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Framing Noether's Theorem

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Abstract

We discuss Noether's theorem from a new perspective and show that the spatial continuous symmetries of a system are on one hand symmetries of the space and on the other hand are dictated by the system's potential energy. The Noether's charges arising from an infinitesimal motion, or a Killing vector field, of the space, are conserved if the Lie derivative of the potential energy by this vector field vanishes. The possible spatial symmetries of a mechanical system are listed according to the potential energy of the external forces.

Keywords

Noether's Theorem, Continuous Spatial Symmetries, Conserved Momenta

1. Introduction

The Noether's theorem [1]-[6], proved by Emmy Noether in 1915, relates conserved physical quantities of a system to its corresponding symmetries and vice versa. The theorem is utilized in quantum field theory, quantum mechanics, and classical mechanics. In what follows

- We give a brief account of necessary background to Noether's theorem, namely, the Lagrange and Hamilton formulation of mechanics [7] [8] [9].
- Derive the theorem in its simplest form which describes discrete mechanical systems.
- Brief the concepts of space's symmetries and its connection to the Lie algebra spanned by the set of infinitesimal generators of the group of symmetries [10] [11] [12] [13]. We shall consider only continuous spatial symmetries.
- Show that the system's continuous symmetries occur when its kinetic and potential energies are invariant under the corresponding transformations; and as a consequence, the systems' symmetries are already symmetries of the space.
- Link conserved momenta to the system's potential energy and show that a

system is invariant under a group of transformations if the Lie derivative of the potential energy by the infinitesimal generator of the group is identically zero.

- Apply the new theorem to examples of mechanical systems and specify the possible conserved momenta according to the form of potential energy.

The current work forms a new insight in Noether's theorem by which the continuous spatial symmetries of all systems are limited to a few types, which are determined by the potential energy out of the symmetries of the space. Noether's theorem links systems' symmetry to the invariance of the corresponding Lagrangian. Because different systems have in general different Lagrangians, one may expect that systems may possess symmetries that are not included in the symmetries of the space. However, the current work dismisses utterly such expectations and it sets clear the method by which systems' symmetries, when exist, are determined. It is shown that only systems, that have one of a few specific forms of potential energy, do exhibit symmetries.

2. Essentials of Analytic Mechanics

The Lagrangian function $\mathcal{L}$ of a system of N particles is the difference between its kinetic and potential energies:

$$T - V = \mathcal{L}\left(x_1, \cdots, x_s, \dot{x}_1, \cdots, \dot{x}_s, t\right); \tag{2.1}$$

It is a function in the generalized coordinates x_i, generalized velocities $\dot{x}_i = \mathrm{d}x_i/\mathrm{d}t$, and possibly time t. The index s is the number of the system's degrees of freedom.

The Hamilton principle states that: Out of the curves connecting the initial position $x_i\left(t_1\right)$ and the final position $x_i\left(t_2\right)$, the system follows the path $x_i = x_i\left(t\right)$ along which the action

$$S = \int_{t_1}^{t_2} \mathcal{L}\mathrm{d}t \tag{2.2}$$

is stationary. This implies that the action does not change if the actual path $x_i\left(t\right)$ is replaced by an infinitesimally adjacent curve $x_i\left(t\right) + \delta x_i\left(t\right)$. Mathematically,

$$0 = \delta S = \delta\int_{t_1}^{t_2} \mathcal{L}\mathrm{d}t = \int_{t_1}^{t_2} \delta\mathcal{L}\mathrm{d}t = \int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial x_i}\delta x_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{x}_i\right)\mathrm{d}t, \tag{2.3}$$

where the summation convention is used. We appeal here to a basic rule in the calculus of variation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x_i = \delta\frac{\mathrm{d}x_i}{\mathrm{d}t} = \delta\dot{x}_i. \tag{2.4}$$

and integrate the second term in the integrand by parts:

$$\delta S = \left[\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\delta x_i\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\right)\delta x_i\mathrm{d}t = 0. \tag{2.5}$$

The first term on the right hand-side vanishes on the account of

$\delta x_i(t_1) = \delta x_i(t_2) = 0$. Focusing now on the integral which vanishes for arbitrary δx_i and taking all δx_i, except one at a time, equal to zero, yields Lagrange's equations

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0 \quad (i = 1, 2, \cdots, s). \tag{2.6}$$

The solution of Lagrange's Equations (2.6) gives the path of motion $x_i = x_i(t)$.

The generalized momenta are defined by

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = p_i(x, \dot{x}, t) \quad (i = 1, 2, \cdots, s), \tag{2.7}$$

where x and $\dot{x}$ stand collectively for the generalized coordinates and generalized velocities respectively. In terms of the generalized momenta we write (2.6) in the form

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial x_i} = f_i \quad (i = 1, \cdots, s) \tag{2.8}$$

The quantities $f_i = \partial \mathcal{L}/\partial x_i$ are called the generalized forces in the directions of the generalized coordinates x_i.

