Stability of Stochastic Logistic Model with Ornstein-Uhlenbeck Process for Cell Growth of Microorganism in Fermentation Process

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Abstract

Current research is concerned with the stability of stochastic logistic equation with Ornstein-Uhlenbeck process. First, this research proves that the stochastic logistic model with Ornstein-Uhlenbeck process has a positive solution. After that, it also introduces the sufficient conditions for stochastically stability of stochastic logistic model for cell growth of microorganism in fermentation process for positive equilibrium point by using Lyapunov function. In addition, this research establishes the sufficient conditions for zero solution as mentioned in **Appendix A** due to the cell growth of microorganism μ_{max}, which cannot be negative in fermentation process. Furthermore, for numerical simulation, current research uses the 4-stage stochastic Runge-Kutta (SRK4) method to show the reality of the results.

Keywords

Stability, Fermentation Process, Ornstein-Uhlenbeck Process, Logistic Model, Lyapunov Function, 4-Stage Stochastic Runge-Kutta Method

1. Introduction

Stochastic differential equations (SDEs) have been intensively used to model the natural phenomena in the last decades and these equations play a prominent applied role in various fields [1] [2]. Most SDEs do not have explicit solution but these equations can be solved numerically to approximate their solutions [3]. We use the SRK method to approximate the solution numerically ([3], p. 69) [4] [5] [6] [7]. Whereas, Rüemelin in [8] proposed the $\mathcal{S}$-stage stochastic Runge-Kutta explicit method. Moreover, Xiao in [9] introduced the High strong order stochastic Runge-Kutta methods for stochastic differential equations in case of

Stratonovich since our model is in Stratonovich sense.

This research deals with the stability of stochastic logistic model with Ornstein-Uhlenbeck process. This process is a batter model of Brownian motion ([10], p. 86). The Ornstein-Uhlenbeck process is used to model stock price, currency exchange rates and velocity.

The classical Brownian motion was introduced by Scottish Botanist Robert Brown in (1827). He described this motion based on random movements of pollen grains in liquid or gas [11]. This theory was further supported by Norbert Wiener (1923) who explained full mathematical theory of Brownian motion which existed as a rigorously defined mathematical objective to recognize his contribution [12]. Brownian motion is a simple continuous stochastic process which is extensively used to model phenomena in various fields such as industry, dynamic process, physics, finance and fermentation process [3] [11] [12]. This Brownian motion is the cause of instability (turbulence) in fermentation process [1].

Fermentation process converts sugar into alcohol with the help of yeast [13]. Madihah, in [14] used the logistic model to elaborate the cell growth of *Clostridium acetobutylicum* p. 262 in fermentation process proposed by Verhulst (1838). This model is inadequate to describe the cell growth of microorganism in fermentation process because fermentation process contains random fluctuations. Bahar and Mao in [2] introduced new stochastic logistic model for population dynamics. The stochastic logistic model was used in [15] to elucidate the cell growth of *Clostridium acetobutylicum* p. 262 in fermentation process. This cell growth of microorganism in fermentation process is affected by environmental noise, a batter model for modelling environmental noise is Ornstein-Uhlenbeck process. However, no specific research was conducted on stability stochastic logistic model with Ornstein-Uhlenbeck process for cell growth of microorganism in fermentation process. Nevertheless, there are some previous research works which have been done on stability of logistic equations with white noise, Lévy Jumps and other different aspects [16]-[22]. None of these researches were investigated the stability of stochastic logistic model with Ornstein-Uhlenbeck process for cell growth of microorganism in fermentation process. This research establishes the sufficient conditions for stochastic logistic equation with Ornstein-Uhlenbeck process for positive equilibrium point and zero solution by using Lyapunov function. Since, the Ornstein-Uhlenbeck process is a better model of Brwonian motion ([10], p. 86). In addition, our current study will indicate that the Ornstein-Uhlenbeck process is unfavorable for stability of cell growth in fermentation process. Moreover, we apply the SRK4 method to evaluate the numerical solution.

This paper is organized in five main sections; Introduction, Preliminaries and Models Description, Main Results, Numerical Simulation and Conclusion.

2. Preliminaries and Models Description

Throughout this paper; the notation $E\left[x(t)\right]$ is the expectation of $\left(x(t)\right)$, T shows the terminal time, t is time, t_0 illustrates the initial time, $x(t)$ corresponds to the highest cell size, x_0 is initial cell size, $\mu_{\max}$ denotes the maxi-

mum specific growth rate, $x_{\max}$ illustrates a carrying capacity, σ indicates the random fluctuation, $W(t)$ shows Brownian motion and b is a constant.

The simplest mathematical model to illustrate the exponential phase for cell growth in fermentation process is:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \mu_{\max} \times x(t). \tag{1}$$

The solution of Equation (1) is:

$$x(t) = x_0 \mathrm{e}^{t \times \mu_{\max}}, \tag{2}$$

where t is time, x_0 corresponds to the initial cell size and $\mu_{\max}$ denotes the maximum specific growth rate. If $\mu_{\max} > 0$, Equation (1) is strongly ascending, and if, $\mu_{\max} < 0$, strongly descending. Hence, Equation (1) is not adequate to model stationary phase in fermentation. Therefore, Pierre Francois Verhulst (1838) introduced new model containing stationary phase in fermentation process. Thus, the exponential growth model (1) is augmented by the inclusion of multiplicative factor of $1 - \dfrac{x(t)}{x_{\max}}$. Hence, the logistic ordinary differential equation is:

$$\mathrm{d}x(t) = \mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t, \tag{3}$$

where $x_{\max}$ is a carrying capacity of a microbial species and Equation (3) can be solved analytically by determining the solution is:

$$x(t) = \frac{x_{\max}x_0 \mathrm{e}^{\mu_{\max}t}}{x_{\max} + x_0\left(-1 + \mathrm{e}^{\mu_{\max}t}\right)}, \tag{4}$$

