

The Normal Meromorphic Functions Family Concerning Higher Order Derivative and Shared Set by One-Way

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Abstract

Let F be a meromorphic functions family on the unit disc Δ , If for every $f \in F$ (the zeros of f is a multiplicity of at least k) and if $f(z) = 0$ then $|f^{(k)}(z)| \leq c$ and $\bar{E}_{f^{(k)}}(S) \subseteq \bar{E}_f(S)$ ($S = \{a, b\}$), then F is normal on Δ .

Keywords: Meromorphic Function, Normality Criterion, Shared Set by One-Way, Higher Order Derivative

1. Introduction and Results

First, we introduce the following definition:

Definition: For a, b are two distinct complex values, we have set $S = \{a, b\}$ and

$$\begin{aligned} \bar{E}_f(S) &= \bar{E}_f(a, b) \\ &= \{z : (f(z) - a)(f(z) - b) = 0, z \in D\} \end{aligned}$$

If $\bar{E}_f(S) = \bar{E}_g(S)$, we call that f and g share S in D ; If $\bar{E}_f(S) \subseteq \bar{E}_g(S)$, we call that f and g share S by One-way in D .

For the normal meromorphic functions family concerning one-way sharing set, W. H. Zhang proved the following result [1]:

Theorem A. Let F be a family of meromorphic functions in the unit disc Δ , a and b be two distinct nonzero complex values. $S = \{a, b\}$, If for every $f \in F$, all of whose zeros is multiple, $\bar{E}_{f^{(k)}}(S) = \bar{E}_f(S)$, then F is normal on Δ .

W. H. Zhang continued considering the relation between normality and the shared set, and proved the next result [2]:

Theorem B. Let F be meromorphic functions family in the unit disk Δ , a and b be two distinct nonzero complex values. If for every $f \in F$, all of whose zeros is multiplicity $k+1$ at least (k is a positive integer), $\bar{E}_{f^{(k)}}(S) = \bar{E}_f(S)$, then F is normal in Δ .

In 2008, F. J. Lv got following theorem in [3]:

Theorem C. Let F be a family of meromorphic

function in the unit disk Δ , a and b is two distinct nonzero complex values, k is positive integer, $S = \{a, b\}$. If for every $f \in F$, all of whose zeros have multiplicity $k+1$ at least, $\bar{E}_{f^{(k)}}(S) \subseteq \bar{E}_f(S)$, then F is normal in Δ .

In this paper, we continue to discuss about normality theorem of meromorphic functions families concerning higher order derivative and shared set by one-way, and obtain main results as follow.

Theorem. Let F be a meromorphic functions family on the unit disc Δ , a and b is two distinct nonzero complex values, k is positive integer, $S = \{a, b\}$, If for every $f \in F$, all of whose zeros have multiplicity k at least, If

1) exists $c > 0$,
 such that $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq c$;

2) $\bar{E}_{f^{(k)}}(S) \subseteq \bar{E}_f(S)$;

then F is normal in Δ .

Remark: Through the following example, all of whose zeros have multiplicity k at least is necessary.

Example: Let

$$S = \{1, -1\}, \omega_1 \neq \omega_2 \text{ and } \omega_i^k = 1, i = 1, 2.$$

a family of meromorphic function in the unit disk Δ be $F = \{f_n(z)\}$ there

$$f_n(z) = n(e^{\omega_1 z} - e^{\omega_2 z}), n = 1, 2, 3, \dots$$

Obviously $f_n(z) = f_n^{(k)}(z)$ and

$$f_n^{(l)}(z) = n(\omega_l^l e^{\omega_l z} - \omega_l^l e^{\omega_l z}), \quad (l = 1, 2, 3, \dots, k-1),$$

Hence $z = 0$ satisfy 1) and 2) and all of whose zeros of $f_n(z)$ have multiplicity $k-1$ at most, Since

$$f^\#(0) = \frac{|f'(0)|}{1+|f(0)|^2} = n|\omega_1 - \omega_2| \rightarrow \infty (n \rightarrow \infty),$$

F is not normal at $z = 0$ by Marty Theorem.

2. Lemmas

Lemma 1 [4]. Let F be meromorphic functions families in the unit disk Δ , all of whose zeros have multiplicity k at least. If for every $f \in F$, there exists $A > 0$, such that $|f^{(k)}(z)| \leq A$ when $f(z) = 0$. If F is not normal in Δ , then for every $0 \leq \alpha \leq k$, there exists $0 < r < 1$, $|z_n| < r$, $f_n \in F$, $\rho_n \rightarrow 0$, such

that $g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^\alpha}$ converges locally and uni-

formly to a noncontant meromorphic function $g(\zeta)$, and $g^\#(\zeta) \leq g^\#(0) = kA + 1$. where

$$g^\#(\zeta) = \frac{|g'(\zeta)|}{1+|g(\zeta)|^2}$$

is said to be Spherical derivative of g .

Lemma 2 [5]. Let $f(z)$ be nonconstant meromorphic function in C

$$\Theta(a, f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\bar{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}, \quad (a \neq \infty),$$

$$\Theta(\infty, f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{T(r, f)}$$

then $\sum_{a \in C} [\Theta(a, f)] + \Theta(\infty, f) \leq 2$.

