Advances in Pure Mathematics, 2012, 2, 220-225
http://dx.doi.org/10.4236/apm.2012.24032 Published Online July 2012 (http://www.SciRP.org/journal/apm)

Minimal Surfaces and Gauss Curvature of Conoid in Finsler Spaces with (α, β)-Metrics*

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Received December 26, 2011; received March 18, 2012; accepted March 26, 2012

ABSTRACT

In this paper, minimal submanifolds in Finsler spaces with (*α*, *β*)-metrics are studied. Especially, helicoids are also minimal in (*α*, *β*)-Minkowski spaces. Then the minimal surfaces of conoid in Finsler spaces with (*α*, *β*)-metrics are given. Last, the Gauss curvature of the conoid in the 3-dimension Randers-Minkowski space is studied.

Keywords: Isometrical Immersion; Minimal Submanifold; (*α*, *β*)-Metric; Conoid Surface; Gauss Curvature

1. Introduction

In recent decades, geometry of submanifolds in Finsler geometry has been rapidly developed. By using the Busemann-Hausdorff volume form, Z. Shen [1] introduced the notions of mean curvature and normal curvature for Finsler submanifolds. Being based on it, Bernstein type theorem of minimal rotated surfaces in Randers-Minkowski space was considered in [2]. Later, Q. He and Y. B. Shen used another important volume form, *i.e.*, Holmes-Thompson volume form, to introduce notions of another mean curvature and the second fundamental form [3]. Thus, Q. He and Y. B. Shen constructed the corresponding Bernstein type theorem in a general Minkowski space [4].

The theory of minimal surfaces in Euclidean space has developed into a rich branch of differential geometry. A lot of minimal surfaces have been found in Euclidean space. Minkowski space is an analogue of Euclidean space in Finsler geometry. A natural problem is to study minimal surfaces with Busemann-Hausdorff or Holmes-Thompson volume forms. M. Souza and K. Tenenblat first studied the minimal surfaces of rotation in Randers-Minkowski spaces, and used an ODE to characterize the BH-minimal rotated surfaces in [5]. Later, the nontrivial HT-minimal rotated hypersurfaces in quadratic (*α*, *β*)-Minkowski space are studied [6]. N. Cui and Y. B. Shen used another method to give minimal rotational hypersurface in quadratic Minkowski (*α*, *β*)-space [7]. However, these examples only consider the special (*α*, *β*)-metrics either Randers or quadratic. Therefore, what is the case with the general (*α*, *β*)-metric?

The main purpose of this paper is to study the conoid in (*α*, *β*)-space. It includes minimal submanifolds in Finsler spaces with general (*α*, *β*)-metric ($F=\alpha\phi\left(\frac{\beta}{\alpha}\right)$) and the Causs curvature in Randers-Minkowski 3-space. We present the equations that characterize the minimal hypersurfaces in general (*α*, *β*)-Minkowski space. We prove that the conoid in Minkowski 3-space with metric $F=\alpha\phi\left(\frac{\beta}{\alpha}\right)$ is minimal if and only if it is a helicoid or a plane under some conditions. Finally, similar to [7], we give the Gauss curvature of conoid in Randers-Minkowski 3-space and point out that the Gauss curvature is not always nonpositive on minimal surfaces.

2. Preliminaries

Let M be an *n*-dimensional smooth manifold. A Finsler metric on M is a function $F:TM\to[0,\infty)$ satisfying the following properties: 1) F is smooth on $TM\setminus\{0\}$; 2) $F(x,\lambda y)=\lambda F(x,y)$ for all $\lambda>0$; 3) The induced quadratic form g is positively definite, where

$$g:=g_{ij}(x,y)\mathrm{d}x^i\otimes \mathrm{d}x^j,\qquad g_{ij}:=\frac{1}{2}\left[F^2\right]_{y^iy^j}. \tag{1}$$

Here and from now on, $[F]_{y^i}$, $[F]_{y^iy^j}$ mean $\frac{\partial F}{\partial y^i}$, $\frac{\partial^2 F}{\partial y^i\partial y^j}$, and we shall use the following convention of index ranges unless otherwise stated:

$$1\le i,j,\cdots\le n;\quad 1\le \alpha,\beta,\cdots\le m(>n).$$

*Project supported by NNSFC (no. 10971239, no. 10771160) and the Natural Science Foundation of Shanghai (no. 09ZR1433000).

The projection $\pi: TM \to M$ gives rise to the pull-back bundle $\pi^* TM$ and its dual $\pi^* T^* M$, which sits over $TM \setminus \{0\}$. We shall work on $TM \setminus \{0\}$ and rigidly use only objects that are invariant under positive re-scaling in *y*, so that one may view them as objects on the projective sphere bundle *SM* using homogeneous coordinates.