where $x(t)$ is the highest cell size. Madihah, in [14] used Equation (3) to elaborate cell growth of *Clostridium acetobutylicum* p. 262 in batch fermentation process. Murray in [23] and May in [24] were proved that Equation (3) is stable. This model is inadequate to describe the cell growth of *Clostridium acetobutylicum* p. 262 in fermentation process because fermentation process suffers from random fluctuation. Bahar and Mao in [2] introduced new stochastic logistic equation to model uncontrolled fluctuation in stationary phase at fermentation process by using new perturbation very randomly which is:

$$\rho \to \rho + \sigma\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \tag{5}$$

where $\rho = \dfrac{\mu_{\max}}{x_{\max}}$ is diffusion coefficient. Model (3) with perturbation (5) is:

$$\mathrm{d}x(t) = \mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t + \sigma x^2(t)\,\mathrm{d}W(t),\ t \in [0, T], \tag{6}$$

where $W(t)$ is m-dimensional Weiner process and σ is corresponding to the random fluctuation. Bazli, in [15] used Equation (6) to evaluate the cell growth of *Clostridium acetobutylicum* p. 262 in fermentation process.

Remarks 2.1: Liu *et al.*, in [21] studied the stability of stochastic logistic with

white noise which cannot model random fluctuations in stationary phase. Therefore, this research introduces new stochastic logistic model to cope with this problem and established the sufficient condition for positive equilibrium point. The new model is Equation (6) with Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process is:

$$y''(t) = -by'(t) + \mathrm{d}W(t) \tag{7}$$

where $y(t)$ stands for positive Brownian motion at time t, $b > 0$ shows the coefficient friction, σ indicates the diffusion coefficient and $W(t)$ is white noise.

By substituting

$$y'(t) = W(t), \tag{8}$$

into Equation (7) hence, the Ornstein-Uhlenbeck process becomes;

$$\mathrm{d}W(t) = -bW(t)\mathrm{d}t + \sigma \mathrm{d}W(t). \tag{9}$$

Substituting Equation (9) into Equation (6) yields

$$\mathrm{d}x(t) = \mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\mathrm{d}t - \sigma x^2(t)bW(t)\mathrm{d}t + x^2(t)\sigma^2\mathrm{d}W(t). \tag{10}$$

Equation (10) is stochastic logistic model with Ornstein-Uhlenbeck process. □

3. Main Results

First, it is needed to prove that Equation (10) has a unique positive solution and later on we will pay attention on stability.

Theorem 3.1: For all $t \geq 0$ and for any $0 < x(0) = x_0$ Equation (10) has a unique positive solution.

Proof: For any given initial value $x_0 \in R_+^n$, the coefficients of Equation (10) are locally Lipschitz continuous. Hence, there is a unique locally solution $x(t), t \in [0, i_e)$ Where i_e shows the explosion time Arnold in [25] and Friedman [26]. To show that the $x(t), t \in [0, i_e)$ is a unique positive solution. So, it is really necessary to determine that $i_e \to \infty$. Let $k_0 \in \left[\frac{1}{k_0}, k_0\right], k_0 > 0$ and $k > k_0$ is sufficiently large for each component of x_0, the stopping time for every integer $k > k_0$ is:

$$t_k = \inf\left\{t \in [0, i_e) : x(t) \text{ is not in } \left(\frac{1}{k}, k\right)\right\}.$$

k_0 is sufficiently large means that the t_k is increasing as $k \to \infty$. Hence, $t_\infty = \lim_{k \to \infty} t_k$, whereas $t_\infty \leq i_e$.

If we can show that $i_\infty = \infty$ a.s., then $i_\infty = \infty$ a.s. and $x(t) \in R_+^n$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $i_\infty = \infty$ a.s. For if this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0,1)$ such that

$$P\{i_\infty \leq T\} \geq \epsilon.$$

Therefore, there exist an integer $k_1 \geq k_0$

$$P\{i_\infty \leq T\} \geq \epsilon, \ \forall \ k \geq k_1 \tag{11}$$

Define $V\big(x(t)\big) = \left(\sqrt{x(t)} - 1 - \dfrac{1}{2}\ln x(t) \right)$ for all $x(t) \in [0, i_e)$ and the nonnegativity of this function can be seen from

$$\sqrt{u} - 1 - \frac{1}{2}\ln u$$

Itô ([27], p. 95) [28] [29] [30] ([31], p. 54) formula shows that

$$
\begin{aligned}
\mathrm{d}\big(V\big(x(t)\big)\big) = & \underbrace{\frac{1}{2\sqrt{x(t)}} - \frac{1}{2x(t)}\left[\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t \right]}_{A} \\
& - \underbrace{\left[\sigma x^2(t) b W(t)\,\mathrm{d}t + \sigma^2 x^2(t)\,\mathrm{d}W(t) \right]}_{A} \\
& + \underbrace{0.5\left[\frac{0.25}{\sqrt{x^3(t)}} - \frac{1}{2x^2(t)}\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t \right)\right.}_{B} \\
& \underbrace{\left. - \sigma x^2(t) b W(t)\,\mathrm{d}t + \sigma^2 x^2(t)\,\mathrm{d}W(t) \right]^2}_{B}
\end{aligned}
\tag{12}
$$

