

Some Steiner Symmetry Results in Overdetermined Boundary Value Problem*

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Received June 17, 2011; revised August 2, 2011; accepted August 10, 2011

Abstract

In this paper, we use the moving planes method to prove that the domain Ω and the solution u are Steiner symmetric if u is a positive solution of the overdetermined boundary value problem in Ω .

Keywords: Steiner Symmetry, Moving Planes Method, Overdetermined Boundary Value Problem

1. Introduction

We will present some symmetry results in overdetermined boundary value problem

$$\Delta u + f(x, u, |\nabla u|) = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

$$\frac{\partial u}{\partial \nu} = c(x') \quad \text{on } \partial\Omega$$

where $x = (x_1, x')$, $x' = (x_2, x_3, \dots, x_n)$ and ν is the outward unit normal to $\partial\Omega$. Here Ω is a bounded domain of class C^2 in R^n , f is a function of class C^1 and satisfies certain conditions, c is a differentiable function. Obviously the constant normal derivative of u on $\partial\Omega$ is a special case of this problem.

For the motion of a viscous incompressible fluid moving in straight parallel streamlines through a pipe with planar section Ω or the torsion of a solid straight bar of given cross section Ω , both of them can be described by the following overdetermined problem

$$-\Delta u = 1 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

$$\frac{\partial u}{\partial \nu} = \text{const} \quad \text{on } \partial\Omega$$

where Ω is a bounded domain of class C^2 in R^n such that there exists a function u satisfying the above problem, and ν is the unit normal to $\partial\Omega$. Using the moving planes method, J. Serrin proved that if the above problem has a solution u in $C^2(\overline{\Omega})$, then Ω must be a ball. For more detail, see [1]. Later there are many au-

thors who investigated the overdetermined the problem. B. Gidas, W. M. Ni and L. Nirenberg proved symmetry of positive solutions of elliptic equations under the Dirichlet boundary condition via moving planes method,

$$-\Delta u = f(u) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

See [2]. The paper [3] extended the result of [2], but it did not contain the gradient. A. Colesanti considered the positive solutions of a more general p-Laplacian equation under the overdetermined conditions which has only one critical point via moving planes method too,

$$\text{div}\left((|\nabla u|)^{p-2} \Delta u\right) + f(u, |\nabla u|) = 0 \quad \text{in } \Omega \setminus \{P\}$$

$$u = 0 \quad \text{on } \partial\Omega$$

$$\frac{\partial u}{\partial \nu} = c > 0 \quad \text{on } \partial\Omega.$$

See [4]. Recently there have been found several other new approaches in studying symmetry problem, such as continuous Steiner symmetrization, domain derivatives and some geometry method, see [5-9]. The moving planes method is a classical technique and is very useful in dealing with symmetry problems, so we still use it to extend the results of [2,3] to the overdetermined boundary problem containing gradient.

We shall prove that Ω and u are Steiner symmetric with respect to the hyperplane $\{x_1 = \lambda\}$ for some real number λ . To be more specifically, a set A is Steiner symmetric with respect to the hyperplane $\{x_1 = \mu\}$ if it is symmetric with respect to the hyperplane $\{x_1 = \mu\}$ and convex in the x_1 direction. The

*The Project-sponsored by SRF for ROCS, SEM.

function u is called Steiner symmetric with respect to the hyperplane $\{x_1 = \mu\}$, we mean that the level sets of u are Steiner symmetric with respect to the hyperplane $\{x_1 = \mu\}$. Throughout this paper, we consider Ω as a bounded domain of class C^2 in R^n unless otherwise stated.

Section 2 of this paper is devoted to the preliminary results. In Section 3, we will present our main results and proofs.

2. Preliminary Results

In this section we will introduce the notations in the moving planes method and four results.

Let μ be a real number, Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $\lambda_1 \equiv \max\{x_1 | x \in \overline{\Omega}\}$. We choose a hyperplane perpendicular to the x_1 direction in the process of moving a plane.

Consider the hyperplane $T_\mu := \{x \in R^n | x_1 = \mu\}$. For sufficiently large positive number μ , T_μ is disjoint from Ω . As we decrease μ , at some moment, T_μ begins to intersect Ω and from that moment T_μ will cut off from Ω an open cap $\Sigma_\mu = \{x \in \Omega | x_1 > \mu\}$. Let Σ'_μ denote the reflection of Σ_μ across the plane T_μ . At the beginning of the process, Σ'_μ will be contained in Ω , until one of the following events occurs:

- A1) Σ'_μ becomes internally tangent to $\partial\Omega$ at some point P not on T_μ ,
- A2) T_μ reaches a position where it is orthogonal to $\partial\Omega$ at some point Q .

