

On the First and Second Locating Zagreb Indices of Graphs

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Abstract

By the distance or degree of vertices of the molecular graph, we can define graph invariant called topological indices. Which are used in chemical graph to describe the structures and predicting some physicochemical properties of chemical compound? In this paper, by introducing two new topological indices under the name first and second Zagreb locating indices of a graph G , we establish the exact values of those indices for some standard families of graphs included the firefly graph.

Keywords

Zagreb Indices, First and Second Locating Indices, Join Graph, Firefly Graph

1. Introduction

Topological indices play a significant role mainly in chemistry, pharmacology, etc. (see [1]-[7]). Many of the topological indices of current interest in mathematical chemistry are defined in terms of vertex degrees of the molecular graph. Two of the most famous topological indices of graphs are the first and second Zagreb indices which have been introduced by Gutman and Trinajstić in [8], and defined as $M_1(G) = \sum_{u \in V(G)} (d(u))^2$ and $M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$, respectively. The Zagreb indices have been studied extensively due to their numerous applications in the place of existing chemical methods which need more time and increase the costs. Many new reformulated and extended versions of the Zagreb indices have been introduced for several similar reasons (cf. [9]-[17]).

One of the present authors Saleh [18] has recently introduced a new matrix representation for a graph G by defining the locating matrix $\mathbf{Lo}(G)$ over G . We will redefine this representation as in the following.

Definition 1 ([18]) Let $G=(V,E)$ be a connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. A locating function of G denoted by $\mathbf{L}(G)$ is a function $\mathbf{L}(G):V(G) \rightarrow (\mathbb{Z}^+ \cup \{0\})^n$ such that $\mathbf{L}(v_i) = \mathbf{v}_i = (d(v_1, v_i), d(v_2, v_i), \dots, d(v_n, v_i))$, where $d(v_i, v_j)$ is the distance between the vertices v_i and v_j in G . The vector \mathbf{v}_i is called the locating vector corresponding to the vertex v_i , where $\mathbf{v}_i \cdot \mathbf{v}_j$ is actually the dot product of the vectors \mathbf{v}_i and \mathbf{v}_j in the integers space $(\mathbb{Z}^+ \cup \{0\})^n$ such that v_i is adjacent to v_j .

The above locating function and huge applications of Zagreb indices motivated us to introduce two new topological indices, namely *first* and *second locating indices*, based on the locating vectors.

Definition 2. Let $G=(V,E)$ be a connected graph with a vertex set $V = \{v_1, v_2, \dots, v_n\}$ and an edge set $E(G)$. Then we define the *first* and *second locating indices* as

$$M_1^L(G) = \sum_{v_i \in V(G)} (\mathbf{v}_i)^2 \text{ and } M_2^L(G) = \sum_{v_i, v_j \in E(G)} \mathbf{v}_i \cdot \mathbf{v}_j,$$

respectively.

All graphs in this paper will be assumed simple, undirected and connected unless stated otherwise. For graph theoretical terminologies, we refer [19] to the readers.

2. Some Exact Values in Terms of Locating Indices

In this section, by considering Definition 2, we determine the first and second locating indices for the standard graphs $K_n, C_n, K_{n,m}, W_n, P_n$, and also for the join graph $G \cong G_1 + G_2$ such that G_1 and G_2 are both connected graphs with diameter 2 and G will be assumed as C_3, C_5 -free graphs.

Theorem 3. Let $G \cong K_n$ be the complete graph with a vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, where $n \geq 2$. Then $M_1^L(K_n) = n(n-1)$ and $M_2^L(K_n) = \frac{n(n-1)(n-2)}{2}$.

Proof. Let \mathbf{v}_i be a locating vector corresponding to the vertex $v_i \in V(G)$. Then $\mathbf{v}_i = (a_1, a_2, \dots, a_n)$ such that $a_i = 0$ and $a_{i+1} = 1$. Thus $(\mathbf{v}_i)^2 = n-1$. But we have total n vertices in $V(G)$, and so $M_1^L(K_n) = n(n-1)$, as required. On the other hand, for any two locating vectors \mathbf{v}_i and \mathbf{v}_j , where $i \neq j$, we definitely have $\mathbf{v}_i \cdot \mathbf{v}_j = n-2$. Hence $M_2^L(K_n) = \frac{n(n-1)(n-2)}{2}$. ■

In the next two Theorems, we investigate the cycle C_n depends on the status of n .

