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Relative Widths of Some Sets of l_p^m *

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Abstract

In this paper, the relative widths of some sets in l_p^m are studied. Relative widths is the further development of Kolmogorov widths and it is a new problem in approximation theory which aroused some mathematics workers great interest recently. We present some basic propositions of relative widths and investigate relative widths of some sets (ball or ellipsoid) of l_p^m.

Keywords: Kolmogorov Widths, Relative Widths

1. Introduction

In 1984, V. N. Konovalov in [1] first proposed the definition of relative widths which is in the sense of Kolmogorov. Let W and V be centrally symmetric sets in a Banach space X. The Kolmogorov n-dimensional widths of W relative to V in X (shortly, *relative widths*) is

$$K_n(W,V,X) := \inf_{L^n} \sup_{f\in W} \inf_{g\in V\cap L^n} \|f-g\|_X,$$

where the infimum is taken over all n-dimensional subspaces L^n of X, $n\in N$. When $V=X$ the relative widths coincides with the n-dimensional Kolmogorov widths (shortly, $n-K$ *widths*) of W in X, which we denote by $d_n(W,X)$. Of course,

$$K_n(W,V,X) \ge d_n(W,X)$$

for any set V, and if $V_1 \subseteq V_2$, then $K_n(W,V_1,X) \ge K_n(W,V_2,X)$.

Y. N. Subbotin and S. A. Telyakovskii in [7-9], V. M. Tikhomirov in [11], V. F. Babenko in [2-4], V. N. Konovalov in [1,5,6], V. T. Shevaldin in [10] etc. gained many results in this field. And some Chinese mathematics workers such as Yongping Liu, Lianhong Yang in [15-17] and Weiwei Xiao in [12-14] also did some work on relative widths.

Let l_p^m, $1\le p\le\infty$, denote space of vectors $\boldsymbol{x}=(x_1,\cdots,x_m)$ with norm

$$\|\boldsymbol{x}\|_p = \left(|x_1|^p+\cdots+|x_m|^p\right)^{1/p},\ 1\le p<\infty$$

$$\|\boldsymbol{x}\|_\infty = \max\{|x_1|,\cdots,|x_m|\},\ p=\infty.$$

Let $B_p := \{\boldsymbol{x}\in l_p^m : \|\boldsymbol{x}\|_p \le 1\}$ be the unit ball in l_p^m. Let $\boldsymbol{D} = diag\{D_1,\cdots,D_m\}$ be an $m\times m$ real diagonal matrix. Without loss of generality we assume that $D_1\ge D_2\ge\cdots\ge D_m>0$. Let M be a positive real number, set

$$M\mathcal{D}_p = \left\{\boldsymbol{Dx} : \boldsymbol{x}\in R^m,\ \|\boldsymbol{x}\|_p \le M\right\},$$

obviously it is ellipsoid in l_p^m. When $M=1$, we denote it by $\mathcal{D}_p$.

Theorem A: [19] For $1\le p\le\infty$, $1\le m<n$,

$$d_n\left(\mathcal{D}_p, l_p^m\right) = D_{n+1},$$

Similar to the proof in [18] we can get the following proposition.

Proposition 1.

1) If W is a finite set of m elements, then for the linear spanning subspace $lin(W)$ one has

$$K_n(W, lin(W), X) = K_n(lin(W), lin(W), X) = 0$$

for $n\ge m$.

2) If $W_1 \subset W$, then

$$K_n(W_1,V,X) \le K_n(W,V,X).$$

3) For any scalar α, and any W and V, one has

$$K_n(\alpha W, \alpha V, X) = |\alpha| K_n(W,V,X).$$

4) $K_0(W,V,X) \ge K_1(W,V,X) \ge K_2(W,V,X) \ge \cdots$.

5) Let $W = K_0 + \Gamma_m$, where K_0 is a bounded set and

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Γ_m is a subspace of dimension m. If $n < m$, then $K_n(W,V,X) = \infty$.

6) For the convex hull $co(W)$, if for each subspace X_n of dimension n, $co(W) \cap X_n$ is a locally sequentially compact and closed subset, then

$$K_n(W, co(W), X) = K_n(co(W), co(W), X).$$

7) If Y is a subspace of X and $W \subset Y \subseteq X$, $V \subset Y$, then

$$K_n(W,V,X) \le K_n(W,V,Y).$$

Theorem 1 *For* $m > n \in N$, $1 \le p \le \infty$, $D_1 > D_{n+1}$, *the smallest number* M *which makes the equalities*

$$K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) = d_n(\mathcal{D}_p, l_p^m) = D_{n+1}, \qquad (1)$$

hold is $M_0 := 1 - \frac{D_{n+1}}{D_1}$, and

$$K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) = \begin{cases} (1-M)D_1, & 0 < M < M_0, \\ D_{n+1}, & M \ge M_0. \end{cases}$$