To simplify A and B separately yields

$$
\begin{aligned}
A = & \frac{1}{2\sqrt{x(t)}}\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t - \frac{1}{2\sqrt{x(t)}}\sigma x^2(t) b W(t)\,\mathrm{d}t \\
& + \frac{1}{2\sqrt{x(t)}}\sigma^2 x^2(t)\,\mathrm{d}W(t) - \frac{1}{2x(t)}\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t \\
& + \frac{1}{2x(t)}\sigma x^2(t) b W(t)\,\mathrm{d}t - \frac{1}{2x(t)}\sigma^2 x^2(t)\,\mathrm{d}W(t)
\end{aligned}
$$

$$
\begin{aligned}
B = & \frac{0.25}{\sqrt{x^3(t)}} - \frac{1}{2x^2(t)}\left[\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t \right)^2 \right. \\
& + 2\left(\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)\,\mathrm{d}t \right)\left(-\sigma x^2(t) b W(t)\,\mathrm{d}t + \sigma^2 x^2(t)\,\mathrm{d}W(t) \right) \\
& \left. + \left(-\sigma x^2(t) b W(t)\,\mathrm{d}t + \sigma^2 x^2(t)\,\mathrm{d}W(t) \right)^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
B = & \frac{0.25}{\sqrt{x^3(t)}} - \frac{1}{2x^2(t)}\left[\left(\mu_{\max}^2 x^2(t)\,\mathrm{d}t \times \mathrm{d}t - 2\frac{\mu_{\max}^2}{x_{\max}}x^2(t)\,\mathrm{d}t \times \mathrm{d}t \right.\right. \\
& \left. + \left(\frac{\mu_{\max}}{x_{\max}}x(t)\,\mathrm{d}t \right)^2 \right) + 2\left(\mu_{\max}x(t)\,\mathrm{d}t - \frac{\mu_{\max}}{x_{\max}}x^2(t)\,\mathrm{d}t \right) \\
& \times \left(-\sigma x^2(t) b W(t)\,\mathrm{d}t + \sigma^2 x^2(t)\,\mathrm{d}W(t) \right) + \left(-\sigma x^2(t) b W(t)\,\mathrm{d}t \right)^2 \\
& \left. + 2\left(-\sigma x^2(t) b W(t)\,\mathrm{d}t \right)\left(\sigma^2 x^2(t)\,\mathrm{d}W(t) \right) + \left(\sigma^2 x^2(t)\,\mathrm{d}W(t) \right)^2 \right]
\end{aligned}
$$

An application of these facts $dt \times dt = 0, dt \times dw(t) = 0$ and $dW(t) \times dW(t) = dt$ ([32], p. 87) and somewhat lengthy calculation we have

$$B = \frac{0.125}{\sqrt{x^3(t)}} - \frac{0.25}{x^2(t)}\left(\sigma^4 x^4(t)dt\right)$$

$$\Rightarrow B = \frac{0.125\sigma^4 x^4(t)dt}{\sqrt{x^3(t)}} - \frac{0.25\sigma^4 x^4(t)dt}{x^2(t)}$$

By substituting the value of A and B into Equation (12) yields

$$d\left(V\left(x(t)\right)\right) = \left[\frac{1}{2\sqrt{x(t)}}\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt - \frac{1}{2\sqrt{x(t)}}\sigma x^2(t)bW(t)dt\right.$$
$$+ \frac{1}{2\sqrt{x(t)}}\sigma^2 x^2(t)dW(t) - \frac{1}{2x(t)}\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)dt$$
$$\left. + 0.5\sigma x(t)bW(t)dt - 0.5\sigma^2 x(t)dW(t)\right]$$
$$+ \frac{0.125\sigma^4 x^4(t)dt}{\sqrt{x^3(t)}} - \frac{0.25\sigma^4 x^4(t)dt}{x^2(t)} \tag{13}$$

$$d\left(V\left(x(t)\right)\right) = \left[0.5x^{0.5}(t)\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right) - 0.5x^{-0.5}(t)\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)\right.$$
$$- 0.5\sigma x(t)bW(t) + 0.5\sigma x(t)bW(t) + 0.125\sigma^4 x^{2.5}(t)$$
$$\left. - 0.125\sigma^4 x^2(t)\right]dt - 0.5\sigma^2 x^2(t)dW(t)dt - 0.5\sigma^2 x(t)dW(t) \tag{14}$$

$$d\left(V\left(x(t)\right)\right) = \left[0.5x^{0.5}(t)\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right) - 0.5x^{-0.5}(t)\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)\right.$$
$$\left. + 0.5\sigma x(t)bW(t)\left(1 - x^{0.5}(t)(t)\right) + 0.25\sigma^4 x^2\left(0.5x(t) - 1\right)\right]dt \tag{15}$$
$$+ 0.5\sigma^2 x(t)dW(t)\left(x^{0.5}(t) - 1\right)$$

Worth mentioning,

$$0.5x^{0.5}(t)\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right) - 0.5x^{-0.5}(t)\mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)$$
$$+ 0.25\sigma^4 x^2\left(0.5x(t) - 1\right) \tag{16}$$

$$\leq 0.5x^{0.5}(t)\breve{\mu}_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right) - 0.5x^{-0.5}(t)\breve{\mu}_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)$$
$$+ 0.25\sup_{t \in R^+}\left|\sigma^4\right|x^2(t)\left(0.5x(t) - 1\right). \tag{17}$$

If $x(t) \leq x_{\max}$ then $\left(1 - \frac{x(t)}{x_{\max}}\right) \geq 0$, it follows the boundedness σ. Therefore, there is nonnegative number Q_1 which independent of x and t.

