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A Study on Exact Travelling Wave Solutions of Generalized *KdV* Equations

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Abstract

In this paper, generalized *KdV* equations are investigated by using a mathematical technique based on the reduction of order for solving differential equations. The compactons, solitons, solitary patterns and periodic solutions for the equations presented in this paper are obtained. For these generalized *KdV* equations, it is found that the change of the exponents of the wave function u and the coefficient a, positive or negative, leads to the different physical structures of the solutions.

Keywords

Generalized *KdV* Equations, Compactons, Solitons, Physical Structures

1. Introduction

Late in the 19th century, Korteweg and de Vries developed a theory to describe weakly nonlinear wave propagation in shallow water. The classical Korteweg-de Vries (*KdV*) equation is usually written as

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

After a long time, the *KdV* equation has been found to be involved in a wide range of physics phenomena, especially those exhibiting shock waves, travelling waves, and solitons. Certain theoretical physics phenomena in the quantum mechanics domain can be explained by means of *KdV* model.

As is well known, the classical *KdV* equation has been played a central role in the study of nonlinear phenomena, especially solitons phenomena which exist due to a balance between weak nonlinearity and dispersion. As one of the most fundamental equations of solitons phenomena, Equation (1) has caused great attention from many researchers, all forms of modified *KdV* equations have been studied extensively (see [1]-[10]).

Tzirtzilakis, *et al.* [1] discussed second and third order approximations of water wave equations of *KdV* type. Analytical expression for solitary wave solutions for some special equations was derived. By using a Fourier pseudospectral method combined with a finite-difference scheme, a detailed numerical study of these solutions obtained in [1] was carried out. The stability of these solitary wave solutions was also established.

Rosenau and Hyman [2] introduced and studied a class of *KdV* equations—$K(m,n)$ equation. They discovered that the solitary solutions of these equations, for certain *m* and *n*, have compact support, namely they vanish outside a finite core region. Solitons with finite wavelength are called compactons.

In [3], Rosenau subsequently studied the model

$$u_t + a\left(u^{n+1}\right)_x + \left[u\left(u^n\right)_{xx}\right]_x = 0, \quad n \geq 1, \tag{2}$$

where $a > 0$. This model emerged in nonlinear lattices and was used to describe the dispersion of dilute suspensions for $n = 1$. But Rosenau [3] only got general formulas in terms of the cosine for model (2). With the use of new ansatze methods, Wazwaz [4] examined model (2) for two cases, $a > 0$ and $a < 0$. And the exact travelling solutions in terms of sine, cosine function, the hyperbolic function sinh and cosh were derived.

Wazwaz investigated variants of the *KdV* equations respectively in [5] and [6] as follows:

$$u_t + au\left(u^n\right)_x + \left[u\left(u^n\right)_{xx}\right]_x = 0, \quad n > 1, \tag{3}$$

$$u_t + au^n\left(u\right)_x + \left[u^n\left(u\right)_{xx}\right]_x = 0, \quad n \geq 3, \tag{4}$$

where *a* is a nonzero constant. The compactons and solitary pattern solutions were presented.

The present work aims to extend the work made by Wazwaz [5] [6]. We desire to seek another method to solve nonlinear equations. For this purpose, the wave variable $\xi = \mu x - ct$ is introduced to carry the PDEs into ODEs. By using this variable replacement method, some new exact solutions including solitons can be obtained. In fact, the method in this paper is efficient to solve many nonlinear equations. It avoids tedious algebra and guesswork and also can be used in higher dimensional space.

In this paper, we will discuss generalized *KdV* equations, Equaiton (3) and Equaiton (4) and the following equations with negative exponents:

$$u_t + au\left(u^{-n}\right)_x + \left[u\left(u^{-n}\right)_{xx}\right]_x = 0, \quad n > 1, \tag{5}$$

$$u_t + au^{-n}\left(u\right)_x + \left[u^{-n}\left(u\right)_{xx}\right]_x = 0, \quad n \geq 3, \tag{6}$$

where *a* is a nonzero constant. In the sense of ignoring the constants of integration resulted from solving Equations (3)-(6), the exact travelling solutions have been obtained which contain the main results made in [5] [6] as special cases.

