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A Study on the Conversion of a Semigroup to a Semilattice

Bahman Tabatabaie, Seyed Mostafa Zebarjad
Department of Mathematics, Shiraz University, Shiraz, Iran
E-mail: *s_zebarjad@yahoo.com*
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Abstract

The main aim of the current research has been concentrated to clarify the condition for converting the inverse semigroups such as S to a semilattice. For this purpose a property the so-called $E^*-\text{unitary}$ has been defined and it has been tried to prove that each inverse semigroups limited with $E^*-\text{unitary}$ show the specification of a semilattice.

Keywords: Semigroup, Semilattice, $E^*-\text{unitary}$

1. Introduction

1.1. Literature Survey

Literature survey done by the authors show that a special class of semigroups possessing is formed by the $E^*-\text{unitary}$ inverse semigroups, sometimes also called $0-E^*-\text{unitary}$, which was defined by Szendrei [1] and has been intensely studied in the semigroup literature. See, for example, Kellendonk's topological groupoid is Hausdorff when S is $E^*-\text{unitary}$ [2], and the related class of $E-\text{unitary}$ inverse semigroups have also been shown to provide Hausdorff groupoids [3]. In the current research the authors try to prove that each inverse semigroups limited with $E^*-\text{unitary}$ show the specification of a semilattice. For this purpose, firstly we present elementary concepts as follows.

1.2. Preliminary Definitions and Propositions

A groupoid is a set *G* together with a subset $G^2 \subseteq G\times G$, a product map $(a,b)\mapsto ab$.

From G^2 to G, and an inverse map $a\mapsto a^{-1}$ (so that $(a^{-1})^{-1}=a$) from G onto G such that:

1) if $(a,b),(b,c)\in G^2$, then $(ab,c),(a,bc)\in G^2$ and $(ab)c=a(bc)$.

2) $(b,b^{-1})\in G^2$ for all $b\in G$, and if $(a,b)\in G^2$ then $a^{-1}(ab)=b$ and $(ab)b^{-1}=a.$

Note that G^2 is nothing but the set of all pairs (x,y) in $G\times G$ for which xy is defined, and G^2 is called the set of *composable pairs* of the groupoid G [3].

If $x\in G, d(x)=x^{-1}x$ is the *domain* of x and $r(x)=xx^{-1}$ is its *range*. The pair (x,y) is composable if and only if the range of y is the domain of x. $G^0=d(G)=r(G)$ is the *unit space* of G, its elements are units in sense that $x\mathrm{d}(x)=x$ and $r(x)=x$ [4].

By an inverse semigroup we mean a semigroup S such that for each a in S, there exists a unique element a^* in S with the following properties:

$$aa^*a=a,\ \text{ and }\ a^*aa^*=a^*$$

It is well known that the correspondence $a\mapsto a^*$ is an involutive anti-homomorphism, *i.e.*, $(ab)^*=b^*a^*$ for all a and b in S. It is very common to denote it by $E(S)$, the set of all idempotent elements of S, it means that $a^2=a$ for all a in $E(S)$. It is clear that $a^*=a$ for all a in $E(S)$.

A very important example of an inverse semigroup is given by $S=I(X)$ the set of all partial one-to-one maps on a set X. So each element of $I(X)$ is a bijection form a subset U of X onto another subset V of X. The set $I(X)$ is a semigroup where the multiplication rule is given by composition of partial maps with the largest possible domain.

For example, if $\theta_1,\theta_2\in I(X)$ with $\theta_1:U_1\to V_1$ and $\theta_2:U_2\to V_2$, then

$$\theta_1\theta_2:\theta_2^{-1}(V_2\cap U_1)\to\theta_1(V_2\cap U_1)$$

is given by:

$$\theta_1\theta_2(a)=\theta_1(\theta_2(a)).$$

The element θ_1^* is taken to be θ_1^{-1}. It is easily checked that $I(X)$ is an inverse semigroup [3,5].

We recall that a relation $\le$ on a set X is called a partial ordering of X if for all $a,b,c \in X$:

1) $a \le a$

2) $a \le b$ and $b \le a$ implies $a = b$

3) $a \le b$ and $b \le c$ implies $a \le c$.

The following example is of great importance to us. Define $e \le f\left(e, f \in E(S)\right)$ to mean $ef = fe = e$. It is clear that $\le$ is a partial ordering of $E(S)$. We shall call $\le$ the natural partial ordering of $E(S)$.