If $0 < x(t) < x_{\max}$, obviously there is a positive number Q_2 which is inde-

pendent of x and t, and it follow bounded σ, such that

$$
\begin{aligned}
& 0.5x^{0.5}(t)\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)-0.5x^{-0.5}(t)\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right) \\
& +0.25\sigma^4 x^2\left(0.5x(t)-1\right) \\
& \leq 0.5x^{0.5}(t)\hat{\mu}_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)-0.5x^{-0.5}(t)\hat{\mu}_{\max}\left(1-\frac{x(t)}{x_{\max}}\right) \\
& +0.25\sup_{t\in R^+}\left|\sigma^4\right|x^2\left(0.5x(t)-1\right)\leq Q_2.
\end{aligned}
\tag{18}
$$

On the other hand, we show that there is a nonnegative number Q which independent of x and t such that

$$
\begin{aligned}
& 0.5x^{0.5}(t)\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)-0.5x^{-0.5}(t)-\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right) \\
& +0.25\sigma^4 x^2\left(0.5x(t)-1\right)\leq Q.
\end{aligned}
\tag{19}
$$

By substituting the inequality (19) into Equation (13) yields

$$
\begin{aligned}
\mathrm{d}\left(V\left(x(t)\right)\right) \leq & Q\mathrm{d}t+0.5\sigma x(t)bW(t)\left(1-x^{0.5}(t)(t)\right)\mathrm{d}t \\
& +0.5\sigma^2 x(t)\mathrm{d}W(t)\left(x^{0.5}(t)-1\right).
\end{aligned}
\tag{20}
$$

where $W(t)$ is Brownian motion, and taking integral and expectation

$$
E\left(V\left(x(i_n\wedge T)\right)\right)\leq V\left(x(0)\right)+QE\left(i_n\wedge T\right)\leq V\left(x(0)\right)+QT.
\tag{21}
$$

By taking into account, the inequality (11) then we have

$$
P\left(i_n\leq T\right)\geq\epsilon.
$$

For every $\epsilon\in\left(i_n\leq T\right)$, $x\left(i_n,\omega\right)=n$ or $\left(i,\omega\right)=\dfrac{1}{n}$. Therefore, $V\left(x(t)\right)$ is no less than either

$$
\min\left\{\sqrt{k}-1-\frac{1}{2}\ln(k),\frac{1}{\sqrt{k}}-1+\frac{1}{2}\ln(k)\right\}.
$$

Based on (20) we have

$$
\begin{aligned}
V\left(x(0)\right)+QT & \geq E\left[\{i_n\leq T\}(\omega)V\left(x(i_n)\right)\right] \\
& \geq\epsilon\min\left\{\sqrt{k}-1-\frac{1}{2}\ln(k),\frac{1}{\sqrt{k}}-1+\frac{1}{2}\ln(k)\right\}.
\end{aligned}
$$

where $\{i_n\leq T\}(\omega)$ illustrates the function $\{i_n\leq T\}$. Letting n tends to infinity leads to dissidence

$$
\infty>V\left(x(0)\right)+QT=\infty
$$

Therefore, we need to have $i_\infty=\infty$. Eventually, our claim is proved truly. $\square$

In the next section we prove that Equation (10) with Ornstein-Uhlenbeck process is globally asymptotically stable in positive solution which illustrates the highly randomness in stationary phase.

Theorem 3.2: Equation (10) is globally asymptotically stable in positive equi-

librium points $x_{\max}$, under following assumptions a.s.

H1: For

1) $0 \le t_0, 0 < x_0 < \sigma < \mu_{\max} < b < x_{\max}$

2) $0 \le t_0, 0 < x_0 < \sigma < b < x_{\max} < \mu_{\max}$

and the

$$\limsup_{t \to \infty} \frac{\int_0^t x(s)\,\mathrm{d}s}{t} = \breve{b} \text{ and } \liminf_{t \to \infty} \frac{\int_0^t x(s)\,\mathrm{d}s}{t} = \hat{b}.$$

where, $\breve{b}$ and $\hat{b}$ are positive constant then the solution $x(t)$ satisfies the $\lim_{t \to \infty} x(t) = x_{\max}$.

Proof: An application Itô's formula ([27], p. 95) [28] [29] [30] for Equation (10) and somewhat lengthy calculation leads to

$$x(t) = \frac{x_{\max}}{1 + \left[x_0 \exp\left(\sigma b \int_0^t x(s) W(s)\,\mathrm{d}s + \sigma^2 \int_0^t x(s)\,\mathrm{d}W(s) \right) \right.} \\ \left. \times \frac{1}{\exp\left(-\frac{\mu_{\max}}{x_{\max}} \int_0^t x(s)\,\mathrm{d}s - \frac{\sigma^4}{2} \int_0^t x^2(s)\,\mathrm{d}s - \mu_{\max} \int_0^t \mathrm{d}s \right) \left(\frac{x_{\max}}{x_0} - 1 \right)} \right] \tag{22}$$

For the convenience we set the denominator of Equation (22) to Q and according to the ([10], p. 69), we have:

$$Q = 1 + e^{\left(-\int_0^t \mu_{\max}\,\mathrm{d}s - \frac{\mu_{\max}}{x_{\max}} \int_0^t x(s)\,\mathrm{d}s - \int_0^t \sigma b W(s)\,\mathrm{d}W(s) \times \int_0^t x(s)\,\mathrm{d}W(s) \right)} \\ \times e^{\left(\sigma^2 \int_0^t x(s)\,\mathrm{d}W(s) - \frac{1}{2}\sigma^4 \int_0^t x^2(s)\,\mathrm{d}s \right)}.$$