The Hamiltonian function is defined by

$$H(x, p, t) = p_i \dot{x}_i - \mathcal{L} \tag{2.9}$$

Taking the differential of both sides and benefiting from (2.7) we get

$$\begin{aligned} &\frac{\partial H}{\partial x_i}\mathrm{d}x_i + \frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial t}\mathrm{d}t \\ &= \dot{x}_i \mathrm{d}p_i - \frac{\partial \mathcal{L}}{\partial x_i}\mathrm{d}x_i - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t = \dot{x}_i \mathrm{d}p_i - \dot{p}\mathrm{d}x_i - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \end{aligned}$$

Equating the coefficients of the differentials on both sides yields

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \ \dot{p}_i = -\frac{\partial H}{\partial x_i}, \ \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \quad (i = 1, 2, \cdots, s) \tag{2.10}$$

The first two equations in (2.10) are Hamilton's equations of motion. We shall assume throughout this work that the potential energy is a function of the generalized coordinates and possibly of time, $V = V(x, t)$, and that the kinetic energy is a quadratic form in the generalized velocities, $T = \frac{1}{2} m g_{ij} \dot{x}_i \dot{x}_j$, where $g_{ij}(x)$ is the covariant space's metric tensor. The first term on the right hand-side of (2.9) is

$$\dot{x}_i p_i = \dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \dot{x}_i \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{2} m g_{kj} \dot{x}_k \dot{x}_j\right) = m g_{kj} \dot{x}_k \dot{x}_j = 2T. \tag{2.11}$$

Substituting in (2.9) yields

$$H = 2T - \mathcal{L} = T + V. \tag{2.12}$$

To obtain H as function in x and p we express the kinetic energy in terms of the generalized momenta p_i through expressing the generalized velocities $\dot{x}_i$ in terms of p_i:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{2} m g_{kj} \dot{x}_k \dot{x}_j\right) = m g_{ij} \dot{x}_j$$

$$\rightarrow g^{ki} p_i = g^{ki}\left(m g_{ij} \dot{x}_j\right) = m\dot{x}_k \tag{2.13}$$

Substituting in the expression of the kinetic energy we get

$$T = \frac{1}{2} m g_{ij}\left(\frac{1}{m} g^{ki} p_k\right)\left(\frac{1}{m} g^{rj} p_r\right) = \frac{1}{2m} g^{rk} p_r p_k \tag{2.14}$$

where g^{rk} is the contravariant metric tensor.

A physical observable is any function (differentiable) of the form $F(x, p, t)$. The rate of change of *F*, or its equation of motion, is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial x_i}\dot{x}_i + \frac{\partial F}{\partial p_i}\dot{p}_i + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x_i} + \frac{\partial F}{\partial t} \equiv \{F, H\} + \frac{\partial F}{\partial t} \tag{2.15}$$

The quantity $\{F, H\}$ is called the Poisson bracket of the observables F and *H*. If F does not depend on time explicitly then $\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}$, and F is a constant of motion if and only if its Poisson bracket with the Hamiltonian H vanishes. If F depends explicitly on time then it is a constant of motion if $\{F, H\} + \frac{\partial F}{\partial t} = 0$. Taking in particular $F = H$, we have by (2.15)

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = \frac{\partial V}{\partial t} \tag{2.16}$$

It follows that the total energy $H = T + V$ is a constant of motion if and only if the potential energy does not depend on time explicitly.

3. Noether's Theorem

A coordinate x_i that does not appear in $\mathcal{L}$ is called ignorable (or cyclic); it signifies the absence of the generalized force $f_i = \partial \mathcal{L} / \partial x_i$ in its direction [7] [8] [9], and by Lagrange equation (2.6), the conjugate momentum p_i is conserved. The absence of a coordinate x_i from $\mathcal{L}$ indicates symmetry in the system with respect to this coordinate. Noether's theorem generalizes the latter observation to include a broader type of symmetries. In Noether theorem, a symmetry means a transformation of the generalized coordinates, generalized velocities, and possibly of the time, that leaves the Lagrangian unchanged.

Noether's theorem states that [1]-[6]: If the transformations

$$x_i(t) \rightarrow x_i(t) + \epsilon \eta_i(t), \dot{x}_i(t) \rightarrow \dot{x}_i(t) + \epsilon \dot{\eta}_i(t), t \rightarrow t \quad (i = 1, \cdots, s) \tag{3.1}$$

where ϵ is a small number, is a symmetry transformations for some functions $\eta_i(t)$, then the quantity

$$p_i \eta_i(t) \quad (\text{sum on } i) \tag{3.2}$$

is a constant of motion, *i.e.*, it is conserved.

Proof: We will not assume here that δx_i vanishes at the extremes, and hence $\delta S \neq 0$ in (2.5). However, the change of the action resulting from the variation

(3.1) is still given by the right-hand side of (2.5). Since the motion obeys Lagrange Equation (2.6), the second term in (2.5) vanishes, giving

$$\delta S = \left[\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \delta x_i(t) \right]_{t_1}^{t}, \tag{3.3}$$

where we took the end point at an arbitrary instant of time t. If however, the transformation (3.1) is a symmetry transformation then $\mathcal{L}$ does not change by the transformation and $\delta S = 0$, which combined with $\delta x_i(t) = \epsilon \eta_i(t)$, proves Noether theorem.