In $\pi^* T^* M$ there is a global section $\omega = [F]_{y^i} \mathrm{d}x^i$, called the Hilbert form, whose dual is $l = l^i \frac{\partial}{\partial x^i}$, $l^i = y^i / F$, called the distinguished field.The volume element $\mathrm{d}V_{SM}$ of *SM* with respect to the Riemannian metric $\hat{g}$, the pull-back of the Sasaki metric on $TM \setminus \{0\}$, can be expressed as

$$\mathrm{d}V_{SM} = \Omega \mathrm{d}\tau \wedge \mathrm{d}x, \tag{2}$$

where

$$\Omega := \det\left(\frac{g_{ij}}{F}\right), \quad \mathrm{d}x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \tag{3}$$

$$\mathrm{d}\tau := \sum_{i=1}^{n} (-1)^{i-1} y^i \mathrm{d}y^1 \wedge \cdots \wedge \widehat{\mathrm{d}y^i} \wedge \cdots \wedge \mathrm{d}y^n. \tag{4}$$

The *volume form* of a Finsler *n*-manifold (*M*, *F*) is defined by

$$\mathrm{d}V_M := \sigma(x) \mathrm{d}x, \quad \sigma(x) := \frac{1}{c_{n-1}} \int_{S_x M} \Omega \mathrm{d}\tau, \tag{5}$$

where c_{n-1} denotes the volume of the unit Euclidean $(n-1)$-sphere S^{n-1}, $S_x M = \{[y] \| y \in T_x M\}$.

Let (*M*, *F*) and $(\tilde{M}, \tilde{F})$ be Finsler manifolds, and $f: M \to \tilde{M}$ be an immersion. If $F(x,y) = \tilde{F}(f(x), \mathrm{d}f(y))$ for all $(x,y) \in TM \setminus \{0\}$, then *f* is called an *isometric immersion*. It is clear that

$$g_{ij}(x,y) = \tilde{g}_{\alpha\beta}(\tilde{x}, \tilde{y}) f_i^\alpha f_j^\beta, \tag{6}$$

for the isometric immersion $f: (M,F) \to (\tilde{M}, \tilde{F})$, where $\tilde{x}^\alpha = f^\alpha(x)$, $\tilde{y}^\alpha = f_i^\alpha y^i$, $f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}$. Let $(\pi^* TM)^\perp$ be the orthogonal complement of $\pi^* TM$ in $\pi^*(f^{-1} T\tilde{M})$ with respect to $\tilde{g}$, and set

$$h^\alpha = f_{ij}^\alpha y^i y^j - f_k^\alpha G^k + \tilde{G}^\alpha, \qquad h_\alpha = \tilde{g}_{\alpha\beta} h^\beta, \quad h = \frac{h^\alpha}{F^2} \frac{\partial}{\partial \tilde{x}^\alpha}, \tag{7}$$

where $f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$, G^k and $\tilde{G}^\alpha$ are the geodesic coefficients of *F* and $\tilde{F}$ respectively. We can see that $h \in (\pi^* TM)^\perp$ (see (1.14) in [3]), which is called the *normal curvature*. Recall that for an isometric immersion $f: (M,F) \to (\tilde{M}, \tilde{F})$, we have (see formulae (2.14) and (3.14) of Chapter V in [8])

$$G^k = \phi_\beta^k \left(f_{ij}^\beta y^i y^j + \tilde{G}^\beta\right), \tag{8}$$

where $\phi_\beta^k = f_l^\alpha g^{lk} \tilde{g}_{\alpha\beta}$. From (2.7), it follows that

$$h^\beta = p_\alpha^{\perp\beta} \left(f_{ij}^\alpha y^i y^j + \tilde{G}^\alpha\right), \tag{9}$$

where $p_\alpha^{\perp\beta} := \delta_\alpha^\beta - f_i^\beta \phi_\alpha^i$. Set

$$\mu = \frac{1}{c_{n-1}\sigma} \left(\int_{S_x M} \frac{h_\alpha}{F^2} \Omega \mathrm{d}\tau\right) \mathrm{d}\tilde{x}^\alpha, \tag{10}$$

which is called the *mean curvature* form of *f*. An isometric immersion $f: (M,F) \to (\tilde{M}, \tilde{F})$ is called a *minimal immersion* if any compact domain of *M* is the critical point of its volume functional with respect to any variation vector field. Then *f* is minimal if and only if $\mu = 0$.