2. The Generalized *KdV* Equations with Positive Exponents

2.1. Exact Travelling Wave Solutions for Equation (3)

Firstly, we assume that the travelling wave solutions of Equation (3) take the form

$$u(x,t)=u(\xi),\quad \xi=\mu x-ct, \tag{7}$$

in which $\mu\neq 0$, $c\neq 0$.

Notice that

$$\frac{\partial}{\partial t}=-c\frac{\mathrm{d}}{\mathrm{d}\xi},\quad \frac{\partial}{\partial x}=\mu\frac{\mathrm{d}}{\mathrm{d}\xi},\quad \frac{\partial^3}{\partial x^3}=\mu^3\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}. \tag{8}$$

Substituting (7) and (8) into Equation (3) gives the following nonlinear ODE

$$-cu_\xi+a\mu u\left(u^n\right)_\xi+\mu^3\left[u\left(u^n\right)_{\xi\xi}\right]_\xi=0. \tag{9}$$

Integrating Equation (9) once and setting the constant of integration to be zero, we find

$$\mu^3u\left(u^n\right)_{\xi\xi}+\frac{an\mu}{n+1}u^{n+1}-cu=0. \tag{10}$$

Considering $u\neq 0$, we get

$$\mu^3\left(u^n\right)_{\xi\xi}+\frac{an\mu}{n+1}u^n-c=0. \tag{11}$$

Set $V=u^n$, then

$$\mu^3V_{\xi\xi}+\frac{an\mu}{n+1}V-c=0. \tag{12}$$

Letting $\frac{\mathrm{d}V}{\mathrm{d}\xi}=Z$, we get $\frac{\mathrm{d}^2V}{\mathrm{d}\xi^2}=Z\frac{\mathrm{d}Z}{\mathrm{d}V}$. So Equation (12) becomes

$$\mu^3Z\frac{\mathrm{d}Z}{\mathrm{d}V}+\frac{an\mu}{n+1}V-c=0. \tag{13}$$

By using the separating variants method, we have

$$\frac{\mu^3}{2}Z^2=cV-\frac{an\mu}{2(n+1)}V^2. \tag{14}$$

That is

$$\left(\frac{\mathrm{d}V}{\mathrm{d}\xi}\right)^2=\frac{V}{\mu^3}\left(2c-\frac{an\mu}{n+1}V\right). \tag{15}$$

Case 1. $a>0$: Solving Equation (15) gives

$$V=\frac{2c(n+1)}{an\mu}\sin^2\left(\frac{1}{2\mu}\sqrt{\frac{an}{n+1}}\xi\right), \tag{16}$$

and

$$V=\frac{2c(n+1)}{an\mu}\cos^2\left(\frac{1}{2\mu}\sqrt{\frac{an}{n+1}}\xi\right). \tag{17}$$

Hence, we limit the domain of ξ, obtain the following compacton solutions:

$$u(x,t)=\begin{cases}\left\{\dfrac{2c(n+1)}{an\mu}\sin^2\left[\dfrac{1}{2\mu}\sqrt{\dfrac{an}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, & |\xi|\le 2|\mu|\pi\Big/\sqrt{\dfrac{an}{n+1}},\\ 0, & \text{otherwise,}\end{cases} \tag{18}$$

and

$$u(x,t)=\begin{cases}\left\{\dfrac{2c(n+1)}{an\mu}\cos^2\left[\dfrac{1}{2\mu}\sqrt{\dfrac{an}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, & |\xi|\le |\mu|\pi\Big/\sqrt{\dfrac{an}{n+1}},\\ 0, & \text{otherwise.}\end{cases} \tag{19}$$