We denote the hyperplane T_μ when it reaches either one of these positions by T_λ and we call Σ_λ (an open cap by T_λ) the maximal cap. Clearly its reflection Σ'_μ is contained in Ω .

Throughout this paper, the following notations will be used: $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$.

The following two lemmas are due to J. Serrin. We will present the lemma, for detailed proof, see [8].

Lemma 2.1 Let Ω be a domain with C^2 boundary and T be a hyperplane containing the normal to $\partial\Omega$ at some point Q . Let Ω^* denote the portion of Ω lying on some particular side of T .

Suppose that w is of class C^2 in the closure of Ω^* and satisfies the elliptic inequality

$$Lw = \sum_{i,j=1}^n a_{ij}(x)w_{ij} + \sum_{i=1}^n b_i(x)w_i \leq 0, \quad x \in \Omega^*$$

where the coefficients are uniformly bounded. We assume that the matrix a_{ij} is uniformly definite

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq k|\xi|^2, \quad k = \text{const} > 0,$$

and that

$$\sum_{i,j=1}^n |a_{ij}\xi_i\xi_j| \leq K(|\xi \cdot \eta| + |\xi| \cdot |d|), \quad K = \text{const} > 0,$$

where $\xi = (\xi_1, \dots, \xi_n)$ is an arbitrary real vector,

$\eta = (\eta_1, \dots, \eta_n)$ is the unit normal to the plane T , and d is the distance from T . Suppose also $w \geq 0$ in Ω^* and $w = 0$ at Q . Let s be any direction at Q which enters Ω^* nontangentially. Then

$$\frac{\partial w}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial s^2} > 0 \quad \text{at } Q,$$

unless $w \equiv 0$.

Let Ω be a domain in R^n , $u \in C^2(\overline{\Omega})$ and $u \neq 0$. Consider a linear differential operator of second order of the form

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x)u_{ij} + \sum_{i=1}^n b_i(x)u_i + c(x)u$$

where L is uniformly elliptic and the coefficients of L are bounded. Here and in what follows, we use summation convention.

The following lemma is a refinement of Hopf's boundary point lemma (See [5]).

Lemma 2.2 Under the above assumptions, suppose that $Lu \leq 0$ and $u \geq 0$ in Ω ,

1) If $x_0 \in \partial\Omega$, $u(x_0) = 0$ and Ω satisfies an interior ball conditions at x_0 , then we have

$$\frac{\partial u}{\partial \nu}(x_0) < 0$$

where the vector $\nu = (\nu_1, \dots, \nu_n)$ denotes the outer (unit normal) to $\partial\Omega$ at x_0 .

2) $u \geq 0$ in Ω .

Lemma 2.2 is well known for us in case $c(x) < 0$. Here, we note the fact that it holds without the restriction $c < 0$.

Now we will present another key proposition which describes the property near the boundary.

Proposition 2.3 Let Ω be a bounded domain of class C^2 in R^n . Let f be of class C^1 . Assume that $u \in C^2(\Omega)$ is a solution of the problem

$$-\Delta u = f(u, |\nabla u|) \quad \text{in } \Omega,$$

$$u > 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

If λ is a real number which defines the maximal cap, then for each point $x_0 \in \partial\Omega \cap \{x_1 > \lambda\}$, it holds either $u_1(x_0) < 0$ or else $u_{11}(x_0) > 0$, that is, u is a strictly decreasing function of x_1 near x_0 .

Proof. Note that $\nu_1 > 0$ at x_0 . By the hypothesis, it

is obvious that $\frac{\partial u}{\partial \nu} \leq 0$ on $\partial\Omega \cap \{x_1 > \lambda\}$, and so

$$u_1 \leq 0 \text{ on } \partial\Omega \cap \{x_1 > \lambda\}.$$

First, suppose $f(0,0) \geq 0$. Then we get

$$\begin{aligned} 0 &= \Delta u + f(u, |\nabla u|) \\ &= \Delta u + f(u, |\nabla u|) - f(0,0) + f(0,0) \\ &\geq \Delta u + f(u, |\nabla u|) - f(0,0) \end{aligned}$$

Denote

$$\begin{aligned} p &= (p_1, p_2, \dots, p_n) = (u_{x_1}, u_{x_2}, \dots, u_{x_n}), \\ f[u] &\equiv f(u, |\nabla u|). \end{aligned}$$