3. Minimal Hypersurfaces of (*α*, *β*)-Spaces

Here and from now on, we consider general (*α*, *β*)-metric. Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$,

$$\alpha = \sqrt{a_{ij}(x) y^i y^j}, \quad \beta = b_i(x) y^i,$$

$$\|\beta\|_\alpha = \sqrt{a^{ij} b_i b_j} = b(0 < b < b_0).$$

If $\phi(s) = 1 + s$, then *F* is a Randers metric. If α is an Euclidean metric and β is parallel with respect to α, *F* is a locally Minkowski metric and (*M*, *F*) is called an (*α*, *β*)-Minkowski metric. By [9], *F* is a Finsler metric if and only if $\phi(s)$ satisfies

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0. \tag{11}$$

Let

$$A = \det(a_{ij}), \quad g = \det(g_{ij}), \quad \Omega = \frac{g}{F^n}. \tag{12}$$

It have been proved ([9]) that

$$g = \phi(s)^n H(s) A, \tag{13}$$

where

$$H(s) = \phi(\phi - s\phi')^{n-2} \left[\phi - s\phi' + (b^2 - s^2)\phi''\right].$$

In the following part, we will discuss minimal hypersurfaces in Minkowski space with (*α*, *β*)-metric. Let $f: (M,F) \to (\tilde{M}, \tilde{F})$ be an isometric immersion, $\tilde{F} = \tilde{\alpha}\phi(s) = \tilde{\alpha}\phi\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)$, where

$$\tilde{\alpha}=\sqrt{\tilde{a}_{\alpha\beta}\tilde{y}^{\alpha}\tilde{y}^{\beta}},\quad \tilde{\beta}=\tilde{b}_{\alpha}\tilde{y}^{\alpha}.$$

Since f is an isometric immersion, we get

$$F=f^{*}\tilde{F}=\alpha\phi\left(\frac{\beta}{\alpha}\right),$$

where

$$\alpha=\sqrt{a_{ij}y^{i}y^{j}},\quad a_{ij}=\tilde{a}_{\alpha\beta}f_{i}^{\alpha}f_{j}^{\beta},$$
$$\beta=b_{i}y^{i},\quad b_{i}=\tilde{b}_{\alpha}f_{i}^{\alpha}.$$

Note that (M, F) is a hypersurface of $\left(\tilde{M},\tilde{F}\right)$, let $n=n^{\alpha}\tilde{e}_{\alpha}$ be the unit normal vector field of $f\left(M\right)$ with respect to $\tilde{\alpha}$ and $n=\tilde{n}^{\alpha}\tilde{e}_{\alpha}$ be the unit normal vector field of M with respect to $\tilde{g}$, respectively. That is

$$\sum_{\alpha}n^{\alpha}f_{i}^{\alpha}=0,\quad \tilde{g}_{\alpha\beta}\tilde{n}^{\alpha}f_{i}^{\beta}=0,$$

$$\tilde{\alpha}\left(n,n\right)=\tilde{a}_{\alpha\beta}n^{\alpha}n^{\beta}=1,\quad \tilde{g}\left(n,n\right)=\tilde{g}_{\alpha\beta}\tilde{n}^{\alpha}\tilde{n}^{\beta}=1.$$

There exist a function $\lambda\left(x,y\right)$ on SM, such that

$$\tilde{g}_{\alpha\beta}\tilde{n}^{\beta}=\lambda\tilde{a}_{\alpha\beta}n^{\beta},$$

where $\lambda=\tilde{g}\left(n,n\right)=\left(\tilde{a}\left(n,n\right)\right)^{-1}$. Then

$$\tilde{n}^{\alpha}=\lambda\tilde{g}^{\alpha\beta}\tilde{a}_{\beta\gamma}n^{\gamma}. \tag{14}$$

From above, we know that f is minimal if and only if

$$n^{\alpha}\int_{S_xM}\frac{h_{\alpha}}{F^{2}}\Omega\mathrm{d}\tau=0. \tag{15}$$

From (3.3) and (3.4), and in a similar way as in [5], we can get

$$\begin{aligned}h_{\alpha}&=\tilde{g}_{\alpha\gamma}h^{\gamma}=\tilde{g}_{\alpha\gamma}\left[\left(f_{ij}^{\beta}y^{i}y^{j}+\tilde{G}^{\beta}\right)\tilde{g}_{\beta\delta}\tilde{n}^{\delta}\right]\tilde{n}^{\gamma}\\&=\lambda^{2}\left[\left(f_{ij}^{\beta}y^{i}y^{j}+\tilde{G}^{\beta}\right)\tilde{a}_{\beta\delta}n^{\delta}\right]\tilde{a}_{\alpha\gamma}n^{\gamma}.\end{aligned}$$

$$g=\frac{A}{\lambda^{2}\tilde{A}}\tilde{g}=\frac{A}{\lambda^{2}\tilde{A}}\phi^{n+1}\tilde{H}\tilde{A}=\frac{\phi^{n+1}\tilde{H}A}{\lambda^{2}}.$$

