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Note on Laguerre Transform in Two Variables

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Abstract

An attempt is made to investigate the some new properties of Laguerre transform in two variables [1].

Keywords: Laguerre Transform, Laguerre Polynomials in Two Variables, Laguerre Transform in Two Variables

1. Introduction

Debnath [2] introduced the Laguerre transform and derived some properties. He also discussed the applications in study of heat conduction [3] and to the oscillations of a very long and heavy chain with variable tension [4].

Glaeske generalized Laguerre transform of one variable as Laguerre-Pinney transformation [5], Wiener-Laguerre transformation [6] and derived its properties. Debnath *et al.* [7] reported all these work in their book.

Recently Shukla *et al.* [1] introduced the Laguerre Transform of $f(x,y)$ as

$$F_n(\alpha,\beta) = \mathrm{L}\{f(x,y), x\to\alpha, y\to\beta, n\} = \int_0^\infty\int_0^\infty \mathrm{e}^{-(x+y)} x^\alpha y^\beta K_n^{(\alpha,\beta)}(x,y)\, f(x,y)\,\mathrm{d}x\mathrm{d}y \quad (1.1)$$

where $f(x,y)$ be a Riemann integrable function defined on the set $S=\mathbb{R}^+\times\mathbb{R}^+$, $\alpha>-1$, $\beta>-1$, n is non-negative integer and

$$K_n^{(\alpha,\beta)}(x,y) = \sum_{r=0}^{n} \frac{(-xy)^r}{r!(-n)_r} L_{n-r}^{(\alpha+r,\beta+r)}(x,y) \quad (1.2)$$

Ragab [8] introduced Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$, which is defined as

$$L_n^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!} \cdot \sum_{k=0}^{n} \frac{L_{n-k}^{(\alpha)}(x)(-y)^k}{k!\Gamma(\alpha+n-k+1)\Gamma(\beta+k+1)} \quad (1.3)$$

Ragab [8] also obtained,

$$K_n^{(\alpha,\beta)}(x,y) = L_n^{\alpha}(x) L_n^{\beta}(y) \quad (1.4)$$

Therefore, the equivalent definition for the Laguerre Transform of $f(x,y)$ is

$$\mathrm{L}\{f(x,y)\} = F_n(\alpha,\beta) = \int_0^\infty\int_0^\infty \mathrm{e}^{-(x+y)} x^\alpha y^\beta L_n^{\alpha}(x) L_n^{\beta}(y)\, f(x,y)\,\mathrm{d}x\mathrm{d}y \quad (1.5)$$

We also used following theorems based on Shukla *et al.* [1]:

Theorem 1: If $K_n^{(\alpha,\beta)}(x,y)$ is defined as (1.2), then

$$\int_0^\infty\int_0^\infty \mathrm{e}^{-(x+y)} x^\alpha y^\beta K_n^{(\alpha,\beta)}(x,y) K_m^{(\alpha,\beta)}(x,y)\,\mathrm{d}x\mathrm{d}y = \delta_n\delta_{mn} \quad (1.6)$$

where δ_{mn} (Kronecker delta symbol) is defined as

$$\delta_{mn} = \begin{cases} 0, & m\neq n \\ 1, & m=n \end{cases},$$

$$\delta_n = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n!)^2}, \alpha>-1 \text{ and } \beta>-1.$$

Srivastava and Manocha[9] reported following results:

$$\sum_{m=0}^{\infty} \frac{m!(\lambda)_m}{(\alpha+1)_m(\beta+1)_m} L_m^{\alpha}(x) L_m^{\beta}(y) t^m = (1-t)^{-\lambda} \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!(\alpha+1)_m(\beta+1)_m}\left(\frac{xyt}{1-t}\right)^m \cdot \psi_2\left[\lambda+m;\alpha+m+1,\beta+m+1;\frac{xt}{t-1},\frac{yt}{t-1}\right], |t|<1 \quad (1.7)$$

where ψ_2 is defined as:

$$\psi_2[\alpha;\beta,\beta';x,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\beta)_m(\beta')_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.8}$$