Theorem 4. For an even integer $n \geq 2$, let $G \cong C_n$. Then

$$M_1^L(C_n) = \frac{n(n^2 + 2)}{12} \text{ and } M_2^L(C_n) = \frac{n^2(n-2)^2}{12}.$$

Proof. By labeling the vertices of the cycle C_n as $\{v_1, v_2, \dots, v_n\}$ in the anti-clockwise direction, we obtain

$$\begin{aligned}
 \mathbf{v}_1 &= \left(0, 1, 2, 3, \dots, \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \dots, 1\right), \\
 \mathbf{v}_2 &= \left(1, 0, 1, 2, \dots, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}-1, \dots, 2\right), \\
 \mathbf{v}_3 &= \left(2, 1, 0, 1, \dots, \frac{n}{2}-2, \frac{n}{2}-1, \frac{n}{2}, \dots, 3\right), \\
 &\vdots \\
 \mathbf{v}_n &= \left(1, 2, 3, \dots, \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \frac{n}{2}-2, \dots, 0\right),
 \end{aligned}$$

and hence $\mathbf{v}_i^2 = 2\sum_{i=1}^{\frac{n}{2}} i^2 - \frac{n^2}{4}$. It is not difficult to see that each \mathbf{v}_i has the same components within different location, and so each \mathbf{v}_i^2 has the same sum as the form of $\mathbf{v}_i^2 = \frac{n(n+1)(n+2)-3n^2}{12}$. Therefore $M_1^{\mathcal{L}}(C_n) = \frac{n(n^2+2)}{12}$. In addition, by the symmetry,

$$\mathbf{v}_i \cdot \mathbf{v}_{i+1} = 2\sum_{i=2}^{\frac{n}{2}} i(i-1) = 2\left(\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)(n+1)}{6} - 1\right) - 2\left(\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2} - 1\right) = \frac{n(n-2)^2}{12}$$

which gives $M_2^{\mathcal{L}}(C_n) = \frac{n^2(n-2)^2}{12}$. ■

Theorem 5. For an odd integer $n \geq 3$, let $G \cong C_n$. Then

$$M_1^{\mathcal{L}}(C_n) = \frac{n^2(n^2-1)}{12} \quad \text{and} \quad M_2^{\mathcal{L}}(C_n) = \frac{n(n-1)(n-2)(n+3)}{12}.$$

Proof. With a similar procedure as in the proof of Theorem 4, we get

$$\begin{aligned}
 \mathbf{v}_1 &= \left(0, 1, 2, 3, \dots, \frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}-2, \dots, 1\right), \\
 \mathbf{v}_2 &= \left(1, 0, 1, 2, \dots, \frac{n-1}{2}-1, \frac{n-1}{2}, \frac{n-1}{2}-1, \dots, 2\right), \\
 \mathbf{v}_3 &= \left(2, 1, 0, 1, \dots, \frac{n-1}{2}-2, \frac{n-1}{2}-1, \frac{n-1}{2}, \dots, 3\right), \\
 &\vdots \\
 \mathbf{v}_n &= \left(1, 2, 3, \dots, \frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}-2, \dots, 0\right)
 \end{aligned}$$

which implies

$$\mathbf{v}_i^2 = 2\sum_{i=1}^{\frac{n-1}{2}} i^2 = \frac{n(n^2-1)}{12},$$

and so $M_1^{\mathcal{L}}(C_n) = \frac{n^2(n^2-1)}{12}$. Also, by the symmetry,

$$\begin{aligned}
 v_i \cdot v_{i+1} &= 2 \sum_{i=2}^{\frac{n-1}{2}} i(i-1) + \frac{(n-1)^2}{4} \\
 &= \left(2 \frac{\frac{n-1}{2} \left(\frac{n-1}{2} + 1 \right) \left(2 \frac{n-1}{2} + 1 \right)}{6} - 1 - \left(2 \frac{\frac{n-1}{2} \left(\frac{n-1}{2} + 1 \right)}{2} - 1 \right) + \frac{(n-1)^2}{4} \right) \\
 &= \frac{(n-1)(n-2)(n+3)}{12}
 \end{aligned}$$

which gives the exact value of $M_2^L(C_n)$ as depicted in the statement of theorem. ■

Now we will take into account the complete bipartite graphs to determine the locating indices.