Then (3.5) is equivalent to

$$n^{\alpha}a_{\alpha\beta}\int_{S_xM}\frac{\left(\phi-s\phi'\right)^{n-1}\left[\phi-s\phi'+\left(\tilde{b}^{2}-s^{2}\right)\phi''\right]}{\tilde{\alpha}^{n+2}}\cdot\left(f_{ij}^{\beta}y^{i}y^{j}+\tilde{G}^{\beta}\right)\mathrm{d}\tau=0. \tag{16}$$

If $\tilde{F}=\tilde{\alpha}\phi\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)$ is an (α, β)-Minkowski metric, then $\tilde{G}^{\beta}=0$. In Minkowski-(α, β) space, f is minimal if and only if

$$f_{ij}^{\beta}n^{\beta}\int_{S_xM}\frac{y^{i}y^{j}\left(\phi-s\phi'\right)^{n-1}\left[\phi-s\phi'+\left(\tilde{b}^{2}-s^{2}\right)\phi''\right]}{\tilde{\alpha}^{n+2}}\mathrm{d}\tau=0. \tag{17}$$

Theorem 1 *Let (M, F) be a hypersurface of $\left(\tilde{M},\tilde{F}\right)$, and $\tilde{F}=\tilde{\alpha}\phi\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)$ be an (α, β)-Minkowski metric. Then $f:\left(M,F\right)\to\left(\tilde{M},\tilde{F}\right)$ is a minimal immersion if and only if*

$$f_{ij}^{\beta}n^{\beta}\int_{S_x}y^{i}y^{j}\left(\phi(\tilde{\beta})-\tilde{\beta}\phi'\left(\tilde{\beta}\right)\right)^{n-1}\cdot\left[\phi(\tilde{\beta})-\tilde{\beta}\phi'\left(\tilde{\beta}\right)+\left(\tilde{b}^{2}-\tilde{\beta}^{2}\right)\phi''\left(\tilde{\beta}\right)\right]\mathrm{d}\tau=0, \tag{18}$$

where S_x is a sphere such that $\alpha=1$.

Now, we consider the conoid in 3-dimensional (α, β)-Minkowski space paralleling to x^3-axis. Set $\tilde{F}=\tilde{\alpha}\phi\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)$, where

$$\tilde{\alpha}=\sqrt{\left(\tilde{y}^{1}\right)^{2}+\left(\tilde{y}^{2}\right)^{2}+\left(\tilde{y}^{3}\right)^{2}},\quad \tilde{\beta}=\tilde{b}\tilde{y}^{3},$$

and $\tilde{b}$ is a constant. Let $f=\left(u\cos v,u\sin v,h\left(v\right)\right)$, where $h\left(v\right)$ is a unknown function. Then

$$\left(f_{i}^{\alpha}\right)_{2\times3}=\begin{pmatrix}\cos v&\sin v&0\\-u\sin v&u\cos v&h'\end{pmatrix},$$

$$\begin{aligned}\begin{pmatrix}\tilde{y}^{1}&\tilde{y}^{2}&\tilde{y}^{3}\end{pmatrix}&=\begin{pmatrix}y^{1}&y^{2}\end{pmatrix}\begin{pmatrix}\cos v&\sin v&0\\-u\sin v&u\cos v&h'\end{pmatrix}\\&=\begin{pmatrix}y^{1}\cos v-uy^{2}\sin v&y^{1}\sin v+uy^{2}\cos v&y^{2}h'\end{pmatrix}.\end{aligned}$$

Assume that $y^{1}=\cos\theta,\ y^{2}=\sqrt{\frac{1}{u^{2}+\left(h'\right)^{2}}}\sin\theta$, $\theta\in\left[0,2\pi\right]$, then

$$\begin{aligned}\tilde{\alpha}&=\sqrt{\left(\tilde{y}^{1}\right)^{2}+\left(\tilde{y}^{2}\right)^{2}+\left(\tilde{y}^{3}\right)^{2}}\\&=\sqrt{\left(y^{1}\right)^{2}+\left(u^{2}+\left(h'\right)^{2}\right)\left(y^{2}\right)^{2}}=1.\end{aligned}$$

Note that the normal vector of the surface is

$$n=\left(\frac{-h'\sin v}{\sqrt{\left(h'\right)^{2}+u^{2}}},\frac{h'\cos v}{\sqrt{\left(h'\right)^{2}+u^{2}}},-\frac{u}{\sqrt{\left(h'\right)^{2}+u^{2}}}\right),$$

and

$$\begin{aligned}&\left(f_{11}^{\alpha}\right)_{1\times3}=\begin{pmatrix}0&0&0\end{pmatrix},\\&\left(f_{12}^{\alpha}\right)_{1\times3}=\left(f_{21}^{\alpha}\right)_{1\times3}=\begin{pmatrix}-\sin v&\cos v&0\end{pmatrix},\\&\left(f_{22}^{\alpha}\right)_{1\times3}=\begin{pmatrix}-u\cos v&-u\sin v&h''\end{pmatrix}.\end{aligned}$$