Taking limit from both sides of above equation yields:

$$\lim_{t \to \infty} Q = 1 + \lim_{t \to \infty} e^{(-t)\left(\frac{\int_0^t \mu_{\max}\,\mathrm{d}s}{t} + \frac{\mu_{\max}}{x_{\max}} \frac{\int_0^t x(s)\,\mathrm{d}s}{t} + \frac{\int_0^t \sigma b W(s)\,\mathrm{d}W(s) \int_0^t x(s)\,\mathrm{d}W(s)}{t} \right)} \\ \times \lim_{t \to \infty} e^{(-t) - \sigma^2 \frac{\int_0^t x(s)\,\mathrm{d}W(s)}{t} + \frac{1}{2}\sigma^4 \frac{\int_0^t x^2(s)\,\mathrm{d}s}{t}}. \tag{23}$$

where,

$$\int_0^t x(s)\,\mathrm{d}W(s) = M(t) \tag{24}$$

is a Martingale with quadratic variation

$$\langle M(t), M(t) \rangle = \int_0^t x^2(s)\,\mathrm{d}(s) \tag{25}$$

Based on ([31], p. 65) the

$$\limsup_{t \to \infty} \frac{M(t)}{t} = 0 \ a.s. \tag{26}$$

and

$$\limsup_{t \to \infty} \frac{\langle M(t), M(t) \rangle}{t} = C \ a.s. \tag{27}$$

where $C \in R^+$ then Equation (23) becomes

$$\lim_{t\to\infty} Q = 1 + \lim_{t\to\infty} e^{(-t)\left(\frac{\int_0^t \mu_{max}\,ds}{t} + \frac{\mu_{max}}{x_{max}}\frac{\int_0^t x(s)\,ds}{t} + \frac{1}{2}\sigma^4 C\right)} \tag{28}$$

$$\lim_{t\to\infty} Q = 1 + \lim_{t\to\infty} e^{(-t)\left(\frac{\int_0^t \mu_{max}\,ds}{t} + \frac{\mu_{max}}{x_{max}}\frac{\int_0^t x(s)\,ds}{t} + \frac{1}{2}\sigma^4 C\right)} = 0 \ a.s. \tag{29}$$

By substituting Equation (29) into Equation (22) we have:

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x_{max}$$

$$\Rightarrow \lim_{t\to\infty} x(t) = x_{max}.$$

Ultimately, prove is completed. □

Remark 3.1: Liu *et al.,* in [21] studied the stability of stochastic logistic model with white noise which illustrates highly randomness in lag phase and the stationary phase is steady means there is no fluctuations. In lag phase microbes are adapted with each other and as a result no growth takes place. But, the model, conditions and approach which are presented in this study quite different and easy respectively. This research considers Equation (6) with Ornstein-Uhlenbeck process which highly presents the randomness in stationary phase. In stationary phase microbes fight with each other to survive (uncontrolled fluctuations reached maximum level). In addition, concentration of toxin reaches a value which is unable to maintain the maximum cell growth of microorganism rate in fermentation process. These agents become the cause of instability at stationary phase in fermentation process. In conclusion, the stochastic logistic model which presented in Lui *et al.,* [21] is not suitable model for cell growth of microorganism in fermentation process. Since, Lui *et al.,* in [21] narrates in zero solution (almost lag phase in fermentation process) the population (cell growth) is extinctive and in positive equilibrium point (almost stationary phase) the population will be permanent. It is contrary to the features phases in fermentation process. To support our theory, we use the SRK4 method for numerical simulations which presented in below section.

4. Numerical Simulation

In this section we consider a strong and accurate numerical method (SRK4) to elaborate the analytical results ([3], p. 69) [4] [5] [6] [7]. Worth mentioning, we cannot use the stochastic Runge-Kutta to approximate the numerical solution of Equation (10), due to it is in Itô sense [33]. Thus, it is convertible to Stratonovich sense by using the below formula

$$\overline{f}(t, x(t)) = f(t, x(t)) - \frac{1}{2} g(t, x(t))\frac{\partial g}{\partial x}(t, x(t)) \tag{30}$$

Hence,

$$dx(t) = \left[\mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)x(t) - \sigma x^2(t)bW(t)\right]dt - \sigma^4 x^3(t) \circ dW(t) \tag{31}$$

where $\circ$ is used to denote the Stratonovich form of SDE (*i.e.* $\circ \mathrm{d}W(t)$. Equation (10) and Equation (31) present some solution under different approach. We use the SRK4 method for numerical approximation. This method was introduced by Rümellin [8]. SRK was developed based on the increment of Wiener process, $\int_{t_n}^{t_{n+1}} \circ \mathrm{d}W(t)$. Furthermore, Xiao in [9] introduced the High strong order stochastic Runge-Kutta methods for stochastic differential equations in case of Stratonovich with scalar noise. Since our model is in Stratonovich sense. Figure 1 and Figure 2 show the stability of Equation (3) and Equation (10) in positive equilibrium for cell growth of microorganism in fermentation process respectively.
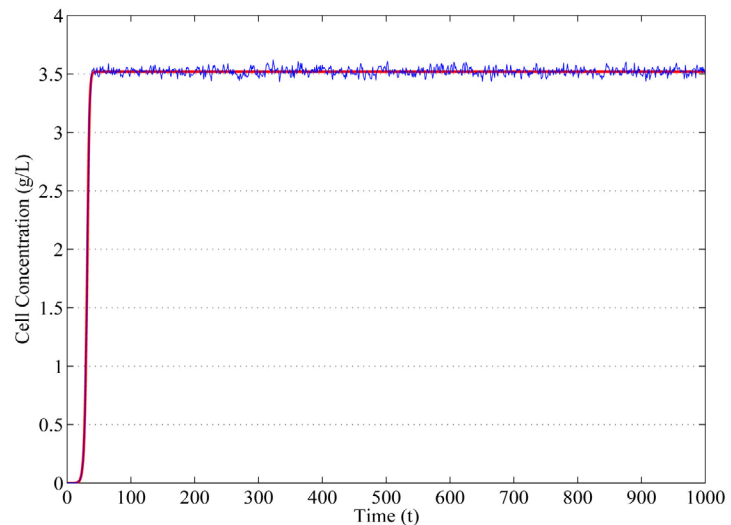
Figure 1. Shows the stability of Equation (10). For deterministic part we use $t_0 = 0.00$, $x(0) = 0.0001$, $x_{\max} = 3.520$, $\mu_{\max} = 0.5$, $\sigma = 0.00$, and $b = 0$ and for stochastic part we chose same value just difference are $\sigma = 0.002$, and $b = 1$ respectively.
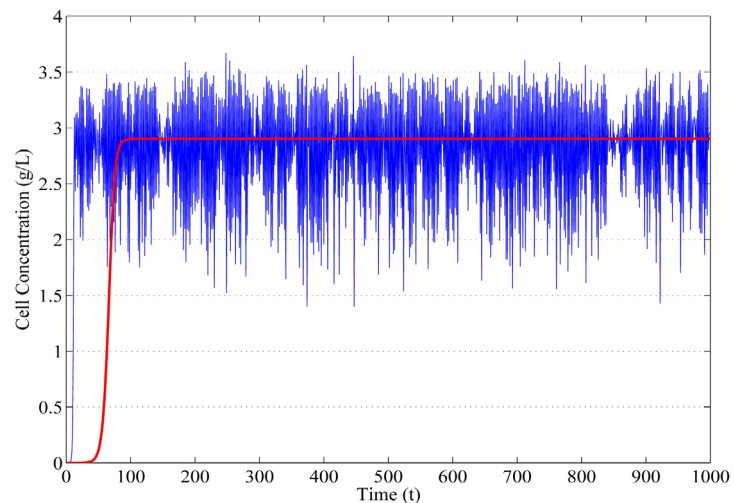

