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Uniformly Stable Positive Monotonic Solution of a Nonlocal Cauchy Problem

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ABSTRACT

In this paper, we study the existence of a uniformly stable positive monotonic solution for the nonlocal Cauchy problem $x'(t)=f(t,x(t)),t\in[0,T]$ with the nonlocal condition $\sum_{j=1}^{m} b_j\, x(\eta_j)=x_1$, where $\eta_j\in(0,a)\subset[0,T]$.

Keywords: Nonlocal Cauchy Problem; Local and Global Existence Nondecreasing Positive Solution; Continuous Dependence; Lyapunov Uniformly Stability

1. Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to (see [1-14] and [15-18]) and references therein.

Here we are concerned with the nonlocal Cauchy problem

$$x'(t)=f(t,x(t)),t\in[0,T], \tag{1}$$

$$\sum_{j=1}^{m} b_j\, x(\eta_j)=x_1,\eta_j\in(0,a)\subset[0,T], \text{ and } \left(\sum_{j=1}^{m} b_j\right)\neq 0. \tag{2}$$

Let X be the class of all continuous functions defined on $[0,T],T<\infty$ with the norm

$$\|x\|=\sup_{t\in[0,T]}|x(t)|, x\in X.$$

Let Y be the class of all continuous functions defined on $[t_0,T],T<\infty$ with the equivalent norm

$$\|x\|=\sup_{t\in[0,T]} e^{-N(t-t_0)}|x(t)|, x\in Y,$$

where $t_0=\max\{\eta_j, j=1,2,\cdots,m\}$, and N is positive arbitrary.

Here we firstly study, in X, the local existence of the solution of the problem (1)-(2) and the continuous dependence of the parameter x, will be proved.

Secondly, we study, in Y, the global existence and Lyapunov uniform stability of the solution of the problem (1)-(2).

2. Integral Equation Representation

Consider the nonlocal Cauchy problem (1)-(2).

Let $f:[0,T]\times R^+\to R^+$ is continuous and satisfies the Lipschitz condition

$$|f(t,x)-f(t,y)|\le k|x-y|,k>0, \quad \text{for all } x,y\in R^+ \tag{3}$$

Lemma 2.1. The solution of the nonlocal Cauchy problem (1)-(2) can be expressed by the integral equation

$$x(t)=B\left(x_1-\sum_{j=1}^{m} b_j\int_0^{\eta_j} f(s,x(s))\,\mathrm{d}s\right)+\int_0^t f(s,x(s)),\mathrm{d}s \tag{4}$$

where $B=\left(\sum_{j=1}^{m} b_j\right)^{-1}$.

Proof. Integrating the Equation (1), we obtain

$$x(t)=x(0)+\int_0^t f(s,x(s))\,\mathrm{d}s. \tag{5}$$

Let $t=\eta_j$ in (5), we obtain

$$x(\eta_j)=x(0)+\int_0^{\eta_j} f(s,x(s))\,\mathrm{d}s, \tag{6}$$

and

$$\sum_{j=1}^{m} b_j x(\eta_j)=\sum_{j=1}^{m} b_j x(0)+\sum_{j=1}^{m} b_j\int_0^{\eta_j} f(s,x(s))\,\mathrm{d}s. \tag{7}$$

Substitute from (2) into (7), we obtain

$$x(0)=B\left(x_1-\sum_{j=1}^{m} b_j\int_0^{\eta_j} f(s,x(s))\,\mathrm{d}s\right). \tag{8}$$

Substitute from (8) into (5), we obtain

$$x(t)=B\left(x_1-\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s\right)+\int_0^t f(s,x(s))\,\mathrm{d}s.$$

Corollary 2.1. The solution of the integral Equation (4) is nondecreasing.

Proof. Let x be a solution of the integral Equation (4), then for $t_1<t_2$, we have

$$\begin{aligned}x(t_1)&=B\left\{x_1-\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s\right\}+\int_0^{t_1}f(s,x(s))\,\mathrm{d}s\\&<B\left\{x_1-\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s\right\}+\int_0^{t_2}f(s,x(s))\,\mathrm{d}s\\&=x(t_2),\end{aligned}$$

which proves that the solution x of the integral Equation (4) is nondecreasing.