Set

$$W^{ij}=\int_{S_x}y^{i}y^{j}\left(\phi\left(\tilde{\beta}\right)-\tilde{\beta}\phi'\left(\tilde{\beta}\right)\right)\cdot\left[\phi\left(\tilde{\beta}\right)-\beta\phi'\left(\tilde{\beta}\right)+\left(\tilde{b}^{2}-\tilde{\beta}^{2}\right)\phi''\left(\tilde{\beta}\right)\right]\mathrm{d}\tau, \tag{19}$$

Then (3.8) is equivalent to

$$\sum_{\alpha=1}^{3}\left(2f_{12}^{\alpha}n^{\alpha}W^{12}+f_{22}^{\alpha}n^{\alpha}W^{22}\right)=0. \tag{20}$$

Since S_x is symmetric with respect to y^1 and $\tilde{\beta}$ is a function only depending on y^2,

$$W^{12}=\int_{S_x}y^1y^2\left(\phi\left(\tilde{\beta}\right)-\tilde{\beta}\phi'\left(\tilde{\beta}\right)\right)\cdot\left[\phi\left(\tilde{\beta}\right)-\tilde{\beta}\phi'\left(\tilde{\beta}\right)+\left(\tilde{b}^2-\tilde{\beta}^2\right)\phi''\left(\tilde{\beta}\right)\right]\mathrm{d}\tau=0,$$

Therefore, (3.10) becomes to

$$uh''W^{22}=0, \forall u.$$

However, $W^{22}=0$ is impossible. Recall that

$$W^{22}=\int_{S_x}\left(y^2\right)^2\frac{\lambda^2 g}{\phi^{n+2}(s)}\mathrm{d}\tau,\quad \phi(s)>0,$$

and y^2 is not identically vanishing, we can obtain $W^{22}>0$. Then $h''=0$,

$$h=cv+d,$$

where *c*, *d* are arbitrary constants.

Theorem 2 *Let* $\left(V^3,\tilde{F}\right)$ *be an* (*α*, *β*)*-Minkowski space*, $\tilde{F}=\tilde{\alpha}\phi\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)$, $\tilde{\beta}=\tilde{b}\tilde{y}^3$, *and* $f=\left(u\cos v,u\sin v,h(v)\right)$ *be a conoid. Then f is minimal if and only if f is a helicoid or a plane.*

Remark 3.1 *From theorem* 2, *we can affirm that a helicoid is minimal not only in Euclidean space but also in* (*α*, *β*) *Minkowski space*, *where* $\tilde{\beta}=\tilde{b}\tilde{y}^3$. *This is an interesting result for minimal surfaces.*

But whether the result hold if the condition $\tilde{\beta}=\tilde{b}\tilde{y}^3$ is not satisfied? Now we consider the following condition:

$$\begin{aligned}\tilde{\beta}&=\tilde{b}_1\tilde{y}^1+\tilde{b}_2\tilde{y}^2+\tilde{b}_3\tilde{y}^3\\&=\left(\tilde{b}_1\cos v+\tilde{b}_2\sin v\right)y^1+\left(\tilde{b}_2\cos v-\tilde{b}_1\sin v\right)y^2,\end{aligned}$$

where $\tilde{b}_1,\tilde{b}_2,\tilde{b}_3$ are not all zeros. To simplify the computation, we only discuss quadratic (*α*, *β*)-metric: $F=\alpha+k\frac{\beta^2}{\alpha}$. Set $B_1=\tilde{b}_1\cos v+\tilde{b}_2\sin v$, $B_2=\tilde{b}_2\cos v-\tilde{b}_1\sin v$. Then (3.8) becomes an equation respect to *u*:

$$\begin{aligned}&C_5(v)u^5+C_4(v)u^4+C_3(v)u^3\\&+C_2(v)u^2+C_1(v)u+C_0(v)=0,\end{aligned} \tag{21}$$

where

$$C_5=\frac{15}{8}B_2^4h'',$$

$$C_4=\frac{15}{2}B_1\tilde{b}_3\left(\tilde{b}_3^2\left(h'\right)^2+B_2^2\right)h'h'',$$