An element b of a partially ordered set X is called an upper bound of a subset Y of X, if $y \le b$ for each y in Y. An upper bound b of Y is called a least upper bound or join of Y, if $b \le c$ for every upper bound c of Y. If Y has a join in X, it is clearly unique. Lower bound and greatest lower bound or meet can be defined similarly.

A partially ordered set X is called a semilattice if every two elements subset $\{a,b\}$ of X has a join and a meet in X; it implies that every finite subset of X has both a join and a meet. The join (or meet) of $\{a,b\}$ will be denoted by $a \wedge b$ (or $a \vee b$)[3].

Definition 1.1 Suppose that *S* is an inverse semigroup and X can be assumed that as a locally compact Hausdorff topological space.

An action of S on X is a semigroup homomorphism as follows:

$$\alpha : S \to I(X)$$

$$a \mapsto \alpha_a$$

such that

1) for every $a \in S$ there is a continuous α_a with open domain in X.

2) the union of the domains of all the α_a coincides with X.

Proposition 1.2 Let S be an inverse semigroup, α an action of S on a set X and $a \in S$, then

$$\alpha_a \alpha_{a^*} \alpha_a = \alpha_a \text{ and } \alpha_{a^*} \alpha_a \alpha_{a^*} = \alpha_{a^*}$$

Proof: Since α is an action of S on X then $\alpha : S \to I(X)$ is a semigroup homomorphism, so for every $a \in S$ we have $\alpha(a)\alpha(a^*)\alpha(a) = \alpha(a)$, then $\alpha_a \alpha_{a^*} \alpha_a = \alpha_a$, and simillary $\alpha_{a^*} \alpha_a \alpha_{a^*} = \alpha_{a^*}$

With regard to the above text one may conclude that, $\alpha_{a^*} = \alpha_a^{-1}$, and if $e \in E(S)$, so α_e is the identity map on its domain.

Since the range of each α_a coincides with the domain of $\alpha_{a^*} = \alpha_a^{-1}$, therefore it can be open as well as its domain. Also it can be mentioned that α_a^{-1}, is continuous, so α_a is necessarily a homeomorphism onto its range.

For every $e \in E(S)$ the domain (and range) of α_e can be denoted by E_e, it means:

$$\alpha_e : E_e \to E_e.$$

It is clear to show that the domains of both α_a and α_{a^*a} is the same, and implies that the domain of α_a is E_{a^*a}. Likewise the range of α_a is given by E_{aa^*}. Thus $\alpha_a : E_{a^*a} \to E_{aa^*}$ is a homeomorphism for every $a \in S$. Briefly if e and f are in $E(S)$ then we have $\alpha_e \alpha_f = \alpha_{ef}$ and $E_e \cap E_f = E_{ef}$.

Proposition 1.3 For each $a \in S$ and $e \in E(S)$ we have: $\alpha_a\left(E_e \cap E_{aa^*}\right) = E_{aea^*}$

Proof: Since N. Sieben [6], R. Exel [7] and Lawson [8] proved it, the authors use their result.

Definition 1.4 Let Σ be the subset of $S \times X$ given by:

$$\Sigma = \left\{(ab) \in S \times X : b \in E_{a^*a}\right\}$$

and for every (a_1,b_1) and (a_2,b_2) in Σ we will say that $(a_1,b_1) \sim (a_2,b_2)$ if $b_1 = b_2$ and there exists an idempotent e in $E(S)$ such that $b_1 \in E_e$, and $a_1 e = a_2 e$.

It is clearly that the relation $\sim$ is an equivalence relation on Σ The equivalence class of (a,b) will be denoted by $[a,b]$.

Let $G = \left\{[a,b] : a \in S, b \in X\right\}$ and put

$$G^2 = \left\{\left([a_1,b_1],[a_2,b_2]\right) \in G \times G : b_1 = \alpha_{a_2}(b_2)\right\}$$

And for every $\left([a_1,b_1],[a_2,b_2]\right) \in G^2$ define:

$$\begin{cases} [a_1,b_1] \cdot [a_2,b_2] = [a_1 a_2, b_2] \\ [a_1,b_1]^{-1} = \left[a_1^*, \alpha_{a_1}(b_1)\right] \end{cases}$$

it is easy to see that G is a groupoid [3] and the unit space $G^{(0)}$ of G naturally identifies with X under the correspondence

$$[e,b] \in G^{(0)} \mapsto b \in X,$$

where e is any idempotent such that $e \in E_S$. We show G semigroup as $G(\alpha, S, X)$.