Corollary 2.2. Let f be satisfies (3). The solution of the integral Equation (4) is positive for $t\in[a,T]$.

Proof. Let x be a solution of the integral Equation (4), and $x_1>0$, for $t\in[a,T]$, we have

$$\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s\le\int_0^t f(s,x(s))\,\mathrm{d}s,\quad \eta_j<t$$

and

$$\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s\le\sum_{j=1}^{m}b_j\int_0^t f(s,x(s))\,\mathrm{d}s.$$

Multiplying by $B=\left(\sum_{j=1}^{m}b_j\right)^{-1}$, we obtain

$$\begin{aligned}B\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s&\le B\sum_{j=1}^{m}b_j\int_0^t f(s,x(s))\,\mathrm{d}s\\&=\int_0^t f(s,x(s))\,\mathrm{d}s\end{aligned}$$

and the solution x of the integral Equation (4) is positive for $t\in[a,T]$. This complete the proof. ■

3. Local Existence of Solution

Theorem 3.1. Let f be satisfies the Lipschitz condition. If $T<k\left(1+|B|\sum_{j=1}^{m}|b_j|\right)^{-1}$ then the nonlocal Cauchy problem (1)-(2) has a unique nondecreasing positive solution.

Proof. Define the operator $T:C[0,T]\to C[0,T]$ by

$$Tx(t)=B\left(x_1-\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s\right)+\int_0^t f(s,x(s))\,\mathrm{d}s. \tag{9}$$

Let $x,y\in C[0,T]$, then

$$\begin{aligned}Tx(t)-Ty(t)&=-B\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s+\int_0^t f(s,x(s))\,\mathrm{d}s+B\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,y(s))\,\mathrm{d}s-\int_0^t f(s,y(s))\,\mathrm{d}s\\&=-B\sum_{j=1}^{m}b_j\int_0^{\eta_j}\{f(s,x(s))-f(s,y(s))\}\,\mathrm{d}s+\int_0^t\{f(s,x(s))-f(s,y(s))\}\,\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}|Tx(t)-Ty(t)|&\le k|B|\sum_{j=1}^{m}|b_j|\int_0^{\eta_j}|x(s)-y(s)|\,\mathrm{d}s+k\int_0^t|x(s)-y(s)|\,\mathrm{d}s\\&\le k|B|\sum_{j=1}^{m}|b_j|\sup_{t\in I}|x(t)-y(t)|\int_0^{\eta_j}\mathrm{d}s+k\sup_{t\in I}|x(t)-y(t)|\int_0^t\mathrm{d}s\\&\le kT|B|\sum_{j=1}^{m}|b_j|\|x-y\|+kT\|x-y\|\le kT\left(1+|B|\sum_{j=1}^{m}|b_j|\right)\|x-y\|\le K\|x-y\|\end{aligned}$$

but if

$$K=kT\left(1+|B|\sum_{j=1}^{m}|b_j|\right)<1,$$

then we get

$$\|Tx-Ty\|\le K\|x-y\|,$$

which proves that the map $T:C[0,T]\to C[0,T]$ is contraction.

Applying the Banach contraction fixed point theorem we deduce that the integral Equation (4) has a unique solution $x\in C[0,T]$.

To complete the proof, we prove that the integral Equation (4) satisfies nonlocal problem (1)-(2).

Differentiating (4), we get

$$x'(t)=f(t,x(t)). \tag{10}$$

Let $t=\eta_j$ in (4), we obtain

$$\begin{aligned}x(\eta_j)&=B\left\{x_1-\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s\right\}\\&\quad+\int_0^{\eta_j}f(s,x(s))\,\mathrm{d}s,\end{aligned}$$

then

$$\sum_{j=1}^{m}b_j x(\eta_j)=x_1.$$

This implies that there exist a unique nondecreasing positive solution $x \in C[0,T]$ of the nonlocal Cauchy problem (1)-(2), This complete the proof. ■