Theorem 2 *For all* $m \in N$ *such that* $m > 1$,

$$K_{m-1}(B_1, B_1, l_\infty^m) = \frac{1}{2}.$$

2. Proof of Theorems

Proof of Theorem 1: For $x_0 = (D_1, 0, \cdots, 0)$, we have

$$\begin{aligned} K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) &= \inf_{L^n \subset l_p^m} \sup_{x \in \mathcal{D}_p} \inf_{y \in M\mathcal{D}_p \cap L^n} \|x - y\|_p \\ &\ge \sup_{x \in \mathcal{D}_p} \inf_{y \in M\mathcal{D}_p} \|x - y\|_p \\ &\ge \inf_{y \in M\mathcal{D}_p} \|x_0 - y\|_p \\ &= D_1(1-M). \end{aligned}$$

That is

$$K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) \ge D_1(1-M), \quad \forall\, 0 < M \le 1. \qquad (2)$$

In order to make the equalities (1) hold, we have that

$$D_{n+1} \ge D_1(1-M),$$

that is $M \ge 1 - \frac{D_{n+1}}{D_1}$. (3)

For $0 < M \le 1 - \frac{D_{n+1}}{D_1}$, we will prove that

$$K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) \le (1-M)D_1. \qquad (4)$$

For each $x = Dz \in \mathcal{D}_p$, $\|z\|_p \le 1$, set $y = (Mx_1, \cdots, Mx_n, 0, \cdots, 0) \in L^n \cap M\mathcal{D}_p$. When $p = \infty$, the inequality (4) is trivial, so we only need to prove the case of $1 \le p < \infty$.

$$\begin{aligned} \|x - y\|_p^p &= (1-M)^p \left(|x_1|^p + \cdots + |x_n|^p\right) + |x_{n+1}|^p + \cdots + |x_m|^p \\ &= (1-M)^p \left(|D_1 z_1|^p + \cdots + |D_n z_n|^p\right) + |D_{n+1} z_{n+1}|^p + \cdots + |D_m z_m|^p \\ &\le (1-M)^p D_1^p \left(|z_1|^p + \cdots + |z_n|^p\right) \\ &\quad + (1-M)^p D_1^p (1-M)^{-p} D_1^{-p} D_{n+1}^p \left(|z_{n+1}|^p + \cdots + |z_m|^p\right) \\ &\le (1-M)^p D_1^p \|z\|_p^p \le (1-M)^p D_1^p. \end{aligned}$$

In fact, when $0 < M \le 1 - \frac{D_{n+1}}{D_1}$, we have $(1-M)^{-1} D_1^{-1} D_{n+1} \le 1$. So we get that inequality (4).

By inequalities (2) and (4), we have that

$$\forall\, 0 < M \le 1 - \frac{D_{n+1}}{D_1}, \quad K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) = (1-M)D_1. \qquad (5)$$

From (3) and (5) we get that the smallest number M which makes the equalities (1) hold is $M_0 = 1 - \frac{D_{n+1}}{D_1}$.

For $M \ge M_0$,

$$K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) \le K_n(\mathcal{D}_p, M_0\mathcal{D}_p, l_p^m) = D_{n+1}. \qquad (6)$$

By Theorem A, for all $M > 0$,

$$K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) \ge d_n(\mathcal{D}_p, l_p^m) = D_{n+1}. \qquad (7)$$

From (6) and (7) we get

$$K_n(\mathcal{D}_p, M\mathcal{D}_p, l_p^m) = D_{n+1}, \quad \forall M \ge M_0.$$

The proof of Theorem 1 is complete.

Proof of Theorem 2: From [6] we know that

$$K_{m-1}(B_1, B_1, l_\infty^m) \ge \frac{1}{2}. \qquad (8)$$

We want to prove that

$$K_{m-1}(B_1, B_1, l_\infty^m) \le \frac{1}{2}. \qquad (9)$$

By proposition (6) we know that

$$K_{m-1}\left(B_1, B_1, l_\infty^m\right) = K_{m-1}\left(W, B_1, l_\infty^m\right), \tag{10}$$

where

$$W = \left\{\left(0,\cdots,0,\left(\pm 1\right)_i, 0,\cdots,0\right) : i = 1,\cdots,m,\ i \text{ represent the } i\text{th coordinate}\right\}.$$

Set

$$L_*^{m-1} := \left\{ \boldsymbol{x} \in R^m : x_1 + x_2 + \cdots + x_m = 0 \right\}.$$

For $a = \left(0,\cdots,0,\left(\pm 1\right)_i, 0,\cdots,0\right) \in W$, set $b = \left(0,\cdots,0,\left(\pm 1/2\right)_i, \left(\mp 1/2\right)_{i+1}, 0,\cdots,0\right) \in L_*^{m-1} \cap B_1$, $i = 1,\cdots,d$, when $i = d$, $i+1$ represent the 1st coordinate, we get $\left\|a-b\right\|_\infty = 1/2$. So we proved

$$K_{m-1}\left(W, B_1, l_\infty^m\right) \le 1/2,$$

which means that inequality (9) is valid. The proof of Theorem 2 is complete.

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