Theorem 6. Let $G \cong K_{n,m}$, where $1 \leq n \leq m$. Then

$$M_1^L(K_{n,m}) = 4(n^2 + m^2) - 4(n + m) + 2nm \text{ and } M_2^L(K_{n,m}) = 2nm(n + m - 2).$$

Proof. For all $1 \leq i \leq n$ and $1 \leq j \leq m$, by labeling the adjacent vertices v_i and v_{n+j} of $K_{n,m}$, the locating vectors v_i of v_i are given by:

$$\begin{aligned}
 v_1 &= \left(0, \overbrace{2, \dots, 2}^{\frac{n-1}{2}}, \overbrace{1, 1, \dots, 1}^m \right), v_2 = \left(2, 0, \overbrace{2, \dots, 2}^{\frac{n-2}{2}}, \overbrace{1, 1, \dots, 1}^m \right), \\
 v_3 &= \left(2, 2, 0, \overbrace{2, \dots, 2}^{\frac{n-3}{2}}, \overbrace{1, 1, \dots, 1}^m \right), \dots, v_n = \left(2, \dots, 2, 0, \overbrace{1, 1, \dots, 1}^m \right), \\
 v_{n+1} &= \left(\overbrace{1, \dots, 1}^n, 0, \overbrace{2, 2, \dots, 2}^{m-1} \right), v_{n+2} = \left(\overbrace{1, \dots, 1}^n, 2, 0, \overbrace{2, \dots, 2}^{m-2} \right), \dots, \\
 v_{n+m} &= \left(\overbrace{1, \dots, 1}^n, \overbrace{2, \dots, 2}^{m-1}, 0 \right).
 \end{aligned}$$

In here, for any $i = 1, 2, \dots, n$, we have $v_i^2 = 4(n-1) + m$ and for any $i = n+1, n+2, \dots, n+m$, we get $v_i^2 = 4(m-1) + n$. Therefore

$$\begin{aligned}
 M_1^L(K_{n,m}) &= n(4(n-1) + m) + m(4(m-1) + n) \\
 &= 4(n^2 + m^2) - 4(n + m) + 2nm.
 \end{aligned}$$

On the other hand, for any two consecutive locating vertices v_i, v_{i+1} in $K_{n,m}$, since $v_i \cdot v_{i+1} = 2(n + m - 2)$, we obtain $M_2^L(K_{n,m}) = 2nm(n + m - 2)$. ■

Since the following consequences of Theorem 6 are very special cases and clear, we will omit their proofs.

Corollary 7. Let $G \cong K_{n,n}$, where $n \geq 1$. Then $M_1^L(K_{n,n}) = 2n(5n - 4)$ and $M_2^L(K_{n,n}) = 2n^2(2n - 2)$.

Corollary 8. Let $G \cong K_{1,m}$. Then $M_1^L(K_{1,m}) = 2m(2m - 1)$ and $M_2^L(K_{1,m}) = 2m(m - 1)$.

The case for wheel graphs will be investigated in the following result.

Theorem 9. Let us consider G as the wheel graph W_n ($n \geq 4$) with $n+1$ vertices. Then we have $M_1^L(W_n) = 4n(n - 2)$ and $M_2^L(W_n) = n(6n - 15)$.

Proof. With a similar approximation as in the previous results, by labeling the

vertices of $V(G)$ in the anticlockwise direction as $v_1, v_2, \dots, v_n, v_{n+1}$ such that v_{n+1} is the center of the wheel, we obtain

$$\begin{aligned} v_1 &= \left(0, \underbrace{1, 2, 2, \dots, 2}_{n-3}, 1, 1 \right), v_2 = \left(1, 0, \underbrace{1, 2, 2, \dots, 2}_{n-3}, 1 \right), \\ v_3 &= \left(2, 1, 0, \underbrace{1, 2, \dots, 2}_{n-2}, 1 \right), \dots, v_n = \left(1, \underbrace{2, 2, \dots, 2}_{n-3}, 1, 0, 1 \right), \\ v_{n+1} &= \left(\underbrace{1, 1, \dots, 1}_n, 0 \right). \end{aligned}$$

Now for any locating vector v_i corresponding to a vertex v_i ($i \in \{1, 2, \dots, n\}$), we have $v_i^2 = 4n + 9$ and $v_{n+1}^2 = n$. Hence $M_1^L(W_n) = 4n(n - 2)$.

For $M_2^L(W_n)$, by labeling the vertices as above, we have

$$\begin{aligned} v_1 &= \left(\mathbf{0}, \underbrace{1, 2, 2, \dots, 2}_{n-3}, 1, \mathbf{1} \right), v_2 = \left(\mathbf{1}, \mathbf{0}, \underbrace{1, 2, 2, \dots, 2}_{n-3}, 1 \right), \\ v_3 &= \left(2, \mathbf{1}, \mathbf{0}, \underbrace{1, 2, \dots, 2}_{n-2}, 1 \right), v_4 = \left(2, 2, \mathbf{1}, \mathbf{0}, \underbrace{1, 2, \dots, 2}_{n-1}, 1 \right), \\ v_5 &= \left(2, 2, 2, \mathbf{1}, \mathbf{0}, \underbrace{1, 2, \dots, 2}_n, 1 \right), \dots, v_n = \left(1, \underbrace{2, 2, \dots, 2}_{n-3}, 2, \mathbf{1}, \mathbf{0}, \mathbf{1} \right), \\ v_{n+1} &= \left(\underbrace{1, 1, \dots, 1}_n, 0 \right). \end{aligned}$$

