We first recall the following definitions from [9].

Definition 4.1. A *r*-lightlike submanifold *M* of a semi-Riemannian product manifold *M* is said to be a screen transversal (ST) lightlike submanifold of $\bar{M}$ if there exists a screen transversal bundle $S(TM^{\perp})$ such that

$$F(RadTM) \subset S(TM^{\perp}).$$

Definition 4.2. A ST-lightlike submanifold M of a semi-Riemannian product manifold M is said to be a radical ST-lightlike submanifold if $S(TM)$ is invariant with respect to F.

We also need the following definition of totally umbilical lightlike submanifolds of a semi-Riemannian manifold.

Definition 4.3. [5] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called totally umbilical in $\bar{M}$, if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ of M, called the transversal curvature vector of M, such that for all $X, Y \in \Gamma(TM)$,

$$h(X,Y) = g(X,Y)H$$

It is known that M is totally umbilical if and only if on each co-ordinate neighborhood U, there exists smooth vector fields $H^{l} \in \Gamma(ltr(TM))$ and $H^{s} \in \Gamma(S(TM^{\perp}))$ such that

$$h^{l}(X,Y) = g(X,Y)H^{l}, h^{s}(X,Y) = g(X,Y)H^{s} \quad \text{and } D^{l}(X,W) = 0 \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

In respect of the integrability of the distributions involved in the definition of totally umbilical radical ST-lightlike submanifolds immersed in a semi-Riemannian product manifold, we have:

Theorem 4.4. Let M be a totally umbilical radical ST-lightlike submanifold of a Semi-Riemannian product manifold. Then the screen distribution $S(TM)$ is always integrable.

Proof. From (2.3) and (3.2), a direct calculation shows that

$$\bar{g}([X,Y],N) = \bar{g}(h^{s}(X,FY) - h^{s}(Y,FX),FN) \tag{4.2}$$

for $X, Y \in \Gamma(S(TM))$ and $N \in (ltr(TM))$. Using (4.1) in (4.2), we get

$$\bar{g}([X,Y],N) = 0,$$

from which our assertion follows.

Theorem 4.5. Let M be a totally umbilical radical ST-lightlike submanifold of a semi-Riemannian product manifold. Then the distribution $RadTM$ is always integrable.

Proof. For $Z, W \in \Gamma(RadTM)$ and $X \in \Gamma(S(TM))$, from (2.3) and (3.2) we get

$$g([Z,W],X) = -\bar{g}(h^{s}(Z,FX),FW) + \bar{g}(h^{s}(W,FX),FZ). \tag{4.3}$$

Taking account of (4.1) in (4.3), we obtain

$$g([Z,W],X) = 0,$$

which proves our assertion.

The necessary and sufficient conditions under which $H^{s} = 0$ is given by the following result.

Theorem 4.6. Let M be a totally umbilical radical ST-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then $h^{*}(X,Y) = 0$ if and only if H^{s} has no components in $F(RadTM)$ for any $X, Y \in \Gamma(S(TM))$.

Proof. Using (2.3) and (3.2), for any $X, Y \in \Gamma(S(TM))$, we obtain

$$\nabla_{X} FY + h^{l}(X,FY) + h^{s}(X,FY) = F\nabla_{X} Y + Fh^{l}(X,Y) + Fh^{s}(X,Y). \tag{4.4}$$

Taking inner product of (4.4) with FN for any $N \in \Gamma(ltr(TM))$ and using the fact that $F^{2} = I$, we get

$$\bar{g}(h^{s}(X,FY),FN) = \bar{g}(\nabla_{X} Y,N). \tag{4.5}$$

From (2.7), (4.1) and (4.5), we have

$$g(X,FY)\bar{g}(H^{s},FN) = \bar{g}(h^{*}(X,Y),N). \tag{4.6}$$

Thus, our assertion follows from (4.6).

It is known that the induced connection on a lightlike submanifold immersed in a semi-Riemannian manifold is not a metric connection. In view of this, it is interesting to see under what condition the induced connection on a totally umbilical radical ST-lightlike submanifold is a metric connection. The following theorem gives the geometric conditions for the induced connection to be a metric connection.