By the mean value theorem of multidimensional calculus,

$$f[u] - f[0] = \sum_{i=1}^n \frac{\partial f}{\partial p_i}.$$

So for certain bounded functions $b_i(x)$, $c(x)$,

$$\begin{aligned} 0 &\geq \Delta u + f(u, |\nabla u|) - f(0,0) \\ &= \Delta u + \sum_{i=1}^n b_i(x) u_i + c(x). \end{aligned}$$

Thus it follows that $\frac{\partial u}{\partial \nu}(x_0) < 0$ by Lemma 2.2. Since

∇u is the opposite direction to the outward normal ν on $\partial\Omega$ and $\nu_1 > 0$ at x_0 , we conclude $u_1(x_0) < 0$.

Now, suppose $f(0,0) < 0$ and $u_1(x_0) = 0$ and $\Delta u(x_0) = -f(0,0)$ by the hypotheses. We claim

$$u_{ij}(x_0) = -f(0,0)\nu_i\nu_j \text{ for each } 1 \leq i, j \leq n \quad (2.1)$$

Since (2.1) is invariant under a rotation of coordinate axes, we may consider a coordinate frame with origin at x_0 , the x_1 axis being directed along the outward normal to $\partial\Omega$ at x_0 (at this moment, $\nu = (1, 0, \dots, 0)$). In this frame we can represent the boundary of Ω locally by the equation

$$x_1 = \varphi(x_2, \dots, x_n), \quad \varphi \in C^2.$$

Since $u = 0$ on $\partial\Omega$, we have

$$u(\phi, x_2, \dots, x_n) = 0 \quad (2.2)$$

Differentiating (2.2) with respect to $x_i (i = 2, \dots, n)$, we obtain

$$u_1\phi_i + u_i = 0 \quad (2.3)$$

Differentiating (2.3) with respect to $x_j (j = 2, \dots, n)$, we obtain

$$u_{ij} + u_{i1}\phi_j + (u_{1j} + u_{11}\phi_j)\phi_i + u_1\phi_{ij} = 0 \quad (2.4)$$

Evaluating (2.4) at x_0 , where $\phi_i = 0$ and $u_1 = 0$, we find

$$u_{ij}(x_0) = 0 \text{ for } 2 \leq i, j \leq n.$$

Thus $u_{11}(x_0) = -f(0,0)$. Since $u_1 \leq 0$ on $\partial\Omega \cap \{x_1 > \lambda\}$ and $u_1(x_0) = 0$, the mapping $(x_2, \dots, x_n) \mapsto u_1(\phi, x_2, \dots, x_n)$ has a maximum at x_0 . Hence $u_{1i}(x_0) = 0$ for $2 \leq i \leq n$. So far we have showed that (2.1) holds for $\nu = (1, 0, \dots, 0)$. Returning to the original coordinate axes, we can see that (2.1) holds for each $1 \leq i, j \leq n$. Since $f(0,0) < 0$ and $\nu_1 > 0$, we conclude $u_{11}(x_0) > 0$.

Proposition 2.3 is proved.

Using the same argument, we obtain the following corollary immediately.

Corollary 2.4 Let Ω be a bounded domain of class C^2 in R^n . Let f be of class C^1 . Assume that $u \in C^2(\Omega)$ is a solution of the problem

$$\begin{aligned} -\Delta u &= f(|\nabla u|) \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

If λ is a real number which defines the maximal cap, then for each point $x_0 \in \partial\Omega \cap \{x_1 > \lambda\}$, it holds either $u_1(x_0) < 0$ or else $u_{11}(x_0) > 0$, that is, u is a strictly decreasing function of x_1 near x_0 .

Remark: In the Proposition 2.3, the C^1 hypothesis on f can be replaced by either of the following conditions:

- 1) $f(|\nabla u|) = f_1(|\nabla u|) + f_2(|\nabla u|)$ for all u , where $f_1 \in C^1$ and f_2 is nondecreasing,
- 2) f is locally Lipschitz continuous.

3. Main Results and Proofs

Now we will present our main results.

Theorem 3.1 Let f be a continuous function defined on R^{n+1} and satisfies the following conditions:

- 1) f is symmetric in x_1 and nonincreasing in x_1 for $x_1 > 0$,
- 2) f_u is continuous for $x \in \Omega$.