Figure 2. Illustrates the stability of Equation (10). For stochastic part we chose the $t_0 = 0.00$, $x(0) = 0.0001$, $x_{\max} = 3.520$, $\mu_{\max} = 1.99$, $b = 1$, and $\sigma = 0.021$. For deterministic part we use $\sigma = 0.00$, $b = 0$, $t_0 = 0.00$, $x(0) = 0.0001$, $x_{\max} = 3.520$, $\mu_{\max} = 0.21$.

The blue line indicates the simple path of Equation (10) and the red line shows the simple path of Equation (3) respectively. In **Figure 1** and **Figure 2**, the difference are between the value of σ, μ_{max}. The blue line indicates the simple path of Equation (10) and the red line shows the simple path of Equation (3) respectively. **Figure 1** and **Figure 2** are plotted under **H1**.

In both **Figure 1** and **Figure 2** lag phase is the luck of random fluctuations due to microbes are adopt with each other. Therefore, no growth takes place in this phase. Contrariwise, in stationary phase microbes fight with each other for food and space and the behavior of cell proliferation of microorganism is different in phases at fermentation process. As time evolves, the system illustrates the intrinsic variability of the competing within species and deviations from exponential growth arise. It happens as a result of the nutrient level and toxin concentration achieves a value which can no longer support the maximum growth rate. So, the stochastic fluctuations mainly affect the logistic growth of microorganism and for more see **Remarks 3.1**.

Remark 4.1: The red line shows the simple path of Equation (3) and the blue line indicates the simple path of Equation (10) respectively. If $\sigma \to \pm\infty$, there is no phases for cell growth which means this random fluctuations will destroy the phases in fermentation process on the other hands the process will tend to infinity.

Remark 4.2: **Figures 1-4** show the stability of Equation (3) and Equation (10) in different equilibrium points. **Figure 3** and **Figure 4** are plotted under H1 option 2. The thick red line illustrates the stability of Equation (10) without Ornstein-Uhlenbeck process. It is thick in stationary phase due to the growth rate μ_{max}, is increased, compare the value of μ_{max} with three others Figures.

5. Conclusion

This research is conducted on stability of stochastic logistic model with Ornstein-Uhlenbeck process for cell growth of microorganism in fermentation.
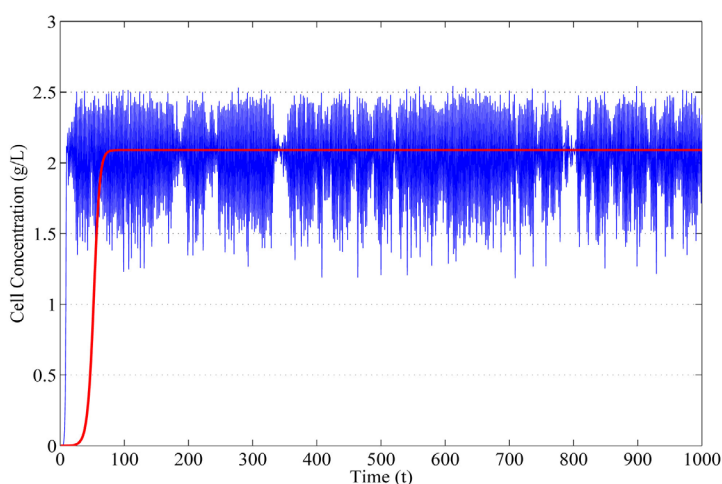

Figure 3. Shows the stability of Equation (10). For deterministic part we use $t_0 = 0.00$, $x(0) = 0.0001$, $x_{max} = 2.09$, $\mu_{max} = 0.2$, $\sigma = 0.00$, and $b = 0$ and for stochastic part we chose same value just difference are $\sigma = 0.0021$, $\mu_{max} = 2.2$ and $b = 1$ respectively.
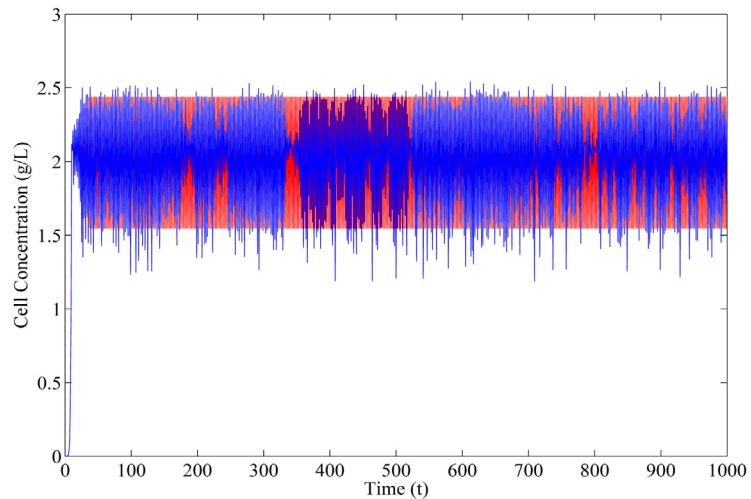
Figure 4. Illustrates the stability of Equation (10). For stochastic part we chose the $t_0 = 0.00$, $x(0) = 0.0001$, $x_{max} = 2.09$, $\mu_{max} = 2.2$, $\mu_{max} = 0.21$, $b = 1$, and $\sigma = 0.02$. For deterministic part we use $\sigma = 0.00$, $b = 0$, $t_0 = 0.00$, $x(0) = 0.0001$, $x_{max} = 2.09$, $\mu_{max} = 2.2$.