Bearing in mind the permutation of components $\mathbf{1}, \mathbf{0}, \mathbf{1}$ in each vector v_i , where $i = 1, 2, \dots, n$, it is easy to see that any two adjacent vertices v_i and v_j ($i, j \in \{1, 2, \dots, n\}$) satisfy $v_i \cdot v_j = 4n - 11$ and $v_i \cdot v_{i+1} = 2n - 4$ for $i = 1, 2, \dots, n$. Hence $M_2^L(W_n) = n(6n - 15)$. ■

The result for determining of locating indices on path graphs can be given as in the following.

Theorem 10. Let $G \cong P_n$ ($n \geq 3$). Then

$$M_1^L(P_n) = \sum_{j=1}^{n-1} \frac{(n-j)(n-j+1)(2n-2j+1)}{3},$$

and

$$M_2^L(P_n) = 2 \sum_{j=1}^{n-1} \frac{(n-j)(n-j+1)(n-j-1)}{3}.$$

Proof. Assume that G is the graph P_n ($n \geq 3$). By labeling the vertices from left to right as v_1, v_2, \dots, v_n according to the locating function, the corresponding vector for each vertex $v_i \in V(G)$ ($i = 1, \dots, n$) will be the form of

$$\begin{aligned} v_1 &= (0, 1, 2, 3, \dots, n-1), v_2 = (1, 0, 1, 2, \dots, n-2), \dots, \\ v_{n-1} &= (n-2, n-1, \dots, 0, 1), v_n = (n-1, n-2, n-3, \dots, 0). \end{aligned}$$

By applying the symmetry on components between the vector pairs v_1, v_n and v_2, v_{n-1}, \dots and so on, we can see that

$$M_1^L(P_n) = 2 \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} i^2 = \sum_{j=1}^{n-1} \frac{(n-j)(n-j+1)(2n-2j+1)}{3}.$$

For $M_2^L(P_n)$, we see that

$$v_1 \cdot v_2 = (0 \cdot 1) + (1 \cdot 0) + \dots + (n-1)(n-2) = \sum_{i=1}^{n-1} i(i-1),$$

$$v_2 \cdot v_3 = (1 \cdot 2) + (0 \cdot 1) + \dots + (n-2)(n-3) = \sum_{i=1}^{n-1} i(i-1),$$

$$v_3 \cdot v_4 = \sum_{i=1}^{n-1} i(i-1),$$

⋮

However, by the symmetry between the components of the vectors as mentioned above, we get

$$M_2^L(P_n) = 2 \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} i(i-1) = 2 \sum_{j=1}^{n-1} \left(\sum_{i=1}^{n-j} i^2 - \sum_{i=1}^{n-j} i \right)$$

which can be rewritten as in the form

$$\begin{aligned} M_2^L(P_n) &= 2 \sum_{j=1}^{n-1} \left(\frac{(n-j)(n-j+1)(2n-2j+1)}{6} - \frac{(n-j)(n-j+1)}{2} \right) \\ &= 2 \sum_{j=1}^{n-1} \frac{(n-j)(n-j+1)(n-j-1)}{3}. \end{aligned}$$

This complete the proof. ■

It is known that from the elementary textbooks the join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2 . In the following theorem we find first and second locating indices for the join graph G .

Theorem 11. *Let $G \cong G_1 + G_2$ such that G_1 and G_2 are both connected graphs with diameter 2 and G is a C_3 or C_5 -free graph. Assume that G_1 has n_1 vertices and m_1 edges while G_2 has n_2 vertices and m_2 edges. Then*

$$M_1^L(G) = 2n_1n_2 + 4(n_1^2 + n_2^2 - n_1 - n_2) - 6(m_1 + m_2),$$

and

$$M_2^L(G) = 2(m_1n_1 + m_2n_2) - 4(m_1 + m_2) + 2n_1n_2(n_1 + n_2 - 2).$$

Proof. Assume that G satisfies the conditions in the statement of theorem. Let us label the vertices of the graph G as

$$v_1, v_2, \dots, v_{n_1}, v_{n_1+1}, v_{n_1+2}, \dots, v_{n_1+n_2},$$

where $v_1, v_2, \dots, v_{n_1} \in V(G_1)$ and $v_{n_1+1}, v_{n_1+2}, \dots, v_{n_1+n_2} \in V(G_2)$. Also let v be the locating vector corresponding to the vertex v such that $v \in V(G_1)$:

$$v = \left(\underbrace{0, 1, \dots, 1}_{\deg(v)}, \underbrace{2, \dots, 2}_{n_1 - \deg(v) - 1}, \underbrace{1, \dots, 1}_{n_2} \right).$$

Then $v^2 = n_2 + 4n_1 - 4 - 3 \deg(v)$.