Theorem 4.7. Let M be a totally umbilical radical ST-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the induced connection ∇ on M is a metric connection if and only if $A_{F\xi} X = 0$ for $X \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$.

Proof. For $X \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$, from (3.2) we have

$$\bar{\nabla}_{X} F\xi = F\bar{\nabla}_{X} \xi. \tag{4.7}$$

Using (2.3), (2.5), (2.8), (2.9) and (4.1) in (4.7), we obtain

$$-A_{F\xi} X + \nabla^{s}_{X} F\xi = f\nabla_{X} \xi + \omega\nabla_{X} \xi + Fh^{l}(X,\xi) + Bh^{s}(X,\xi) + Ch^{s}(X,\xi).$$

Taking tangential components of the above equation and then using (4.1), we arrive at

$$f\nabla_{X} \xi = A_{F\xi} X,$$

which proves our assertion.

Corollary 4.8. Let M be a totally umbilical radical ST-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the distribution $RadTM$ is parallel if and only if $A_{F\xi_2}\xi_1 = 0$ for any $\xi_1, \xi_2 \in \Gamma(RadTM)$.

Proof. From (3.2), for any $\xi_1, \xi_2 \in \Gamma(RadTM)$ we obtain

$$\bar{\nabla}_{\xi_1} F\xi_2 = F\bar{\nabla}_{\xi_1}\xi_2.$$

Using (2.3), (2.5), (2.8), (2.9) and (4.1) in the above equation, we get

$$-A_{F\xi_2}\xi_1 + \nabla^s_{\xi_1} F\xi_2 = f\nabla_{\xi_1}\xi_2 + \omega\nabla_{\xi_1}\xi_2 + Fh^l(\xi_1,\xi_2) + Bh^s(\xi_1,\xi_2) + Ch^s(\xi_1,\xi_2) \quad (4.8)$$

Considering the tangential components of (4.8) and using (4.1), we arrive at

$$-A_{F\xi_2}\xi_1 = f\nabla_{\xi_1}\xi_2,$$

from which our assertion follows.

Lemma 4.9. Let *M* be a totally umbilical ST-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then

$$A_W X = \bar{g}(H^s, W)X$$

for any $X \in \Gamma(S(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Proof. For $X, Y \in \Gamma(TM)$, from (2.6) and (4.1), we have

$$g(A_W X, Y) = g(X, Y)\bar{g}(H^s, W). \quad (4.9)$$

If $X \in \Gamma(RadTM)$, then from (4.9) we infer that $A_W X = 0$. Moreover, if $X \in \Gamma(S(TM))$, then due to non-degeneracy of $S(TM)$, we have

$$A_W X = \bar{g}(H^s, W)X,$$

which proves the assertion.

For the induced connection ∇^t of a totally umbilical radical ST-lightlike submanifold in semi-Riemannian product manifolds to be a metric connection on $tr(TM)$, we have:

Theorem 4.10. Let *M* be a totally umbilical radical ST-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then ∇^t is a metric connection on $tr(TM)$ if and only if $\nabla^s_X N$ has no component in μ for any $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$.

Proof. For $X \in \Gamma(TM)$, $W \in \Gamma(S(TM^\perp))$ and $N \in \Gamma(ltr(TM))$, using (2.2), (2.5), (2.9) and (3.2), we get

$$\bar{g}(\nabla^t_X N, W) = \bar{g}(-A_{FN}X + \nabla^s_X FN + D^l(X, FN), BW + C_1W + C_2W), \quad (4.10)$$

where $BW \in \Gamma(RadTM)$, $C_1W \in \Gamma(ltr(TM))$ and $C_2W \in \Gamma(\mu)$. Using (4.1) and (4.10), we obtain,

$$\bar{g}(\nabla^t_X, N, W) = \bar{g}(-A_{FN}X, C_1W) + \bar{g}(\nabla^s_X FN, C_2W).$$

Considering lemma 4.9, we get

$$\bar{g}(\nabla^t_X, N, W) = \bar{g}(\nabla^s_X FN, C_2W). \quad (4.11)$$

Thus our assertion follows from (4.11) and Theorem 2.3 page 159 of [1].