4. Continuous Dependence of the Solution

Consider the nonlocal Cauchy problem

$$(\tilde{P})\begin{cases} x'(t) = f(t, x(t)), t \in [0,T], \\ \sum_{j=1}^{m} b_j\, x(\eta_j) = \tilde{x}_1, \quad \text{and } \eta_j \in (0,a) \subset [0,T]. \end{cases}$$

Definition 4.1. The solution of the nonlocal Cauchy problem (1)-(2) continuously dependence on x_1 if

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0, \text{ such that } |x_1 - \tilde{x}_1| < \delta,$$
$$\text{then } |x(t) - \tilde{x}(t)| < \varepsilon$$

where $\tilde{x}(t)$ is the solution of the nonlocal Cauchy problem $\tilde{P}$.

Now we have the following theorem

Theorem 4.1. The solution of the nonlocal Cauchy problem (1)-(2) continuously dependence on x_1.

Proof. Let $x(t), \tilde{x}(t)$ are the solutions of (1)-(2) and $\tilde{P}$ respectively.

Then we can get

$$x(t) - \tilde{x}(t) = B(x_1 - \tilde{x}_1) - B\sum_{j=1}^{m} b_j \int_0^{\eta_j} f(s, x(s))\,\mathrm{d}s + B\sum_{j=1}^{m} b_j \int_0^{\eta_j} f(s, \tilde{x}(s))\,\mathrm{d}s$$
$$+ \int_0^t \{ f(s, x(s)) - f(s, \tilde{x}(s)) \}\,\mathrm{d}s$$

$$|x(t) - \tilde{x}(t)| \le |B||x_1 - \tilde{x}_1| + |B|\sum_{j=1}^{m}|b_j| \left|\int_0^{\eta_j} |f(s, x(s)) - f(s, \tilde{x}(s))|\,\mathrm{d}s + \int_0^t |f(s, x(s)) - f(s, \tilde{x}(s))|\right|\mathrm{d}s$$
$$\le |B||x_1 - \tilde{x}_1| + k|B|\sum_{j=1}^{m}|b_j| \sup_{t \in I}\int_0^{\eta_j} |x(s) - \tilde{x}(s)|\,\mathrm{d}s + k\sup_{t \in I}\int_0^t |x(s) - \tilde{x}(s)|\,\mathrm{d}s$$
$$\le |B||x_1 - \tilde{x}_1| + k|B|\sum_{j=1}^{m}|b_j| \sup_{t \in I}|x(t) - \tilde{x}(t)| \int_0^{\eta_j}\mathrm{d}s + k\sup_{t \in I}|x(t) - \tilde{x}(t)|\int_0^t \mathrm{d}s$$

$$\|x - \tilde{x}\| \le |B||x_1 - \tilde{x}_1| + kT|B|\sum_{j=1}^{m}|b_j| \|x - \tilde{x}\| + kT\|x - \tilde{x}\| \le |B||x_1 - \tilde{x}_1| + kT\left(1 + |B|\sum_{j=1}^{m}|bj|\right)\|x - \tilde{x}\|$$

$$\left(1 - kT\left(1 + |B|\sum_{j=1}^{m}|b_j|\right)\right)\|x - \tilde{x}\| \le |B||x_1 - \tilde{x}_1| \|x - \tilde{x}\| \le \left(1 - kT\left(1 + |B|\sum_{j=1}^{m}|b_j|\right)\right)^{-1} |B||x_1 - \tilde{x}_1|.$$

Therefore, for $\delta > 0$ such that

$$|x_1 - \tilde{x}_1| < \delta(\varepsilon),$$

we can find

$$\varepsilon = \left(1 - kT\left(1 + |B|\sum_{j=1}^{m}|b_j|\right)\right)^{-1} |B|\delta$$

such that $\|x - \tilde{x}\| \le \varepsilon$, which complete the proof theorem.

5. Global Existence of Solution

Theorem 5.1. Let f be satisfies the Lipschitz condition, then the nonlocal Cauchy problem (1)-(2) has a unique nondecreasing positive solution.

Proof. Define the operator $T : C[t_0, T] \to C[t_0, T]$ by the Equation (9).