If there exists a positive solution u in $u \in C^2(\overline{\Omega})$ satisfying

$$\begin{aligned} -\Delta u &= f(x, u, |\nabla u|) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} &= c(x') \text{ on } \partial\Omega. \end{aligned}$$

Then we have

- 1) There exists a real number λ such that Ω and u are Steiner symmetric with respect to the hyperplane

$\{x_1 = \lambda\}$.

2) For each $x \in \partial\Omega$, $v_1(x) = 0$ implies that $x_1 = \lambda$, and there exists a C^2 function $\varphi: \Omega' \rightarrow R$ such that $\Omega = \{(x_1, x') \mid 2\lambda - \varphi(x') < x_1 < \varphi(x')\}$, where $\Omega' = \Omega \cap \{x_1 = \lambda\}$.

3) In the case when λ is not zero, f is independent of the variable x_1 in Ω .

Proof. For each real number α , where $\lambda \leq \alpha \leq \lambda_1$, we define functions v and w in Σ_α :

$$v(x) = u(x^\alpha) \text{ and } w = v - u.$$

Then v and w satisfy, respectively

$$-\Delta v = f(x, v, |\nabla v|), \quad x \in \Sigma_\alpha,$$

$$v = u, \quad x \in \partial\Sigma_\alpha \cap T_\alpha,$$

$$v \geq 0, \quad x \in \partial\Sigma_\alpha \cap T_\alpha^c.$$

and

$$-\Delta w = f(x^\alpha, v, |\nabla v|) - f(x, u, |\nabla u|), \quad x \in \Sigma_\alpha \quad (3.1)$$

$$w \geq 0, \quad x \in \partial\Sigma_\alpha$$

By the mean value theorem, (3.1) can be rewritten into

$$\sum_{i,j=1}^n a_{ij}(x) w_{ij} + \sum_{i=1}^n b_i(x) w_i + c(x) = 0,$$

where a_{ij} , b_i and c are certain bounded functions. Now we claim that for each real number $\alpha (\lambda < \alpha < \lambda_1)$,

$$v(x) > u(x) \text{ in } \Sigma_\alpha, \quad (3.2)$$

It follows from Proposition 2.3 that (3.2) holds for each α sufficiently close to λ_1 .

Let $\beta \equiv \inf \{\lambda \leq \alpha \mid (3.2) \text{ holds for each } \alpha < \lambda_1\}$. We will prove that $\beta = \lambda$. Assume $\beta > \lambda$. For any point $x_0 \in \partial\Sigma_\beta \setminus T_\beta$, we have $x_0^\beta \in \Omega$. So we find $0 = u(x_0) < u(x_0^\beta)$.

Hence $w \neq 0$ in Σ_β . Since $w \geq 0$ in Σ_β , applying Lemma 2.2 to $\Omega \equiv \Sigma_\beta$, we obtain

$$w > 0, \quad x \in \Sigma_\beta, \text{ and } w_1 > 0, \quad x \in T_\beta \cap \Omega.$$

So we find $v(x) > u(x)$ in $\Sigma_{\beta-\varepsilon}$ for sufficiently small $\varepsilon > 0$.

We reach a contradiction with the choice of β . Hence $\beta = \lambda$ and this implies that

$$v(x) > u(x) \text{ in } \Sigma_\alpha \text{ for all } \alpha > \lambda,$$

and

$$v(x) \geq u(x) \text{ in } \Sigma_\lambda.$$

By Lemma 2.2, we see that either of the following facts holds:

B1) $w > 0$ at all interior points of Σ_λ , or else

B2) $w \equiv 0$ in Σ_λ .

Now, assume that $w \neq 0$ in Σ_λ , i.e., (B1) holds.

In the case (A1), by Lemma 2.2, we see

$$\frac{\partial w}{\partial \nu}(P^\lambda) < 0,$$

where P^λ is the reflected value of P across T_λ . But

this is a contradiction, since $\frac{\partial u}{\partial \nu}(P^\lambda) = \frac{\partial v}{\partial \nu}(P^\lambda)$. In the

case (A2), since we cannot apply Lemma 2.2 to the function w , we make use of the substitution for w . Let $\tilde{w} = we^{kx_1}$, where k is a constant to be determined. Then we obtain

$$\tilde{w} \geq 0 \text{ in } \Sigma_\lambda, \quad \tilde{w} = 0 \text{ at } Q$$

and

$$\sum_{i,j=1}^n a_{ij}(x) \tilde{w}_{ij} + \sum_{i=1}^n \tilde{b}_i(x) \tilde{w}_i + \tilde{c}x = 0$$

where $\tilde{b}_i = b_i - 2ka_{i1}$, $\tilde{c} = a_{11}k^2 - b_1k + c$.