Theorem 4.11. Let *M* be a totally umbilical radical ST-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then

$$A_{F\xi_1}\xi_2 = A_{F\xi_2}\xi_1$$

for all $\xi_1, \xi_2 \in \Gamma(RadTM)$.

Proof. For any $\xi_1, \xi_2 \in \Gamma(RadTM)$., using product structure on $\bar{M}$, we get

$$F\bar{\nabla}_{\xi_1}\xi_2 = \bar{\nabla}_{\xi_1} F\xi_2,$$

from which we have

$$F\nabla_{\xi_1}\xi_2 + Fh^l(\xi_1,\xi_2) + Fh^s(\xi_1,\xi_2) = -A_{F\xi_2}\xi_1 + \nabla^s_{\xi_1} F\xi_2, \quad (4.12)$$

where we have used (2.3), (2.5) and (3.2). Interchanging ξ_1 and ξ_2 in (4.12) and then subtracting the resulting equation from (4.12), we obtain

$$F\nabla_{\xi_1}\xi_2 - F\nabla_{\xi_2}\xi_1 = -A_{F\xi_2}\xi_1 + A_{F\xi_1}\xi_2 + \nabla^s_{\xi_1} F\xi_2 - \nabla^s_{\xi_2} F\xi_1. \quad (4.13)$$

Taking inner product of (4.13) with $X \in \Gamma(S(TM))$, we get

$$g(\nabla_{\xi_1}\xi_2, FX) - g(\nabla_{\xi_2}\xi_1, FX) = g(A_{F\xi_1}\xi_2 - A_{F\xi_2}\xi_1, X). \quad (4.14)$$

Now, from (2.3) and (4.1), a direct calculation shows that

$$g(\nabla_{\xi_1}\xi_2, FX) = 0, \; g(\nabla_{\xi_2}\xi_1, FX) = 0. \quad (4.15)$$

Using (4.15) in (4.14), we get

$$g(A_{F\xi_1}\xi_2 - A_{F\xi_2}\xi_1, X) = 0. \quad (4.16)$$

Thus our assertion follows from (4.16) together with non-degeneracy of $S(TM)$.

5. Totally Umbilical ST-Anti-Invariant Lightlike Submanifolds

In this section, we study totally umbilical ST-anti-invariant lightlike submanifolds immersed in a semi-Riemannian product manifold. First we recall the following definition from [9].

Definition 5.1. [9] A ST-lightlike submanifold M of a semi-Riemannian product Manifold $\bar{M}$ is said to be a ST-anti-invariant lightlike submanifold of *M* if $S(TM)$ is screen transversal with respect to *F*, *i.e.*,

$$F(S(TM)) \subset S(TM^\perp).$$

The necessary and sufficient conditions for the induced connection ∇ on a totally umbilical ST-anti-invariant lightlike submanifold *M* to be a metric connection is given by the following result.

Theorem 5.2. Let *M* be a totally umbilical ST-anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the induced connection ∇ on *M* is a metric connection if and only if $\nabla^s_X F\xi$ has no component in $F(S(TM))$ for all $X \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$.

Proof. Using (2.3), (2.5), (2.8), (2.9), (3.2) and (4.1), we arrive at

$$-A_{F\xi}X + \nabla^s_X F\xi = \omega\nabla_X \xi + Ch^l(X,\xi) + Bh^s(X,\xi) + Ch^s(X,\xi). \tag{5.1}$$

Taking inner product of (5.1) with *FY* for $Y \in \Gamma(S(TM))$ and then using (4.1), we obtain

$$\bar{g}(\nabla^s_X F\xi, FY) = g(\nabla_X \xi, Y),$$

which proves the assertion.

Theorem 5.3. Let *M* be a totally umbilical ST-anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then *RadTM* is parallel if and only if $\nabla^s_{\xi_1} F\xi_2$ has no component in $F(S(TM))$ for all $\xi_1, \xi_2 \in \Gamma(RadTM)$.