We would now like to give G is a topology. Let $a \in S$ and U be an open subset of E_{a^*a} we define $\psi(a,U)$ as follows:

$$\psi(a,U) = \left\{[a,b] \in G : b \in U\right\}$$

The collection of all $\psi(a,U)$ is the basis of a topology on G, and also the multiplication and inversion operations on G are continuous, therefore G is a topological groupoid.

2. Main Results

Recall from [2] that an inverse semigroup S is naturally equipped with a partial order defined by:

$$a \le b \leftrightarrow a = ba^*a \,\forall\, a \in S$$

Proposition 2.1 Assume that S is an inverse semigroup which is a semilattice. Suppose that α is an action of S on a locally compact Hausdorff space X, such that for each $a \in S$, the domain E_{a^*a} of α_a is closed. Then $G = G(\alpha, S, X)$ is Hausdorff.

Proof: Suppose $[a,c]$ and $[b,d]$ are two distinct elements of $G(\alpha, S, X)$. The aim is to find two disjoint open subsets T_1 and T_2 of $G(\alpha, S, X)$ such that:

$$[a,c] \in T_1, [b,d] \in T_2, T_1 \cap T_2 = \phi$$

We consider two cases:

Case 1): If $(c \ne d)$:

Since X is Hausdorff space then

$$\exists F_1, F_2 \subseteq X\,(\text{open}), c \in F_1, d \in F_2, F_1 \cap F_2 = \phi$$

Now let $T_1 = \psi\left(a, F_1 \cap E_{a^*a}\right)$ and $T_2 = \psi\left(b, F_2 \cap E_{b^*b}\right)$

Since T_1 and T_2 are open set and

$$T_1 = \left\{[a,k] \in G : k \in F_1 \cap E_{a^*a}\right\},$$

$$T_2 = \left\{[b,k] \in G : k \in F_2 \cap E_{b^*b}\right\},$$

It is clearly that:

$$[a,c] \in T_1, [b,d] \in T_2 \text{ and } T_1 \cap T_2 = \phi$$

Case 2): If $(c = d)$:

Since S is a semilattice let $h = a \wedge b$ so

$$\begin{cases} h \le a \to h = ah^*h \\ h \le b \to h = bh^*h \end{cases} \Rightarrow [a,c] = [b,c]$$

Then referring to what proposed in Definition 1.4. $c \notin E_{h^*h}$. But E_{h^*h} is closed then $T_2 = X \setminus E_{h^*h}$ can be open and $c \in T_2$.

Now we can set T as $T_2 \cap E_{a^*a} \cap E_{b^*b}$. But we know that $\psi(a,T) = \{[a,k] : k \in T\}$ and it is clear that $[a,c] \in \psi(a,T), [b,c] \in \psi(b,T)$.

To do so it is enough to prove that $\psi(a,T) \cap \psi(b,T) = \phi$.

Suppose that $[l,k] \in \psi(a,T) \cap \psi(b,T)$ then:

$$\begin{cases} [l,k] \in \psi(a,T) \to [l,k] = [a,k] \to (l,k) \sim (a,k) \\ \qquad \to \exists e \in E(S), k \in E_e, ae = le \\ [l,k] \in \psi(b,T) \to [l,k] = [b,k] \to (l,k) \sim (b,k) \\ \qquad \to \exists f \in E(S), k \in E_f, bf = lf \end{cases}$$

Since $ef \in E(S)$ and $ef = fe$, $\left(k \in E_e \cap E_{ef}\right)$, it can be replaced e and f with ef and finally we have:

$$aef = lef, lef = lfe = bfe = bef$$

Therefore we can find an element $e \in E(S)$ such that $k \in E_e, ae = le, le = be.$ So $(le)^*(le) = ael^*le = lel^*le = ll^*lee = le$, then $le \le a$, and similary $le \le b$, since $h = a \wedge b$ thus $le \le h$, then $le = leh^*h$, hence $l^*le = l^*leh^*h \le h^*h$, and finally

$$k \in E_{l^*l} \cap E_e = E_{l^*le} \subseteq E_{h^*h}$$

But $k \in T$ that is contradicts.

Definition 2.2 A zero in an inverse semigroup S is an element $0 \in S$ such that:

$$oa = a0 = 0 \forall a \in S$$

Definition 2.3 An inverse semigroup S with zero is said to be E^*-unitary if for every $e, a \in S$ one has that $e^2 \ne e \le a \Rightarrow a^2 = a$.

In other words, if an element dominates a nonzero idempotent then that element itself is an idempotent.