Let $x, y \in C[t_0, T]$, then

$$Tx(t) - Ty(t) = -B\sum_{j=1}^{m} b_j \int_0^{\eta_j} f(s, x(s))\,\mathrm{d}s + \int_0^t f(s, x(s))\,\mathrm{d}s + B\sum_{j=1}^{m} bj \int_0^{\eta_j} f(s, y(s))\,\mathrm{d}s - \int_0^t f(s, y(s))\,\mathrm{d}s$$
$$= -B\sum_{j=1}^{m} b_j \int_0^{\eta_j} \{ f(s, x(s)) - f(s, y(s)) \}\,\mathrm{d}s + \int_0^t \{ f(s, x(s)) - f(s, y(s)) \}\,\mathrm{d}s,$$

$$|Tx(t) - Ty(t)| \le k|B|\sum_{j=1}^{m}|b_j| \int_0^{t_0} |x(s) - y(s)|\,\mathrm{d}s + k\int_0^t |x(s) - y(s)|\,\mathrm{d}s$$

$$e^{-N(t-t_0)}|Tx(t) - Ty(t)| \le k|B|\sum_{j=1}^{m}|b_j| e^{-N(t-t_0)} \int_0^{t_0} |x(s) - y(s)|\,\mathrm{d} + ke^{-N(t-t_0)}\int_0^t |x(s) - y(s)|\,\mathrm{d}s$$

$$e^{-N(t-t_0)}\left|Tx(t)-Ty(t)\right| \le k|B|\sum_{j=1}^{m}|b_j|\int_0^{t_0} e^{-N(t-t_0)+N(s-t_0)}\, e^{-N(s-t_0)}\,|x(s)-y(s)|\,ds$$
$$+k\int_0^t e^{-N(t-t_0)+N(s-t_0)}\, e^{-N(s-t_0)}\,|x(s)-y(s)|\,ds$$
$$\le k|B|\sum_{j=1}^{m}|b_j|\|x-y\|\int_0^{t_0} e^{-N(t-s)}\,ds + k\|x-y\|\int_0^t e^{-N(t-s)}ds$$
$$\le k|B|\sum_{k=1}^{m}|\,|b_j|\|x-y\|\left\{\frac{e^{-N(t-t_0)}-e^{-Nt}}{N}\right\}+k\|x-y\|\left\{\frac{1-e^{-Nt}}{N}\right\}$$
$$\le \frac{k}{N}\left(|B|\sum_{j=1}^{m}|b_j|\left(e^{-N(t-t_0)}-e^{-Nt}\right)+\left(1-e^{-Nt}\right)\right)\|x-y\| \le \frac{k}{N}\left(|B|\sum_{j=1}^{m}|b_j|+1\right)\|x-y\|$$

where

$$K=\frac{k}{N}\left(|B|\sum_{j=1}^{m}|b_j|+1\right).$$

Choose N large enough such that $K<1$, then

$$\|Tx-Ty\|\le K\|x-y\|,$$

therefor the map $T: C[t_0,T]\to C[t_0,T]$ is contraction.

Applying the Banach contraction fixed point theorem we deduce that the integral Equation (4) has a unique solution $x\in C[t_0,T]$.

To complete the proof, we prove that the integral Equation (4) satisfies nonlocal problem (1)-(2).

Differentiating (4), we get

$$x'(t)=f(t,x(t)). \tag{11}$$

Let $t=\eta_j$ in (4), we obtain

$$x(\eta_j)=B\left\{x_1-\sum_{j=1}^{m}b_j\int_0^{\eta_j} f(s,x(s))\,ds\right\}$$
$$+\int_0^{\eta_j} f(s,x(s))\,ds,$$

then

$$\sum_{j=1}^{m}b_j x(\eta_j)=x_1.$$

This implies that there exist a unique nondecreasing positive solution $x\in C[t_0,T]$ of the nonlocal Cauchy problem (1)-(2), This complete the proof. ■

6. Lyapunov Uniform Stability of the Solution

Consider here the nonlocal Cauchy problem

$$(\tilde{P})\begin{cases} x'(t)=f(t,x(t)),\ t\in[t_0,T],\\ \sum_{j=1}^{m}b_j\,x(\eta_j)=\tilde{x}_1,\ \text{and } \eta_j\in(0,a)\subset[t_0,T].\end{cases}$$

Definition 6.1. The solution of the nonlocal Cauchy problem (1)-(2) is uniform stable, if
$\forall\varepsilon>0, \exists\delta(\varepsilon)>0,$ such that

$$|x_1-\tilde{x}_1|<\delta(\varepsilon),\ \text{then } |x(t)-\tilde{x}(t)|<\varepsilon.$$

where $\tilde{x}(t)$ is the solution of the nonlocal Cauchy problem $\tilde{P}$.