Similarly, for any vertex $w \in V(G_2)$, the locating vector w corresponding to w :

$$w = \left(\underbrace{1, \dots, 1}_{n_1}, \underbrace{0, 1, \dots, 1}_{\deg(w)}, \underbrace{2, \dots, 2}_{n_2 - \deg(w) - 1} \right).$$

So $w^2 = n_1 + 4n_2 - 4 - 3 \deg(w)$. Therefore, by the above equalities on v^2 and w^2 , we obtain

$$\begin{aligned} M_1^L(G) &= n_1(n_2 + 4n_1 - 4) - 6m_1 + n_2(n_1 + 4n_2 - 4) - 6m_2 \\ &= 2n_1n_2 + 4(n_1^2 + n_2^2 - n_1 - n_2) - 6(m_1 + m_2). \end{aligned}$$

Now, let us make partition to the set of vertices of G as

$$A = \{u \cdot v : u, v \in V(G_1)\},$$

$$B = \{u \cdot v : u, v \in V(G_2)\},$$

$$C = \{u \cdot v : u \in V(G_1), v \in V(G_2)\}.$$

Hence $M_2^L(G)$ can be written as $\sum_{u,v \in A} u \cdot v + \sum_{u,v \in B} u \cdot v + \sum_{u,v \in C} u \cdot v$. To get $\sum_{u,v \in A} u \cdot v$ for any two adjacent vertices $u, v \in V(G_1)$, let us consider

$$\begin{aligned} u &= \left(\underbrace{0, 1, \dots, 1}_{\deg(u)}, \underbrace{2, \dots, 2}_{n_1 - \deg(u) - 1}, \underbrace{1, \dots, 1}_{n_2} \right) \\ v &= \left(\underbrace{1, 0, 2, \dots, 2}_{\deg(u) - 1}, \underbrace{1, \dots, 1}_{n_1 - \deg(u) - 1}, \underbrace{1, \dots, 1}_{n_2} \right). \end{aligned}$$

We then have

$$u \cdot v = 2(\deg(u) - 1) + 2(n_1 - \deg(u) - 1) + n_2 = n_2 + 2n_1 - 4$$

which implies $\sum_{u,v \in A} u \cdot v = m_1(n_2 + 2n_1 - 4)$. With a similar calculation, we get $\sum_{u,v \in B} u \cdot v = m_2(n_1 + 2n_2 - 4)$.

Next, we need to calculate $\sum_{u,v \in C} u \cdot v$. To do that let us take $u \in V(G_1)$ and $v \in V(G_2)$, and then labeling as

$$\begin{aligned} u &= \left(\underbrace{0, 1, \dots, 1}_{\deg(u)}, \underbrace{2, \dots, 2}_{n_1 - \deg(u) - 1}, \underbrace{1, \dots, 1}_{n_2} \right) \\ v &= \left(\underbrace{1, \dots, 1}_{\deg(u) + 1}, \underbrace{1, \dots, 1}_{n_1 - \deg(u) - 1}, \underbrace{0, 1, \dots, 1}_{\deg(v)}, \underbrace{2, \dots, 2}_{n_2 - \deg(v) - 1} \right). \end{aligned}$$

Hence we get

$$\begin{aligned} u \cdot v &= \deg(u) + 2(n_1 - \deg(u) - 1) + \deg(v) + 2(n_2 - \deg(v) - 1) \\ &= 2(n_1 + n_2) - 4 - (\deg(u) - \deg(v)) \end{aligned}$$

and so $\sum_{u,v \in C} u \cdot v = n_1n_2(2(n_1 + n_2) - 4) - 2n_2m_1 - 2n_1m_2$.