Case 2. $a<0$: Solving Equation (15), we get the solitary pattern solutions as follows:

$$u(x,t)=\left\{-\frac{2c(n+1)}{an\mu}\sinh^2\left[\frac{1}{2\mu}\sqrt{-\frac{an}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, \tag{20}$$

and

$$u(x,t)=\left\{\frac{2c(n+1)}{an\mu}\cosh^2\left[\frac{1}{2\mu}\sqrt{-\frac{an}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}. \tag{21}$$

Remark 1. Letting $\mu=1$ in (18) and (19), we have

$$u(x,t)=\begin{cases}\left\{\dfrac{2c(n+1)}{an}\sin^2\left[\dfrac{1}{2}\sqrt{\dfrac{an}{n+1}}(x-ct)\right]\right\}^{\frac{1}{n}}, & |-ct|\le 2\pi\Big/\sqrt{\dfrac{an}{n+1}},\\ 0, & \text{otherwise,}\end{cases} \tag{22}$$

and

$$u(x,t)=\begin{cases}\left\{\dfrac{2c(n+1)}{an}\cos^2\left[\dfrac{1}{2}\sqrt{\dfrac{an}{n+1}}(x-ct)\right]\right\}^{\frac{1}{n}}, & |-ct|\le \pi\Big/\sqrt{\dfrac{an}{n+1}},\\ 0, & \text{otherwise.}\end{cases} \tag{23}$$

which just are the main results for Equation (3) obtained by Wazwaz [5]. In other words, solutions (22), (23) made in [5] are special cases of formulas (18), (19).

2.2. Exact Travelling Wave Solutions for Equation (4)

Following the analysis presented above, we use the wave variable $\xi=\mu x-ct$ into Equation (4) to get the following ODE:

$$\mu^3 u_{\xi\xi}+\frac{a\mu}{n+1}u-cu^{1-n}=0. \tag{24}$$

Letting $Y=\dfrac{\mathrm{d}u}{\mathrm{d}\xi}$, we get $\dfrac{\mathrm{d}^2u}{\mathrm{d}\xi^2}=Y\dfrac{\mathrm{d}Y}{\mathrm{d}u}$. Then

$$\mu^3 Y\frac{\mathrm{d}Y}{\mathrm{d}u}+\frac{a\mu}{n+1}u-cu^{1-n}=0. \tag{25}$$

Solving Equation (25) yields

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^2=\frac{1}{\mu^3}\left(\frac{2c}{2-n}u^{-n}-\frac{a\mu}{n+1}\right)u^2. \tag{26}$$

Setting $W=u^{-n}$, we have

$$u=W^{-\frac{1}{n}}, \tag{27}$$

$$\mathrm{d}u=-\frac{1}{n}W^{-\frac{1}{n}-1}\mathrm{d}W. \tag{28}$$

Substituting (27) and (28) into Equation (26) gives

$$\left(\frac{\mathrm{d}W}{\mathrm{d}\xi}\right)^2=\frac{n^2}{\mu^3}\left(\frac{2c}{2-n}W-\frac{a\mu}{n+1}\right)W^2. \tag{29}$$

Case 1. $a>0$: For this case, solving Equation (29), we get

$$W=\frac{a\mu(2-n)}{2c(n+1)}\sec^2\left(\frac{n}{2\mu}\sqrt{\frac{a}{n+1}}\xi\right), \tag{30}$$

and

$$W=\frac{a\mu(2-n)}{2c(n+1)}\csc^2\left(\frac{n}{2\mu}\sqrt{\frac{a}{n+1}}\xi\right). \tag{31}$$

Therefore, we obtain the following compacton solutions:

$$u(x,t)=\begin{cases}\left\{\frac{2c(n+1)}{a\mu(2-n)}\sin^2\left[\frac{n}{2\mu}\sqrt{\frac{a}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, & |\xi|\le 2|\mu|\pi\Big/n\sqrt{\frac{a}{n+1}},\\ 0, & \text{otherwise},\end{cases} \tag{32}$$

and

$$u(x,t)=\begin{cases}\left\{\frac{2c(n+1)}{a\mu(2-n)}\cos^2\left[\frac{n}{2\mu}\sqrt{\frac{a}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, & |\xi|\le |\mu|\pi\Big/n\sqrt{\frac{a}{n+1}},\\ 0, & \text{otherwise}.\end{cases} \tag{33}$$