Equation (1.7) can be easily written as

$$\sum_{m=0}^{\infty}\frac{m!(\lambda)_m}{(\alpha+1)_m(\beta+1)_m}L_m^{\alpha}(x)L_m^{\beta}(y)t^m =(1-t)^{-\lambda} \cdot F^{(3)}\left[\begin{matrix}\lambda::\ \_;\ \_;\ \_:\ \_;\ \_;\ \_;\\ \_::\ \alpha+1;\ \_;\ \beta+1:\ \_;\ \_;\ \_;\end{matrix}\ \xi,\ \eta,\ \zeta\right] \tag{1.9}$$

where

$$\xi=\frac{xyt}{1-t},\ \eta=\frac{-xt}{1-t},\ \zeta=\frac{-yt}{1-t}$$

and

We used following resultbased on Erdélyi *et al.* [10]:

$$\sum_{m=0}^{\infty}\frac{m!(\lambda)_m}{(\alpha+1)_m(\beta+1)_m}L_m^{\alpha}(x)L_m^{\beta}(y)t^m =(1-t)^{-\lambda}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!(\alpha+1)_m(\beta+1)_m}\left(\frac{xyt}{1-t}\right)^m \times{}_1F_1\left[\begin{matrix}\lambda+m;\\ \alpha+m+1;\end{matrix}\ \frac{xt}{t-1}\right]{}_1F_1\left[\begin{matrix}\lambda+m;\\ \beta+m+1;\end{matrix}\ \frac{yt}{t-1}\right],|t|<1 \tag{1.11}$$

and following results (1.12 and 1.13) based on Rainville [11]:

$$\int_0^{\infty}e^{-x}x^{\alpha}L_n^{\alpha}(x)x^k\mathrm{d}x=0;\ k=0,1,2,\cdots,(n-1) \tag{1.12}$$

$$\sum_{m=0}^{k}\frac{m!L_m^{\alpha}(x)L_m^{\alpha}(y)}{(1+\alpha)_m} =\frac{(k+1)!}{(1+\alpha)_k}\frac{L_{k+1}^{\alpha}(y)L_k^{\alpha}(x)-L_{k+1}^{\alpha}(x)L_k^{\alpha}(y)}{x-y} \tag{1.13}$$

2. Main Results

In this section, some new properties of Laguerre Transforms in two variables [1] have been obtained.

Theorem 1: If

$$f(x,y)=(1-t)^{-\lambda}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!(\alpha+1)_m(\beta+1)_m}\left(\frac{xyt}{1-t}\right)^m \cdot\psi_2\left[\lambda+m;\alpha+m+1,\beta+m+1;\frac{xt}{t-1},\frac{yt}{t-1}\right],|t|<1$$

then

$$\mathrm{L}\{f(x,y),\alpha,\beta,n\} =F_n(\alpha,\beta)=\frac{t^n(\lambda)_n\Gamma(\alpha+1)\Gamma(\beta+1)}{n!} \tag{2.1}$$

Here ψ_2 is a function defined by (1.7).

Proof: Using (1.7) and (1.5), we have

$$F_n(\alpha,\beta)=\int_0^{\infty}\int_0^{\infty}e^{-(x+y)}x^{\alpha}y^{\beta}L_n^{\alpha}(x)L_n^{\alpha}(y) \cdot\sum_{m=0}^{\infty}\frac{m!(\lambda)_m}{(\alpha+1)_m(\beta+1)_m}L_m^{\alpha}(x)L_m^{\alpha}(y)t^m\mathrm{d}x\mathrm{d}y$$

Further using (1.6), we arrived at

$$F_n(\alpha,\beta)=\sum_{m=0}^{\infty}\frac{m!(\lambda)_m}{(\alpha+1)_m(\beta+1)_m}\delta_n\delta_{mn}t^m$$

Using definition of δ_{mn}, we get

$$F_n(\alpha,\beta)=\frac{n!(\lambda)_n}{(\alpha+1)_n(\beta+1)_n}\delta_n t^n$$

Using definition of δ_n, we get

$$F_n(\alpha,\beta)=\frac{t^n(\lambda)_n\Gamma(\alpha+1)\Gamma(\beta+1)}{n!}$$

This completes the proof.