Proof. From (2.3), (2.5), (2.8), (2.9), (3.2) and (4.1), we have

$$-A_{F\xi_2}\xi_1 + \nabla^s_{\xi_1} F\xi_2 = \omega\nabla_{\xi_1}\xi_2 + Ch^l(\xi_1,\xi_2) + Bh^s(\xi_1,\xi_2) + Ch^s(\xi_1,\xi_2)$$

for any $\xi_1, \xi_2 \in \Gamma(RadTM)$. Using (4.1) in the above equation, we get

$$-A_{F\xi_2}\xi_1 + \nabla^s_{\xi_1} F\xi_2 = \omega\nabla_{\xi_1}\xi_2. \tag{5.2}$$

Taking inner product of (5.2) with *FY* for $Y \in \Gamma(S(TM))$, we obtain

$$\bar{g}(\nabla^s_{\xi_1} F\xi_2, FY) = g(\nabla_{\xi_1}\xi_2, Y),$$

from which our assertion follows.

Theorem 5.4. Let *M* be a totally umbilical ST-anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then H^s has no component in $F(ltr(TM))$.

Proof. For $X, Y \in \Gamma(S(TM))$, using (2.3), (2.5) and (3.2) we get

$$-A_{FX}Y + \nabla^s_X FY + D^l(X, FY) = F\nabla_X Y + Fh^l(X,Y) + Fh^s(X,Y) \tag{5.3}$$

Taking inner product of (5.3) with $\xi \in \Gamma(RadTM)$ and then using (4.1) we obtain

$$g(X,Y)\bar{g}(H^s, F\xi) = 0.$$

from which we have our assertion.

Theorem 5.5. Let *M* be a totally umbilical ST-anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then $H^l = 0$ if and only if $\nabla^s_X FX$ has no component in $F(S(TM))$ for all $X \in \Gamma(S(TM))$.

Proof. Using (2.3), (2.5), (2.8), (2.9), (3.2) and (4.1) we get

$$-A_{FX}X + \nabla^s_X FX = \omega\nabla_X X + Ch^l(X,X) + Bh^s(X,X) + Ch^s(X,X) \tag{5.4}$$

for any $X \in \Gamma(S(TM))$. From screen transversal parts of (5.4), we arrive at

$$\nabla^s_X FX = \omega\nabla_X X + Ch^l(X,X) + Ch^s(X,X).$$

Taking inner product of the above equation with $F\xi$ for $\xi \in \Gamma(RadTM)$ and using (2.8), (4.1) we get

$$\bar{g}(\nabla^s_X FX, F\xi) = g(X,X)\bar{g}(H^l, \xi),$$

which proves our assertion.

The following theorem classifies totally umbilical ST-anti-invariant lightlike submanifold immersed in a semi-Riemannian product manifold.

Theorem 5.6. Let *M* be a totally umbilical ST-anti-invariant lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then either H^s has no components in $F(S(TM))$ or $\dim(S(TM)) = 1$.

Proof. Taking inner product of the tangential components of (5.4) with $Z \in \Gamma(S(TM))$ and using (3.1) and (2.9), we get

$$g(-A_{FX}X, Z) = \bar{g}(h^s(X,X), FZ) \tag{5.5}$$

for any $X \in \Gamma(S(TM))$. On the other hand, by virtue of (2.6) we have

$$g(A_{FX}X, Z) = \bar{g}(h^s(X,Z), FX) \tag{5.6}$$

Combining (5.5) and (5.6), we get

$$\bar{g}(h^s(X,X), FZ) = -\bar{g}(h^s(X,Z), FX)$$

Using (4.1) in the above equation, we obtain

$$g(X,X)\bar{g}(H^s, FZ) = -g(X,Z)\bar{g}(H^s, FX). \tag{5.7}$$

Interchanging *X* and *Z* in (5.7) and rearranging the terms, we get

$$\bar{g}(H^s, FX) = -\frac{g(X,Z)}{g(Z,Z)}\bar{g}(H^s, FZ). \tag{5.8}$$

From (5.7) and (5.8), we conclude that

$$\bar{g}(H^s, FX) = -\frac{g(X,Z)^2}{g(X,X)g(Z,Z)}\bar{g}(H^s, FX). \tag{5.9}$$

Thus our assertion follows from (5.9).

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