$$\begin{aligned}C_3=&-\left(3k^2B_1^3B_2+3k^2B_1B_2^3-4kB_1B_2-2k^2\tilde{b}^2B_1B_2\right)h'\\&+\left(\frac{15}{2}B_2\tilde{b}_3^3h'+\frac{45}{4}\tilde{b}_3^2B_2^2+\frac{3}{8}k^2B_1^4h''+\frac{9}{4}k^2\pi B_1^2B_2^2\right.\\&\left.-k\pi B_1^2-\frac{\pi}{2}k^2\tilde{b}^2B_1^2+2k\tilde{b}^2+1\right)h'',\end{aligned}$$

$$C_2=\frac{15}{2}B_2\tilde{b}_3^3\left(h'\right)^5h''-9k^2B_1\tilde{b}_3B_2^2h'+\frac{9}{2}k^2B_2\tilde{b}_3h'h'',$$

$$\begin{aligned}C_1=&\frac{15}{8}\tilde{b}_3^4\left(h'\right)^4h''\\&+B_2\left(3k^2B_1^3-4kB_1-2k^2\tilde{b}^2B_1^2-2k\tilde{b}_3^2\right)\left(h'\right)^3\\&+\left(\frac{3}{8}k^2B_1^4-k\pi B_1^2-k^2\pi\tilde{b}^2B_1^2\right.\\&\left.+\frac{9}{4}k^2\tilde{b}_3B_1^2+2k\tilde{b}^2+1\right)\left(h'\right)^2h'',\end{aligned}$$

$$C_0=-3k^2\tilde{b}_3B_1\left(1+B_1^2\right)\left(h'\right)^4.$$

Since (3.11) is valid for any *u*, we can obtain

$$C_i=0\left(i=0,\cdots,5\right), \forall v.$$

If $\tilde{b}_1\neq0$ or $\tilde{b}_2\neq0$, then $B_1\neq0$ or $B_2\neq0$, such that $h'(v)=0$. Therefore, when $\tilde{b}_1,\tilde{b}_2$ are not all zeros, $h(v)=const$. That is to say a minimal conoid hypersurface is a plane with respect to the given metric above.

Theorem 3 *Let* $\left(V^3,\tilde{F}\right)$ *be an* (*α*, *β*)*-Minkowski space*, *where* $\tilde{F}=\tilde{\alpha}+k\frac{\tilde{\beta}^2}{\tilde{\alpha}}$ $\left\|\tilde{\beta}\right\|=\tilde{b}$ *satisfying* $\tilde{\beta}=\tilde{b}_1\tilde{y}^1+\tilde{b}_2\tilde{y}^2+\tilde{b}_3\tilde{y}^3$ ($\tilde{b}_1,\tilde{b}_2$ *are not all zeros*). *Then a minimal conoid hypersurface in* $\left(V^3,\tilde{F}\right)$ *is a plane.*

4. Gauss Curvature of Conoid in Randers 3-Space

As we all known, the Gauss curvature of a minimal surface is nonpositive everywhere in Euclidean space. Then, a natural problem arises: whether this fact holds for minimal surfaces in Minkowski-Randers 3-space? In this section, we study the Gauss curvature of conoid in Minkowski-Randers 3-space around x^3-axis in the direction $\tilde{\beta}^\#$, that is $\tilde{\beta}^\#=\tilde{b}\tilde{y}^3$. Consider the conoid $f(u,v)=\left(u\cos v,u\sin v,h(v)\right)$, where $u>0$ and $v\in S^1$. Let $e_1=\mathrm{d}f\left(\frac{\partial}{\partial u}\right)$, $e_2=\mathrm{d}f\left(\frac{\partial}{\partial v}\right)$. Then $y=\xi e_1+\eta e_2$ gives a natural coordinates (u,v,ξ,η) on its tangent bundle. In this section we shall use the convention that $1\le i,j\le2$ and $1\le\alpha,\beta\le3$. Besides, the notations $\zeta^1:=u,\zeta^2:=v$ and $y^1:=\xi, y^2:=\eta$ are also used.

Note that the induced 1-form $\beta=f^*\tilde{\beta}$ on the surface is closed. Then the Ricci curvature tensor of $F=f^*\tilde{F}$ is given by ([10], Page 118)

$$Ric = \overline{Ric} + \frac{1}{4F^2}\left(3r_{00}^2 - 2Fr_{00|0}\right), \quad (22)$$

where $\overline{Ric}$ denotes the Ricci curvature tensor of the induced Riemannian metric $\alpha = f^*\tilde{\alpha}$, $r_{00} = b_{i|j}y^iy^j$ and $b_{i|j}$ denote the coefficients of the covarient derivatives of β with respect to α. Then the Gauss curvature of the surface is given by

$$\begin{aligned} K(x,y) &= \frac{Ric(y)}{F^2} \\ &= \overline{K} + \frac{1}{4F^4}\left(3r_{00}^2 - 2Fr_{00|0}\right), \end{aligned} \quad (23)$$

where $x = f(u,v)$, $\overline{K}$ denotes the Gauss curvature with respect to α.