Proposition 2.4 If S is a E^*-unitary inverse semigroup and a, b belong to the defined semigroup S such that $a^*a = b^*b$ and $ae = be$ for some nonzero idempotent $e \le a^*a$ then $a = b.$

Proof: We define $x = aea^*$. So x is nonzero idempotent because:

$$e \le a^*a \Rightarrow e = \left(a^*a\right)^*\left(a^*a\right)e = ea^*aa^*a$$

Then $e = a^*aea^*a$ (because of the ability of idempotent elements for being commute) and we have

$$ba^*x = ba^*aea^* = bb^*bea^* = bea^* = aea^* = x.$$

Therefore, we have $x \le ba^*$. Since S is a E^*-unitary which implies that ba^* is idempotent. Then $ba^* = \left(ba^*\right)^* = ab^*$ so ab^* is idempotent as well.

But, we have

$$bb^* = bb^*bb^* = ba^*ab^* = ab^*ba^* = aa^*aa^* = aa^*$$

Setting $y = ba^*b,$ we have that

$$y^*y = b^*ab^*ba^*b = b^*aa^*aa^*b = b^*aa^*b = b^*bb^*b = b^*b$$

Also $y^*y = a^*a$, while

$$b = bb^*b = by^*y, \text{ and } a = aa^*a = ay^*y,$$

So it is enough to prove that $y^* = ay^*$. We have

$$ay^* = ab^*ab^* = ab^* = ba^* = bb^*ba^* = bb^*ab^* = by^*$$

In what follows we give the main result of this paper.

Theorem 2.5 In condition that S is a E^*-unitary inverse semigroup with zero, then can be appeared as a semilattice.

Proof: For proving the above theorem it is necessary to show that $a \wedge b$ exists for every $a, b \in S$. If there is not nonzero $h \in S$ such that $h \le a, b$, it is obvious that

$a \wedge b = 0$ and it can be satisfied for the proof.

For doing this we can assume that there is a nonzero $h \in S$ in which $h \le a, b$. Our claim is that $ab^*b = ba^*a$.

Suppose that $k = a^*ab^*b$ and considering to our assumption $\left(h^*h \le a^*a, b^*b\right)$, we have $h^*h \le k$.

Substituting $x = ak$ and $y = bk$,

$$\begin{cases} x^*x = ka^*ak = k^2 = k \\ y^*y = kb^*bk = k^2 = k \end{cases} \Rightarrow x^*x = y^*y$$

also

$$xh^*h = akh^*h = ah^*h = h = bh^*h = bkh^*h = yh^*h$$

Using the proposition (2.4) $x = y$ will be achieved and so

$$ab^*b = aa^*ab^*b = ak = x = y = bk$$
$$= ba^*ab^*b = b\left(b^*b\right)a^*a = ba^*a$$

and finally

$$ab^*b = ba^*a \qquad (1)$$

By applying the above argument to a^*, b^*, h^* and knowing that $h^* \ne 0$ and $h^* \le a^*, b^*$ we have

$$a^*bb^* = b^*aa^*$$

so

$$\left(a^*bb^*\right)^* = \left(b^*aa^*\right)^*$$

and therefore Equation (1) can be modified to (2):

$$bb^*a = aa^*b \qquad (2)$$

We have that $h \le a, b$ then $h = ah^*h$ and $h = bh^*h$, then we can show that

$$b^*ah^*h = b^*bh^*h = h^*h$$

Since S is a E^*-unitary and b^*a is dominated by h^*h, we have $\left(b^*a\right)^2 = b^*a$. By applying the same reasoning to a^*, b^* and h^*, $\left(ba^*\right)^2 = ba^*$ can be a result.

Thus

$$\begin{cases} \left(b^*a\right)^* = b^*a \\ \left(ba^*\right)^* = ba^* \end{cases}$$

and hence $ab^*b = ba^*b = bb^*a$

$$ab^*b = bb^*a \qquad (3)$$

By combination of Equations (1) to (3), Equation (4) will be appeared.

$$ab^*b = ba^*a = bb^*a = aa^*b \qquad (4)$$

At the end we try to prove that ab^*b can satisfy the following condition

$$h \le ab^*b \le a, b$$

for every $h \in S$ such that $h \le a, b$.

It is clear that $ab^*b \le a, b$ and as defined before $k = a^*ab^*b$, then we have $h^*h \le k$, and so

$$h = ah^*h = akh^*h = aa^*ab^*bh^*h = ab^*bh^*h = \left(ab^*b\right)h^*h$$

Finally $h \le ab^*b$. It means that ab^*b is the join of a and b and this is the proof of theorem.

3. References

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