After all above calculations, we finally obtain

$$\begin{aligned}
 M_2^L(G) &= m_1(n_2 + 2n_1 - 4) + m_2(n_1 + 2n_2 - 4) \\
 &\quad + n_1n_2(2(n_1 + n_2) - 4) - 2n_2m_1 - 2n_1m_2 \\
 &= 2(m_1n_1 + m_2n_2) - 4(m_1 + m_2) + 2n_1n_2(n_1 + n_2 - 2).
 \end{aligned}$$

Hence the result. ■

3. Locating Indices of Firefly Graphs

We recall that a firefly graph $F_{s,t,n-2s-2t-1}$ ($s \geq 0, t \geq 0$ and $n - 2s - 2t - 1 \geq 0$) is a graph of order n that consists of s triangles, t pendant paths of length 2 and $n - 2s - 2t - 1$ pendent edges that are sharing a common vertex (cf. [20]). Let \mathcal{F}_n be the set of all firefly graphs $F_{s,t,n-2s-2t-1}$. Note that \mathcal{F}_n contains the stars $S_n (\cong F_{0,0,n-1})$, stretched stars ($\cong F_{0,t,n-2t-1}$), friendship graphs ($\cong F_{\frac{n-1}{2},0,0}$) and butterfly graphs ($\cong F_{s,0,n-2s-1}$).

In the next theorem we present the first and second locating indices for the firefly graph. In our calculations, for simplicity, we denote $n - 2s - 2t - 1$ by a single letter l .

Theorem 12. Let $G \cong F_{s,t,l}$ ($s \geq 0, t \geq 0$ and $l \geq 0$) be a firefly graph of order n . Then

$$M_1^L(G) = 4l^2 + 16ls + 26lt - 2l + 16s^2 + 52st - 10s + 38t^2 - 28t,$$

and

$$M_2^L(G) = 2l^2 + 16ls + 13lt - 2l + 24s^2 + 52st - 20s + 22t^2 - 17t.$$

Proof. Let $G \cong F_{s,t,l}$ ($s \geq 0, t \geq 0$ and $l \geq 0$) is a firefly graph of order n . Let us label the vertices with clockwise direction as

$$\begin{aligned}
 &v_1, v_2, \dots, v_{2s+1}, v_{2s+2}, v_{2s+3}, \dots, v_{2s+l+1}, v_{2s+l+2}, \\
 &v_{2s+l+3}, \dots, v_{2s+l+t+1}, v_{2s+l+t+2}, v_{2s+l+t+3}, \dots, v_{2s+2t+l+1},
 \end{aligned}$$

where v_1 is the center of the firefly graph and

$$\begin{aligned}
 &\underbrace{v_2, v_3, \dots, v_{2s+1}}_{2s} : \text{vertices of triangles,} \\
 &\underbrace{v_{2s+2}, v_{2s+3}, \dots, v_{2s+l+1}}_l : \text{vertices of pendent edges,} \\
 &\underbrace{v_{2s+l+2}, v_{2s+l+3}, \dots, v_{2s+l+t+1}}_t : \text{vertices of pendent path of length 1,} \\
 &\underbrace{v_{2s+l+t+2}, v_{2s+l+t+3}, \dots, v_{2s+2t+l+1}}_t : \text{vertices of pendent path of length 2.}
 \end{aligned}$$

Now we calculate the corresponding vectors v_i for each vertex $v_i \in V(G)$, where $i = 1, 2, \dots, 2s + 2t + l + 1$, as in the following:

$$\begin{aligned}
 v_1 &= \left(0, \underbrace{1, 1, \dots, 1}_{2s}, \underbrace{1, 1, \dots, 1}_l, \underbrace{1, 1, \dots, 1}_t, \underbrace{2, 2, \dots, 2}_t \right), \\
 v_2 &= \left(1, 0, \underbrace{1, 2, 2, \dots, 2}_{2s-2}, \underbrace{2, 2, \dots, 2}_l, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right),
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v}_3 &= \left(1, \underbrace{1, 0, 2, 2, \dots, 2}_{2s-2}, \underbrace{2, 2, \dots, 2}_l, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right), \\
 \mathbf{v}_4 &= \left(1, 2, 2, 0, 1, \underbrace{2, 2, \dots, 2}_{2s-4}, \underbrace{2, 2, \dots, 2}_l, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right), \\
 \mathbf{v}_5 &= \left(1, 2, 2, 1, 0, \underbrace{2, 2, \dots, 2}_{2s-4}, \underbrace{2, 2, \dots, 2}_l, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right), \\
 &\vdots \\
 \mathbf{v}_{2s+1} &= \left(1, \underbrace{2, 2, \dots, 2}_{2s-2}, 1, 0, \underbrace{2, 2, \dots, 2}_l, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right), \\
 \mathbf{v}_{2s+2} &= \left(1, \underbrace{2, 2, \dots, 2}_{2s}, 0, \underbrace{2, 2, \dots, 2}_{l-1}, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right), \\
 \mathbf{v}_{2s+3} &= \left(1, \underbrace{2, 2, \dots, 2}_{2s}, 2, 0, \underbrace{2, 2, \dots, 2}_{l-2}, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right), \\
 &\vdots \\
 \mathbf{v}_{2s+l+1} &= \left(1, \underbrace{2, 2, \dots, 2}_{2s}, \underbrace{2, 2, \dots, 2}_{l-1}, 0, \underbrace{2, 2, \dots, 2}_t, \underbrace{3, 3, \dots, 3}_t \right) \\
 \mathbf{v}_{2s+l+2} &= \left(1, \underbrace{2, 2, \dots, 2}_{2s}, \underbrace{2, 2, \dots, 2}_l, 0, \underbrace{2, 2, \dots, 2}_{t-1}, \underbrace{1, 3, 3, \dots, 3}_{t-1} \right) \\
 &\vdots \\
 \mathbf{v}_{2s+l+t+1} &= \left(1, \underbrace{2, 2, \dots, 2}_{2s}, \underbrace{2, 2, \dots, 2}_l, \underbrace{2, 2, \dots, 2}_{t-1}, 0, \underbrace{3, 3, \dots, 3}_{t-1}, 1 \right), \\
 \mathbf{v}_{2s+l+t+2} &= \left(2, \underbrace{3, 3, \dots, 3}_{2s}, \underbrace{3, 3, \dots, 3}_l, 1, \underbrace{3, 3, \dots, 3}_{t-1}, 0, \underbrace{4, 4, \dots, 4}_{t-1} \right), \\
 \mathbf{v}_{2s+l+t+3} &= \left(2, \underbrace{3, 3, \dots, 3}_{2s}, \underbrace{3, 3, \dots, 3}_l, 3, 1, \underbrace{3, 3, \dots, 3}_{t-2}, 0, \underbrace{4, 4, \dots, 4}_{t-2} \right), \\
 &\vdots \\
 \mathbf{v}_{2s+l+2t+1} &= \left(2, \underbrace{3, 3, \dots, 3}_{2s}, \underbrace{3, 3, \dots, 3}_l, \underbrace{3, 3, \dots, 3}_{t-1}, 1, \underbrace{4, 4, \dots, 4}_{t-1}, 0 \right).
 \end{aligned}$$

Suppose that $A, B, C, D \subset V(G)$ such that

$$\begin{aligned}
 A &= \{v_2, v_3, \dots, v_{2s+1}\}, C = \{v_{2s+l+2}, v_{2s+l+3}, \dots, v_{2s+l+t+1}\}, \\
 B &= \{v_{2s+2}, v_{2s+3}, \dots, v_{2s+l+1}\}, D = \{v_{2s+l+t+2}, v_{2s+l+t+3}, \dots, v_{2s+2t+l+1}\}.
 \end{aligned}$$

Therefore we can write

$$M_1^L(G) = \sum_{v \in A} v^2 + \sum_{v \in B} v^2 + \sum_{v \in C} v^2 + \sum_{v \in D} v^2.$$

For the calculation of $\sum_{v \in A} v^2$, we have the cases $v_i^2 = 2s + l + t + 4t = 2s + l + 5t$ and

$$v_i^2 = 2 + 4(2s - 2) + 4l + 4t + 9t = 4l + 8s + 13t - 6,$$

where $i = 2, 3, \dots, 2s + 1$. Hence

$$\begin{aligned} \sum_{v \in A} v^2 &= 2s + l + 5t + 2s(4l + 8s + 13t - 6) \\ &= 2s + l + 5t + 8sl + 16s^2 + 26st - 12s \\ &= l - 10s + 5t + 8ls + 26st + 16s^2. \end{aligned}$$