Case 2. $a<0$: Solving Equation (29), we have the solitary pattern solutions given by

$$u(x,t)=\left\{-\frac{2c(n+1)}{a\mu(2-n)}\sinh^2\left[\frac{n}{2\mu}\sqrt{-\frac{a}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, \tag{34}$$

and

$$u(x,t)=\left\{\frac{2c(n+1)}{a\mu(2-n)}\cosh^2\left[\frac{n}{2\mu}\sqrt{-\frac{a}{n+1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}. \tag{35}$$

3. The Generalized *KdV* Equations with Negative Exponents

In fact, Equation (3) and Equation (4) and Equation (5) and Equation (6) have the symmetric property about n respectively. We replace n by $-n$ in Equation (3) and Equation (4) and the corresponding travelling wave solutions in Section 2. So we have the following results:

3.1. Exact Travelling Wave Solutions for Equation (5)

Case 1. $a>0$: The periodic solutions are given by

$$u(x,t)=\left\{\frac{an\mu}{2c(n-1)}\sec^2\left[\frac{1}{2\mu}\sqrt{\frac{an}{n-1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, \tag{36}$$

and

$$u(x,t)=\left\{\frac{an\mu}{2c(n-1)}\csc^2\left[\frac{1}{2\mu}\sqrt{\frac{an}{n-1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}. \tag{37}$$

Case 2. $a<0$: The soliton solutions have the forms of

$$u(x,t)=\left\{\frac{an\mu}{2c(n-1)}\operatorname{sech}^2\left[\frac{1}{2\mu}\sqrt{-\frac{an}{n-1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, \tag{38}$$

and

$$u(x,t)=\left\{-\frac{an\mu}{2c(n-1)}\operatorname{csch}^2\left[\frac{1}{2\mu}\sqrt{-\frac{an}{n-1}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}. \tag{39}$$

3.2. Exact Travelling Wave Solutions for Equation (6)

Case 1. $a>0$: In this case, we get the following soliton solutions:

$$u(x,t)=\left\{\frac{a\mu(n+2)}{2c(1-n)}\operatorname{sech}^2\left[\frac{n}{2\mu}\sqrt{-\frac{a}{1-n}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, \tag{40}$$

and

$$u(x,t)=\left\{-\frac{a\mu(n+2)}{2c(1-n)}\operatorname{csch}^2\left[\frac{n}{2\mu}\sqrt{-\frac{a}{1-n}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}. \tag{41}$$

Case 2. $a<0$: We have the following periodic solutions:

$$u(x,t)=\left\{\frac{a\mu(n+2)}{2c(1-n)}\sec^2\left[\frac{n}{2\mu}\sqrt{\frac{a}{1-n}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}, \tag{42}$$

and

$$u(x,t)=\left\{\frac{a\mu(n+2)}{2c(1-n)}\csc^2\left[\frac{n}{2\mu}\sqrt{\frac{a}{1-n}}(\mu x-ct)\right]\right\}^{\frac{1}{n}}. \tag{43}$$

4. Conclusions

The method based on the reduction of order is a powerful tool for acquiring some special solutions of nonlinear PDEs. In this paper, we study three types of generalized *KdV* equations with positive and negative exponents by using this mathematical technique. Different from others, this technique carries some partial differential equations into ordinary equations which are easier to be solved. And the analytical expression of travelling wave solutions, containing compactons, solitons, solitary patterns and periodic solutions, are derived.

The obtained results in Section 2 and Section 3 each represent two completely different sets of models, which has been shown that the variation of exponents and coefficient, positive or negative, could cause the quantitative change in the physical structure of the solutions. The physical structures of the compactons solutions and the solitary patterns solutions deepen our understanding of many scientific processes, such as the super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops, and the inertial fusion.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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