For sufficiently large k , we have

$$\sum_{i,j=1}^n a_{ij}(x) \tilde{w}_{ij} + \sum_{i=1}^n \tilde{b}_i(x) \tilde{w}_i + \tilde{c}x \leq 0 \text{ in } \Sigma_\lambda.$$

By Lemma 2.1, it follows that

$$\frac{\partial \tilde{w}}{\partial s} > 0 \text{ or } \frac{\partial^2 \tilde{w}}{\partial s^2} > 0 \text{ at } Q.$$

At the point Q , since $\tilde{w} = 0$, we find

$$\frac{\partial w}{\partial s} = \frac{\partial \tilde{w}}{\partial s} e^{-kx_1} \text{ and } \frac{\partial^2 w}{\partial s^2} = \left(\frac{\partial^2 \tilde{w}}{\partial s^2} - 2ks_1 \frac{\partial \tilde{w}}{\partial s} \right) e^{-kx_1}.$$

Thus

$$\frac{\partial w}{\partial s} > 0 \text{ or } \frac{\partial^2 w}{\partial s^2} > 0 \text{ at } Q.$$

So far we have already showed that (B1) is impossible. And obviously (B2) implies that u is symmetric with respect to T , Ω is Steiner symmetric with respect to T . Meanwhile, $w > 0$ in Σ_α for all $\alpha \geq \lambda$, u is symmetrically decreasing in x_1 . This completes the proof of (1).

By the symmetry of a domain Ω and the definition of the maximal cap, for each $x \in \partial\Omega$, $v_1(x) = 0$ implies $x_1 = \lambda$. So $x_1 > \lambda$ implies $v_1(x) > 0$. So we can see from the implicit function theorem that there exists a C^2 function $\varphi: \Omega' \rightarrow R$ such that $\Omega = \{(x_1, x') \mid 2\lambda - \varphi(x') < x_1 < \varphi(x')\}$. Finally, the symmetry of u implies $\Delta u(x_1, x') = \Delta u(2\lambda - x_1, x')$ in Ω . Thus f is symmetric with respect to the plane $\{x_1 = \lambda\}$ in Ω . If $\lambda \neq 0$, by the hypothesis of f , we can see that f is independent of the variable x_1 in Ω . This completes the proof of the theorem.

Using Lemma 2.1, Lemma 2.2, Proposition 2.3, Cor-

ollary 2.4 and still the same argument, we can obtain the following two corollaries.

Corollary 3.2 Let f be a continuous function defined on R^{n+1} and satisfies the following conditions:

- 1) f is symmetric in x_1 and nonincreasing in x_1 for $x_1 > 0$,
- 2) f_{u_i} is continuous for $x \in \Omega$.

If there exists a positive solution $u \in C^2(\overline{\Omega})$ satisfying

$$-\Delta u = f(x, |\nabla u|) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

$$\frac{\partial u}{\partial \nu} = c(x') \quad \text{on } \partial\Omega.$$

Then we have

1) There exists a real number λ such that Ω and u are Steiner symmetric with respect to the hyperplane $\{x_1 = \lambda\}$.

2) For each $x \in \partial\Omega$, $v_1(x) = 0$ implies that $\{x_1 = \lambda\}$, and there exists a C^2 function $\varphi: \Omega' \rightarrow R$ such that $\Omega = \{(x_1, x') \mid 2\lambda - \varphi(x') < x_1 < \varphi(x')\}$, where $\Omega' = \Omega \cap \{x_1 = \lambda\}$.

3) In the case when λ is not zero, f is independent of the variable x_1 in Ω .

Corollary 3.3 Let f be a function of class C^1 defined on R . If there exists a positive solution $u \in C^2(\overline{\Omega})$ satisfying

$$-\Delta u = f(|\nabla u|) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

$$\frac{\partial u}{\partial \nu} = c(x') \quad \text{on } \partial\Omega.$$

Then we have

1) There exists a real number λ such that Ω and u are Steiner symmetric with respect to the hyperplane $\{x_1 = \lambda\}$.

2) For each $x \in \partial\Omega$, $v_1(x) = 0$ implies that $\{x_1 = \lambda\}$, and there exists a C^2 function $\varphi: \Omega' \rightarrow R$

such that

$$\Omega = \{(x_1, x') \mid 2\lambda - \varphi(x') < x_1 < \varphi(x')\}, \quad \text{where}$$

$$\Omega' = \Omega \cap \{x_1 = \lambda\}.$$

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