Using (1.9), we get

Corollary 1: If

$$F^{(3)}\left[\begin{matrix}(a)::\ (b);\ (b');\ (b''):\ (c);\ (c');\ (c'');\\ (e)::\ (g);\ (g'');\ (g''):\ (h);\ (h');\ (h'');\end{matrix}\ x,\ y,\ z\right] =\sum_{m,r,p=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m+r+p}\prod_{j=1}^{B}(b_j)_{m+r}\prod_{j=1}^{B'}(b'_j)_{r+p}\prod_{j=1}^{B''}(b''_j)_{p+m}\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_r\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{E}(e_j)_{m+r+p}\prod_{j=1}^{G}(g_j)_{m+r}\prod_{j=1}^{G'}(g_j')_{r+p}\prod_{j=1}^{G''}(g_j'')_{p+m}\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_r\prod_{j=1}^{H''}(h''_j)_p}\frac{x^m}{m!}\frac{y^r}{r!}\frac{z^p}{p!} \tag{1.10}$$

$$f(x,y)=(1-t)^{-\lambda}$$

$$\cdot F^{(3)}\left[\begin{matrix}\lambda:: & \_; & \_; & \_: & \_; & \_; & \_; & \\ & & & & & & & \xi,\ \eta,\ \zeta \\ \_:: & \alpha+1; & \_; & \beta+1: & \_; & \_; & \_; & \end{matrix}\right]$$

where $\xi=\frac{xyt}{1-t},\eta=\frac{-xt}{1-t},\zeta=\frac{-yt}{1-t}$

then

$$\mathrm{L}\{f(x,y),\alpha,\beta,n\}$$

$$=F_n(\alpha,\beta)=\frac{t^n(\lambda)_n\Gamma(\alpha+1)\Gamma(\beta+1)}{n!} \quad (2.2)$$

Here F^3 is a function defined by (1.10).

Also, using (1.11) we have

Corollary 2:

If

$$f(x,y)=(1-t)^{-\lambda}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!(\alpha+1)_m(\beta+1)_m}\left(\frac{xyt}{1-t}\right)^m$$

$$\times {}_1F_1\left[\begin{matrix}\lambda+m; & \\ & \frac{xt}{t-1} \\ \alpha+m+1; & \end{matrix}\right]{}_1F_1\left[\begin{matrix}\lambda+m; & \\ & \frac{yt}{t-1} \\ \beta+m+1; & \end{matrix}\right],$$

$(|t|<1)$

then,

$$\mathrm{L}\{f(x,y),\alpha,\beta,n\}$$

$$=F_n(\alpha,\beta)=\frac{t^n(\lambda)_n\Gamma(\alpha+1)\Gamma(\beta+1)}{n!} \quad (2.3)$$

Theorem 2: If $f(x,y)=x^ky^l$, where k and l are positive numbers such that $k=0,1,2,\cdots,(n-1)$ or $l=0,1,2,\cdots,(n-1)$ then

$$\mathrm{L}\{f(x,y)\}=0 \quad (2.4)$$

Using (1.12) we can obtain (2.4).

Theorem 3:

If

$$f(x,y)=\frac{(k+1)!}{(1+\alpha)_k}\frac{L_{k+1}^{\alpha}(y)L_k^{\alpha}(x)-L_{k+1}^{\alpha}(x)L_k^{\alpha}(y)}{x-y}$$

and $F_n(\alpha,\beta)=\mathrm{L}\{f(x,y),\alpha,\beta,n\}$ then,

$$F_n(\alpha,\alpha)=\frac{\Gamma(\alpha+1)\Gamma(n+\alpha+1)}{n!} \quad (2.5)$$

Proof: Using (1.13) and (1.5), we have

$$F_n(\alpha,\alpha)$$

$$=\int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-(x+y)}x^{\alpha}y^{\alpha}K_n^{(\alpha,\alpha)}(x,y)\sum_{m=0}^{k}\frac{m!L_m^{\alpha}(x)L_m^{\alpha}(y)}{(1+\alpha)_m}\mathrm{d}x\mathrm{d}y$$

Further using (1.6), we arrived at

$$F_n(\alpha,\alpha)=\sum_{m=0}^{k}\frac{m!}{(\alpha+1)_m}\delta_n\delta_{mn}$$

Using definition of δ_{mn}, we get

$$F_n(\alpha,\alpha)=\frac{n!}{(\alpha+1)_n}\delta_n$$

Using definition of δ_n, we get (2.5).

3. References

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