Now we have the following theorem

Theorem 6.1. The solution of the nonlocal Cauchy problem (1)-(2) is uniformly stable.

Proof. Let $x(t),\tilde{x}(t)$ are the solutions of (1)-(2) and $\tilde{P}$ respectively.

Then we can get

$$x(t)-\tilde{x}(t)=B(x_1-\tilde{x}_1)-B\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,x(s))\,ds+B\sum_{j=1}^{m}b_j\int_0^{\eta_j}f(s,\tilde{x}(s))\,ds+\int_0^t\left\{f(s,x(s))-f(s,\tilde{x}(s))\right\}ds$$

$$\left|x(t)-\tilde{x}(t)\right|\le|B||x_1-\tilde{x}_1|+|B|\sum_{j=1}^{m}|b_j|\left|\int_0^{\eta_j}\left|f(s,x(s))-f(s,\tilde{x}(s))\right|ds+\int_0^t\left|f(s,x(s))-f(s,\tilde{x}(s))\right|ds\right.$$
$$\le|B||x_1-\tilde{x}_1|+k|B|\sum_{j=1}^{m}|b_j|\left|\int_0^{t_0}\left|x(s)-\tilde{x}(s)\right|ds+k\int_0^t\left|x(s)-\tilde{x}(s)\right|ds\right.$$

$$e^{-N(t-t_0)}\left|x(t)-\tilde{x}(t)\right|\le e^{-N(t-t_0)}|B||x_1-\tilde{x}_1|+k|B|\sum_{j=1}^{m}|b_j|\left|\int_0^{t_0}e^{-N(t-t_0)+N(s-t_0)}\,e^{-N(s-t_0)}\left|x(t)-\tilde{x}(t)\right|ds\right.$$
$$+k\int_0^t e^{-N(t-t_0)+N(s-t_0)}\,e^{-N(s-t_0)}\left|x(t)-\tilde{x}(t)\right|ds$$

$$\|x-\tilde{x}\| \le |B||x_1-\tilde{x}_1| + k|B|\sum_{j=1}^{m}|b_j|\|x-\tilde{x}\|\int_0^{t_0} e^{-N(t-s)}\mathrm{d}s + k\|x-\tilde{x}\|\int_0^{t} e^{-N(t-s)}\,\mathrm{d}s$$

$$\le |B||x_1-\tilde{x}_1| + k|B|\sum_{j=1}^{m}|b_j|\|x-\tilde{x}\|\left\{\frac{e^{-N(t-t_0)}-e^{-Nt}}{N}\right\} + k\|x-\tilde{x}\|\left\{\frac{1-e^{-Nt}}{N}\right\}$$

$$\le |B||x_1-\tilde{x}_1| + \frac{k}{N}\left(|B|\sum_{j=1}^{m}|b_j|\left(e^{-N(t-t_0)}-e^{-Nt}\right)+\left(1-e^{-Nt}\right)\right)\|x-\tilde{x}\|$$

$$\le |B||x_1-\tilde{x}_1| + \frac{k}{N}\left(|B|\sum_{j=1}^{m}|b_j|+1\right)\|x-\tilde{x}\|$$

$$\|x-\tilde{x}\| \le |B|\left[1-\frac{k}{N}\left(|B|\sum_{j=1}^{m}|b_j|+1\right)\right]^{-1}|x_1-\tilde{x}_1|$$

Therefore, $|x_1-\tilde{x}_1|<\delta(\varepsilon)$, $\Rightarrow \|x-\tilde{x}\|<\varepsilon$, which complete the proof of theorem.

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