Denote $z_i^\alpha = \frac{\partial f^\alpha}{\partial \zeta^i}$ and $z_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial \zeta^i \partial \zeta^j}$. Then

$$\left(z_i^\alpha\right)_{2\times3} = \begin{pmatrix} \cos v & \sin v & 0 \\ -u\sin v & u\cos v & h'(v) \end{pmatrix},$$

$$\left(z_{11}^\alpha\right)_{1\times3} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix},$$

$$\left(z_{12}^\alpha\right)_{1\times3} = \left(z_{21}^\alpha\right)_{1\times3} = \begin{pmatrix} -\sin v & \cos v & 0 \end{pmatrix},$$

$$\left(z_{22}^\alpha\right)_{1\times3} = \begin{pmatrix} -u\cos v & -u\sin v & h'' \end{pmatrix}.$$

Noting that the Gauss curvature is computed in Euclidean space as follows:

$$\overline{K} = \frac{LN - M^2}{EG - F^2},$$

where

$$L = z_{11}^\alpha \cdot n^\alpha, \quad M = z_{12}^\alpha \cdot n^\alpha, \quad N = z_{22}^\alpha \cdot n^\alpha,$$

$$E = z_1^\alpha \cdot z_1^\alpha, \quad F = z_1^\alpha \cdot z_2^\alpha, \quad G = z_2^\alpha \cdot z_2^\alpha.$$

By direct computation, we can obtain

$$\overline{K} = -\frac{h'^2}{\left(u^2 + h'^2\right)^2}. \quad (24)$$

Meanwhile, the coefficients of $\alpha = f^*\tilde{\alpha}$ are given by

$$\left(a_{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + (h')^2 \end{pmatrix},$$

$$\left(a^{kl}\right) = \left(a_{kl}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u^2 + (h')^2} \end{pmatrix},$$

where $a_{ij} = z_i^\alpha z_j^\beta \delta_{\alpha\beta}$. It is easy to verify that $\overline{\Gamma}_{ij}^k = a^{kl} z_l^\alpha z_{ij}^\beta \delta_{\alpha\beta}$. By a direct computation, we have

$$\overline{\Gamma}_{ij}^1 = \begin{pmatrix} 0 & 0 \\ 0 & -u \end{pmatrix},$$

$$\overline{\Gamma}_{ij}^2 = \begin{pmatrix} 0 & \frac{u}{u^2 + (h')^2} \\ \frac{u}{u^2 + (h')^2} & \frac{h'h''}{u^2 + (h')^2} \end{pmatrix}.$$

Since $b_i = \tilde{b} z_i^3$,

$$b_{i|j} = b_{i,\zeta^j} - b_s \overline{\Gamma}_{ij}^s = \tilde{b} \begin{pmatrix} 0 & -\frac{h'u}{u^2 + h'^2} \\ -\frac{u}{u^2 + h'^2} & h'' - \frac{h'h''}{u^2 + h'^2} \end{pmatrix}.$$

From $b_{i|j|k} = b_{i|j,\zeta^k} - b_{i|s}\overline{\Gamma}_{jk}^s - b_{s|j}\overline{\Gamma}_{ik}^s$, we have

$$b_{1|1|2} = \frac{2\tilde{b}u^2h'}{\left(u^2 + h'^2\right)^2},$$

$$b_{2|1|1} = b_{1|2|1} = \frac{\tilde{b}u^2h'}{\left(u^2 + h'^2\right)^2},$$

$$b_{1|2|2} = b_{2|1|2} = \frac{\tilde{b}uh''}{\left(u^2 + h'^2\right)^2}\left(h'^2 - u^2\right),$$

$$b_{2|2|1} = -\frac{2\tilde{b}u^3h''}{\left(u^2 + h'^2\right)^2},$$

$$b_{2|2|2} = -\frac{2\tilde{b}h'u^2}{u^2 + h'^2}\left(\frac{1}{u^2 + h'^2} + h''^2\right).$$

Besides,

$$r_{00} = b_{i|j}y^iy^j = -\frac{2\tilde{b}uh'}{u^2 + h'^2}y^1y^2 - \frac{u^2h''}{u^2 + h'^2}\left(y^2\right)^2,$$

$$\begin{aligned} r_{00|0} = b_{i|j|k}y^iy^jy^k &= \frac{4\tilde{b}u^2h'}{\left(u^2 + h'^2\right)^2}\left(y^1\right)^2 y^2 \\ &+ \frac{2\tilde{b}uh''}{\left(u^2 + h'^2\right)^2}\left(h'^2 - 2u^2\right)y^1\left(y^2\right)^2 \\ &- \frac{2\tilde{b}h'u^2}{u^2 + h'^2}\left(\frac{1}{u^2 + h'^2} + h''^2\right)\left(y^2\right)^3 \end{aligned}$$

Then, from (4.2) and (4.3), we obtain the following theorem.