On the other hand, for the calculation of $\sum_{v \in B} v^2$, we have

$$v_i^2 = 1 + 4(2s) + 4(l - 1) + 4t + 9t = 4l + 8s + 13t - 3,$$

where for $i = 2s + 1, 2s + 2, \dots, 2s + l + 1$. Thus

$$\sum_{v \in B} v^2 = l(4l + 8s + 13t - 3) = 4l^2 + 8ls + 13lt - 3l.$$

Thirdly to calculate $\sum_{v \in C} v^2$, we have

$$v_i^2 = 2 + 4(2s) + 4l + 4(t - 1) + 9(t - 1) = 4l + 8s + 13t - 11,$$

where $i = 2s + l + 2, 2s + l + 3, \dots, 2s + l + t + 1$, and so

$$\sum_{v \in C} v^2 = t(4l + 8s + 13t - 11) = 4tl + 8ts + 13t^2 - 11t.$$

Finally, for the case of $\sum_{v \in D} v^2$, we get

$$v_i^2 = 3 + 9(2s) + 9l + 9(t - 1) + 16(t - 1) = 9l + 18s + 25t - 22,$$

where $i = v_{2s+l+t+2}, v_{2s+l+t+3}, \dots, v_{2s+2t+l+1}$. This gives

$$\sum_{v \in D} v^2 = t(9l + 18s + 25t - 22) = 9tl + 18ts + 25t^2 - 22t.$$

By collecting all above calculations, we obtain

$$\begin{aligned} M_1^L(G) &= \sum_{v \in A} v^2 + \sum_{v \in B} v^2 + \sum_{v \in C} v^2 + \sum_{v \in D} v^2 \\ &= l - 10s + 5t + 8ls + 26st + 16s^2 + 4l^2 + 8ls + 13lt - 3l \\ &\quad + 4tl + 8ts + 13t^2 - 11t + 9tl + 18ts + 25t^2 - 22t \\ &= 4l^2 + 16ls + 26lt - 2l + 16s^2 + 52st - 10s + 38t^2 - 28t, \end{aligned}$$

as required.

Before starting to calculate the index $M_2^L(G) = \sum_{v_i u_j \in E(G)} v_i \cdot u_j$, we should remind that for any two adjacent vertices u and v will be denoted by $u \approx v$. Now, let us again consider the same subsets A, B, C and D of $V(G)$. Therefore we firstly have

$$\begin{aligned} \sum_{v_i \in A} v_1 \cdot v_i &= 2s(1 + 2(2s - 2 + l + t) + 3t) = 2s(2l + 4s + 5t - 3) \\ &= 4sl + 8s^2 + 10st - 6s. \\ \sum_{v_i \in B} v_1 \cdot v_i &= l(2(2s + l - 1 + t) + 3t) = l(2l + 4s + 5t - 2) \\ &= 2l^2 + 4sl + 5lt - 2l. \\ \sum_{v_i \in C} v_1 \cdot v_i &= t(1 + 2(2s + l + t - 1) + 2t) = t(2l + 4s + 4t - 1) \\ &= 2tl + 4ts + 4t^2 - t. \end{aligned}$$

Secondly,

$$\begin{aligned}\sum_{\substack{v_i \in A - \{v_1\} \\ v_i \approx v_{i+1}}} v_i \cdot v_{i+1} &= 2s(1 + 4(2s - 2 + l + t) + 9t) = 2s(4l + 8s + 13t - 7) \\ &= 8sl + 16s^2 + 26st - 14s.\end{aligned}$$

Thirdly,

$$\begin{aligned}\sum_{\substack{v_i \in C, v_{i+t} \in D \\ v_i \approx v_{i+t}}} v_i \cdot v_{i+t} &= t(2 + 6(2s + l + t - 1) + 12(t - 1)) \\ &= t(6l + 12s + 18t - 16) = 6tl + 12ts + 18t^2 - 16t.\end{aligned}$$

Again, by collecting all above calculations, we obtain

$$\begin{aligned}M_2^L(G) &= \sum_{v_i \in A} v_i \cdot v_i + \sum_{v_i \in B} v_i \cdot v_i + \sum_{v_i \in C} v_i \cdot v_i + \sum_{\substack{v_i \in A - \{v_1\} \\ v_i \approx v_{i+1}}} v_i \cdot v_{i+1} + \sum_{\substack{u_i \in C, v_{i+t} \in D \\ v_i \approx v_{i+t}}} v_i \cdot v_{i+t} \\ &= 4sl + 8s^2 + 10st - 6s + 2l^2 + 4sl + 5lt - 2l + 2tl + 4ts + 4t^2 - t \\ &\quad + 8sl + 16s^2 + 26st - 14s + 6tl + 12ts + 18t^2 - 16t \\ &= 2l^2 + 16ls + 13lt - 2l + 24s^2 + 52st - 20s + 22t^2 - 17t.\end{aligned}$$

These all above processes complete the proof. ■

Corollary 13. 1) For any friendship graph of order n ,

$$M_1^L(G) = 4n^2 - 13n + 9 \quad \text{and} \quad M_2^L(G) = 6n^2 - 22n + 16.$$

2) For any butterfly graph of order n ,

$$M_1^L(G) = 4n^2 - 10n - 6s + 6 \quad \text{and} \quad M_2^L(G) = 8ns - 24s - 6n + 2n^2 + 4.$$

4. Conclusion

In this paper, two new topological indices based on Zagreb indices are proposed. The exact values of these new topological indices are calculated for some standard graphs and for the firefly graphs. These new indices can be used to investigate the chemical properties for some chemical compound such as drugs, bridge molecular graph etc. For the future work, instead of defining these new topological indices based on the degrees of the vertices, we can redefine them based on the degrees of the edges by defining them on the line graph of any graph. Similar calculations can be computed to indicate different properties of the graph.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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