This research proved that Equation (10) has a unique positive solution (see Theorem 3.1). Moreover, we proved that Equation (10) was stochastically stable in zero solution and positive equilibrium point (see Theorem 3.2 and **Appendix A** Theorem 1) and viewed in **Figure 1** and **Figure 2** and **Appendix A**. For numerical simulation we used SRK4 method to show the reality of this research. If the $\sigma \to \pm\infty$, the result will be infinity which means that there are no phases for cell growth. On the other hand, it will not preserve the stochastically stable. If $x_0 > x_{max} > \mu_{max}$, what will happen? Unfortunately, there are some impediments here which need some further investigation in future perspective.

Acknowledgements

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A

Theorem 1: Equation (10) under the following assumption is globally asymptotically stable in zero solution a.s (almost surely).

(H2): For $0 \le t_0$, $\mu_{max} < 0 < x_0 < \sigma < x_{max}$, then the $\lim_{t \to \infty} x(t) = 0$.

Proof: An application of Itô's formula for Equation (10) ([27], p. 95) leads to

$$
\begin{aligned}
&\mathrm{d}\left(\ln|x(t)|\right) \\
&= \frac{1}{x(t)}\underbrace{\left[\mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)x(t)\mathrm{d}t - \sigma x^2(t)bW(t)\mathrm{d}t + \sigma^2 x^2(t)\mathrm{d}W(t)\right]}_{A} \\
&\quad + \frac{-1}{2x^2(t)}\underbrace{\left[\mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)x(t)\mathrm{d}t - \sigma x^2(t)bW(t)\mathrm{d}t + \sigma^2 x^2(t)\mathrm{d}W(t)\right]^2}_{B}
\end{aligned}
\tag{32}
$$

Equation (32) has two parts A and B.

So,

$$
A = \mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)x(t)\mathrm{d}t - \sigma x^2(t)bW(t)\mathrm{d}t + \sigma^2 x^2(t)\mathrm{d}W(t)
$$

and

$$
B = \left[\mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)x(t)\mathrm{d}t - \sigma x^2(t)bW(t)\mathrm{d}t + \sigma^2 x^2(t)\mathrm{d}W(t)\right]^2 .
$$

To simplify the B we have

$$
B = \left[\mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)\mathrm{d}t - \sigma x(t)bW(t)\mathrm{d}t + \sigma^2 x^2(t)\mathrm{d}W(t)\right]^2
$$

$$
\begin{aligned}
B = &\left(\mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)\mathrm{d}t\right)^2 + 2\left(\mu_{max}\left(1 - \frac{x(t)}{x_{max}}\right)\mathrm{d}t\right) \\
&\times\left(-\sigma x(t)bW(t)\mathrm{d}t + \sigma^2 x^2(t)\mathrm{d}W(t)\right) \\
&+ \left(-\sigma x(t)bW(t)\mathrm{d}t + \sigma^2 x^2(t)\mathrm{d}W(t)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
B = &\left(\mu_{max}^2 \mathrm{d}t \times \mathrm{d}t - 2\frac{\mu_{max}}{x_{max}}x^2(t)\mathrm{d}t \times \mathrm{d}t + \left(\frac{x(t)}{x_{max}}\mathrm{d}t\right)^2\right) \\
&+ 2\left(\mu_{max}\mathrm{d}t - \frac{\mu_{max}}{x_{max}}x^2(t)\mathrm{d}t\right)\times\left(-\sigma x(t)bW(t)\mathrm{d}t + \sigma^2 x(t)\mathrm{d}W(t)\right) \\
&+ \left(-\sigma x(t)bW(t)\mathrm{d}t + \sigma^2 x(t)\mathrm{d}W(t)\right)^2 + \left(-\sigma x(t)bW(t)\mathrm{d}t\right)^2 \\
&+ 2\left(-\sigma x(t)bW(t)\mathrm{d}t\right)\left(\sigma^2 x(t)\mathrm{d}W(t)\right) + \left(\sigma^2 x^2(t)\mathrm{d}W(t)\right)^2
\end{aligned}
$$