Theorem 4 *Let* $\left(V^3, \tilde{F}\right)$ *be an Randers-Minkowski space with* $\tilde{\beta} = \tilde{b}\tilde{y}^3$, *the Gauss curvature of the conoid* $f(u,v) = \left(u\cos v, u\sin v, h(v)\right)$ *at* $x = f(u,v)$ *in direction of* $y = \xi e_1 + \eta e_2$ *is given by*

$$\begin{aligned} K(x,y) = -\Gamma^0 + \frac{1}{4F^4}\Big[&12\tilde{b}^2\Gamma^1\xi^2\eta^2 - 8\tilde{b}F\Gamma^2\xi^2\eta \\ &+ 12\tilde{b}\Gamma^3\xi\eta^3 - 4\tilde{b}\Gamma^4\xi\eta^2 + 4\tilde{b}F\Gamma^5\eta^3 + 3\Gamma^6\eta^4\Big], \end{aligned}$$

where

$$\Gamma^0 = \frac{h'^2}{\left(u^2 + h'^2\right)^2}, \quad \Gamma^1 = \frac{u^2 h'^2}{\left(u^2 + h'^2\right)^2}, \quad \Gamma^2 = \frac{u^2 h'}{\left(u^2 + h'^2\right)^2},$$

$$\Gamma^3 = \frac{u^3 h' h''}{\left(u^2 + h'^2\right)^2}, \quad \Gamma^4 = \frac{u h''}{\left(u^2 + h'^2\right)^2}\left(h'^2 - 2u^2\right),$$

$$\Gamma^5 = \frac{u^2 h'}{u^2 + h'^2}\left(\frac{1}{u^2 + h'^2} + h''^2\right), \quad \Gamma^6 = \frac{u^4 h''}{\left(u^2 + h'^2\right)^2}.$$

Note that a helicoid is minimal if and only if it is a conoid with respect to (α, β)-metrics (where $\tilde{\beta} = \tilde{b} y^3$). Let $h(v) = cv + d$ (c is a constant), then the Gauss curvature of this surface is given by

$$K(x, y) = -\Pi^0 + \frac{1}{F^4}\left[3\tilde{b}^2 \Pi^1 \xi^2 \eta^2 - \tilde{b} F \Pi^2 \eta \left(2\xi^2 + \eta^2\right)\right], \tag{25}$$

where

$$\Pi^0 = \frac{c^2}{\left(u^2 + c^2\right)^2}, \quad \Pi^1 = \frac{c^2 u^2}{\left(u^2 + c^2\right)^2}, \quad \Pi^2 = \frac{c u^2}{\left(u^2 + c^2\right)^2}.$$

However, for a given point $x = f(u, v)$, in which directions of $T_x S$, $K(x, y) > 0$, $K(x, y) = 0$, $K(x, y) < 0$?

1) If $\eta = 0$, then $K(x, y) < 0$ for any $c \neq 0$;

2) If $\xi = 0$, Since

$$F = \alpha + \beta = \sqrt{a_{ij} y^i y^j} + b_i y^i = \left(\sqrt{u^2 + c^2}\,|\eta| + c\tilde{b}\eta\right),$$

Equation (4.4) becomes

$$K(x, y) = -\frac{1}{\left(u^2 + c^2\right)^2}\left[c^2 + \frac{2c\tilde{b}u^2}{\left(\sqrt{u^2 + c^2}\,|\eta| + c\tilde{b}\eta\right)^3}\eta^3\right].$$

If $c > 0$, let $\eta < 0$, then

$$K(x, y) = -\frac{1}{\left(u^2 + c^2\right)^2}\left[c^2 + \frac{2c\tilde{b}u^2}{\left(c\tilde{b} - \sqrt{u^2 + c^2}\right)^3}\right].$$

we can also make $c^2 + \frac{2c\tilde{b}u^2}{\left(c\tilde{b} - \sqrt{u^2 + c^2}\right)^3} < 0$, then $K(x, y) > 0$; Otherwise, let $\eta > 0$, then

$$K(x, y) = -\frac{1}{\left(u^2 + c^2\right)^2}\left[c^2 + \frac{2c\tilde{b}u^2}{\left(c\tilde{b} + \sqrt{u^2 + c^2}\right)^3}\right].$$

we can make $c^2 + \frac{2c\tilde{b}u^2}{\left(c\tilde{b} + \sqrt{u^2 + c^2}\right)^3} < 0$, then $K(x, y) > 0$. In sum, the Gauss curvature is not nonpositive anywhere.

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