3. Proof of Theorem

Suppose that F be not normal in Δ , then by Lemma 1 we get that there exists $f_n \in F$, $z_n \in \Delta$ and $\rho_n \rightarrow 0^+$

such that $g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k}$ converges locally and

uniformly to a noncontant meromorphic function $g(\zeta)$ [6], and $g^\#(\zeta) \leq g^\#(0) = kc + 1$. We claim that the following conclusions hold.

- 1) zeros of $g(\zeta)$ have multiplicity k at least, and $g(\zeta) = 0 \Rightarrow |g^{(k)}(\zeta)| \leq c$;
- 2) $g^{(k)}(\zeta) \neq a, g^{(k)}(\zeta) \neq b$;

In fact, suppose that there exists $\zeta_0 \in \Delta$, such that

$g(\zeta_0) = 0$, there exists $\zeta_n \rightarrow \zeta_0$,

$$g_n(\zeta_n) = \frac{f_n(z_n + \rho_n \zeta_n)}{\rho_n^k} = 0 \quad \text{for sufficiently large } n,$$

Hence $f_n(z_n + \rho_n \zeta_n) = 0$.

Therefore, the following conclusions is obviously

$$f_n^{(j)}(z_n + \rho_n \zeta_n) = 0, \quad (j = 1, 2, 3, \dots, k-1)$$

and $|f_n^{(k)}(z_n + \rho_n \zeta_n)| \leq c$,

Hence

$$g_n^{(j)}(\zeta_n) = \rho_n^{j-k} f_n^{(j)}(z_n + \rho_n \zeta_n) = 0, \quad (j = 1, 2, 3, \dots, k-1)$$

and $|g_n^{(k)}(\zeta_n)| \leq c$,

$$\text{So } g^{(j)}(\zeta_0) = \lim_{n \rightarrow \infty} g_n^{(j)}(\zeta_n) = 0, \quad (j = 1, 2, 3, \dots, k-1)$$

and $|g^{(k)}(\zeta_0)| \leq c$,

Hence, zeros of $g(\zeta)$ have multiplicity k at least, and $g(\zeta) = 0 \Rightarrow |g^{(k)}(\zeta)| \leq c$,

Therefore, conclusion (1) is hold

In what follow, we complete the proof of the claim 2): Suppose that there exists ζ_0

Such that $g^{(k)}(\zeta_0) = a$, by Hurwitz theorem, there exists $\zeta_n \rightarrow \zeta_0$ such that

$$g_n^{(k)}(\zeta_n) = f_n^{(k)}(z_n + \rho_n \zeta_n) = a \quad \text{for sufficiently large } n.$$

By conditions:

$$\bar{E}_{f^{(k)}}(S) \subseteq \bar{E}_f(S),$$

$$f_n(z_n + \rho_n \zeta_n) = a \text{ (or } b),$$

Hence

$$g(\zeta_0) = \lim_{n \rightarrow \infty} g_n(\zeta_n) = \lim_{n \rightarrow \infty} \frac{f_n(z_n + \rho_n \zeta_n)}{\rho_n^k} = \infty,$$

this is a contradictions for $g^{(k)}(\zeta_0) = a$.

If $g^{(k)}(\zeta) \equiv a$, then $g(\zeta)$ is polynomial of degree k .

Because, zeros of $g(\zeta)$ have multiplicity k at least,

$$\text{then } g(\zeta) = \frac{a}{k!} (\zeta - \zeta_0)^k$$

$$\text{and } g'(\zeta) = \frac{a}{(k-1)!} (\zeta - \zeta_0)^{k-1}$$

Obviously

$$g^\#(0) = \frac{\left| \frac{a}{(k-1)!} (-\zeta_0)^{k-1} \right|}{1 + \left| \frac{a}{k!} (-\zeta_0)^k \right|} \leq \begin{cases} \frac{k}{2}, & |\zeta_0| > 1 \\ \frac{|a|}{(k-1)!}, & |\zeta_0| \leq 1 \end{cases}$$

Because $|g^{(k)}(\zeta_0)| = |a| \leq c$, then

$\frac{|a|}{(k-1)!} \leq \frac{c}{(k-1)!} < kc+1$, Suppose $c \geq 1$ is not general,

therefore $\frac{k}{2} < kc+1$, hence $|g \#(0)| < kc+1$, this is a con-

tradictions for conditions of $g(\zeta)$. Hence $g^{(k)}(\zeta) \neq a$. Similar to prove $g^{(k)}(\zeta) \neq b$.

Therefore

$$\Theta(a, g^{(k)}) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{g^{(k)} - a}\right)}{T(r, g^{(k)})} = 1 \quad \text{and}$$

$$\Theta(b, g^{(k)}) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}\left(r, \frac{1}{g^{(k)} - b}\right)}{T(r, g^{(k)})} = 1$$

Since

$$\begin{aligned} T(r, g^{(k)}) &\geq N(r, g^{(k)}) \\ &= k \overline{N}(r, g) + N(r, g) \geq (k+1) \overline{N}(r, g) \end{aligned}$$

So

$$\Theta(\infty, g^{(k)}) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}(r, g^{(k)})}{T(r, g^{(k)})} \geq 1 - \frac{1}{k+1} = \frac{k}{k+1}$$

Therefore

$$\Theta(a, g^{(k)}) + \Theta(b, g^{(k)}) + \Theta(\infty, g^{(k)}) \geq 2 + \frac{k}{k+1} > 2$$

this is a contradictions for Lemma 2. The proof of Theorem is completed.

4. References

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