Using these facts $\mathrm{d}t \times dt = 0, \mathrm{d}t \times \mathrm{d}w(t) = 0$ and $\mathrm{d}W(t) \times \mathrm{d}W(t) = \mathrm{d}t$ ([32], p. 87) and somewhat lengthy calculation we have

$$
B = \sigma^4 x^4(t)\mathrm{d}t.
$$

By substituting the value of A and B into Equation (32) yields:

$$d\left(\ln|x(t)|\right) = \frac{1}{x(t)}\left[\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)dt - \sigma x^2(t)bW(t)dt \right.$$
$$\left. + \sigma^2 x^2(t)dW(t)\right] + \frac{-1}{2x(t)^2}\left(\sigma^4 x^4(t)dt\right) \tag{33}$$

$$d\left(\ln|x(t)|\right) = \left[\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)dt - \sigma x(t)bW(t)dt \right.$$
$$\left. + \sigma^2 x(t)dW(t)\right] - \frac{1}{2}\left(\sigma^4 x^2(t)dt\right) \tag{34}$$

$$x(t) = e^{\mu_{\max}\int_0^t\left(1-\frac{x(s)}{x_{\max}}\right)ds - \sigma b\int_0^t x(s)W(s)ds + \sigma^2\int_0^t x(s)dW(s)-\frac{1}{2}\sigma^4\int_0^t x^2(s)ds} \tag{35}$$

where $W(t)$ is Brownian motion and taking expectation from both sides of above equation. We know the expectation of Brownian motion is zero ([10], Section 3.2.3, p. 43), we get

$$E\left[x(t)\right] = E\left[\exp\left(-\frac{\mu_{\max}}{x_{\max}}\int_0^t x(s)ds - \frac{\sigma^4}{2}\int_0^t x^2(s)ds + \mu_{\max}\int_0^t ds\right)\right] \tag{36}$$

Takes limit

$$\lim_{t\to\infty} E\left[x(t)\right] = \lim_{t\to\infty} E\left[\exp\left(-\frac{\mu_{\max}}{x_{\max}}\int_0^t x(s)ds - \frac{\sigma^4}{2}\int_0^t x^2(s)ds + \mu_{\max}\int_0^t ds\right)\right] \tag{37}$$

$$\lim_{t\to\infty} E\left[x(t)\right] = \lim_{t\to\infty} E\left[\exp\left(-\frac{\mu_{\max}}{x_{\max}}\frac{\int_0^t x(s)ds}{t} - \frac{\sigma^4}{2}\frac{\int_0^t x^2(s)ds}{t} + \mu_{\max}\frac{\int_0^t ds}{t}\right)\right] \tag{38}$$

For $0 \le t \le T$ and based on ([10], p. 66 and p. 69) the $\int_0^T x^2(s)ds < \infty$ and $\int_0^T x(s)ds < \infty$. Therefore

$$\lim_{t\to\infty} E\left[e^{t\left(\mu_{\max}\frac{\int_0^t ds}{t}\right)}\right]. \tag{39}$$

Under **H2** yields

$$\lim_{t\to\infty} E\left[e^{t\left(\mu_{\max}\frac{\int_0^t ds}{t}\right)}\right] = 0. \tag{40}$$

Therefore, Equation (38) becomes

$$\lim_{t\to\infty} x(t) = 0.$$

Ultimately, our claim is proved. □

For plotting **Figure 5** we use $t_0 = 0.00$, $x(0) = 0.0001$, $x_{\max} = 3.5250$, $\mu_{\max} = -0.0091$, $\sigma = 0.00$, $\sigma = 3.5$ and $b = 0$, $b = 90$ and for **Figure 6** we use $t_0 = 0.00$, $x(0) = 0.0001$, $x_{\max} = 0.50$, $\mu_{\max} = -1.99$, $\sigma = 0.00$, $\sigma = 11$ and $b = 0$, $b = 100$. The blue line indicates the simple path of Equation (10) and the red line shows the simple path of Equation (3).
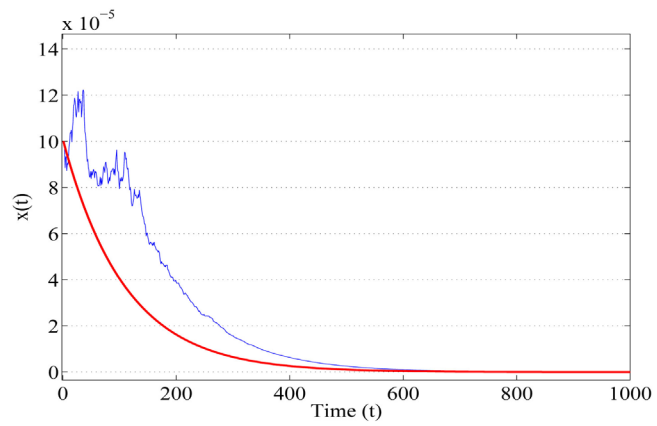
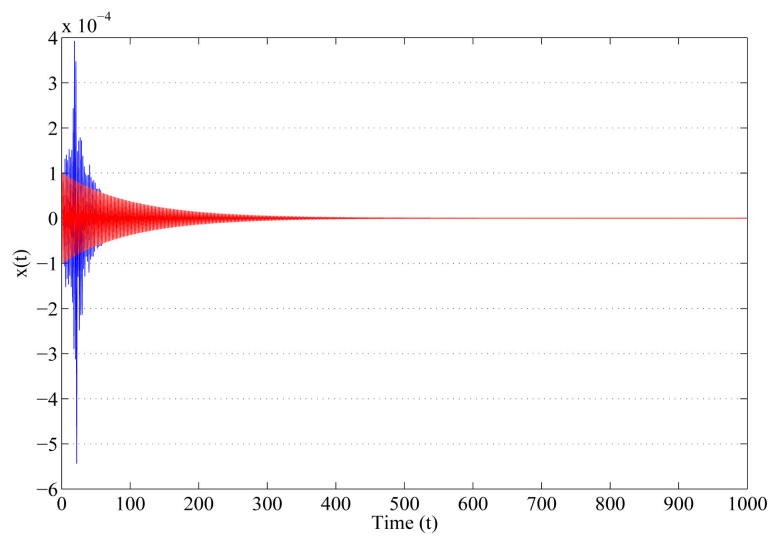**Figure 5.** Shows the stability of Equation (10) with different value of parameters.

Figure 6. Illustrates the stability of Equation (10) with different value of parameters.

Note: The thick red line shows the stability of Equation (10) without Ornstein-Uhlenbeck process. It is thick due to the growth rate μ_{max}, is increased, compare the value of μ_{max} with **Figure 5**.