Noether's theorem is often stated as follows: Whenever we have a continuous symmetry of the Lagrangian $\mathcal{L}$ there will be an associated conservation law.

4. Continuous Space's Symmetries

We view the physical space E^3 as possessing a Euclidean geometry structure, which allows for erecting a rectangular Cartesian frame $S \equiv oxyz$ whose origin o can be chosen at any point of space, and whose axes' directions are at our disposal. The distance between each two points in E^3 is the Euclidean norm of the displacement vector, and the angle between two vectors is determined through the inner product as naturally defined in E^3. The freedom in choosing the origin and the directions of the coordinate axes, which manifests space's homogeneity and isotropy, is equivalent to say that the hypothetical process of translating the space by an arbitrary vector as well as the process of rotating it about an arbitrary axis by an arbitrary angle preserve the length of the displacement vector between each two points and the angle between two vectors. The latter two facts are expressed by the equivalent statement: lengths and angles ("lengths" is sufficient) are invariant under translations and rotations. Because the space looks geometrically equivalent to itself after a translation or a rotation, the latter imaginary processes are symmetry transformations of the space.

A symmetry transformation, or a motion, of a space (or manifold) is a coordinate transformation that preserves the metric of the space (manifold); it maps the space isometrically on itself [10] [11] [12] [13] with the distance between each two points is unchanged. The most general continuous symmetry transformations of the Euclidean space E^3,

$$\overline{x}_k = f_k(x_1, x_2, x_3; \alpha_1, \cdots, \alpha_6), \ (k = 1, 2, 3), \tag{4.1}$$

with $(\alpha_1, \cdots, \alpha_6 \in R)$, forms a 6-parameter Lie group. The Lie group of isometries of the space (4.1), also called the group of motions of E^3 [12] [13], gives rise to 6 linearly independent metric preserving vector fields, or Killing fields, $X_\mu (\mu = 1, 2, \cdots, 6)$, which are its infinitesimal generators [12]. The set $\{X_\mu\}$ is a basis for the Lie algebra LA associated with the Lie group (4.1). Every element of LA can be expressed as a linear combination of the elements of $\{X_\mu\}$, and LA is closed under taking linear combinations of any number of elements, as well as, taking the commutator of any two elements. On fixing five parameters in (4.1) one obtains a one-parameter group (1-PG for short) of symmetries of the space.

In rectangular Cartesian coordinates Equation (4.1) becomes the orthogonal transformations

$$\bar{x}_k = R_{kj} x_j + \beta_k = f_k(\boldsymbol{r}; \alpha, \beta) \quad (k = 1,2,3) \tag{4.2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ and $R_{kj}(\alpha)$ is an arbitrary orthogonal matrix with determinant +1. In this case it is straight forward to list six "independent" 1-PGs of symmetries, or motions, whose generators (apart from multiplicative constants) are the components of a free particle's momentum on the three coordinate axes and angular momentum about them. A convenient way to determine the symmetries of a space or a manifold, especially when it is not Euclidean, is to specify the corresponding Lie algebra. A basis of the latter is composed of the infinitesimal generators of motion, or Killing vector fields $X = \eta_i \partial/\partial x_i$ determined by Killing equations [12] [13]

$$\eta_k g_{ij,k} + \eta_{k,i} g_{kj} + \eta_{k,j} g_{ik} = 0 \quad (i, j = 1,2,3) \tag{4.3}$$

where g_{ij} are the covariant components of the metric tensor, comma denotes differentiation with respect to the variable following it, and sum is implied on the repeated index *k*.

Space's symmetries are specified solely through geometry and do not involve time, whereas, physical quantities pertaining to physical systems embody, in general, time in their very definitions. Homogeneity of time implies that physical laws under the same conditions are the same regardless of the starting instant of time. Combining the last statement with the homogeneity of space we conclude that the outcomes of any experiment and their probabilities depend neither on the experiment's location in space nor on the chosen zero of time. Under the same conditions, all exactly similar experiments have the same results regardless of where or when they are performed. The predictive physical laws cannot therefore involve location in space or initial instant of time but they may, of course, involve distances and periods of time.

5. Continuous System's Symmetry

A coordinate transformation

$$\bar{x}_i = f_i(x_1, x_2, x_3; \alpha) \text{ or } \bar{\boldsymbol{r}} = \boldsymbol{f}(\boldsymbol{r}, \alpha) \tag{5.1}$$

where α is a real parameter, induces a unitary transformation $U_f(\alpha)$ in the Hilbert space [14] [15] of absolutely square-integrable functions, $L^2(E^3)$; it is defined by

$$(U_f \psi)(\bar{\boldsymbol{r}}) \equiv \bar{\psi}(\bar{\boldsymbol{r}}) = \psi(\boldsymbol{r}), \ \psi \in L^2(E^3) \tag{5.2}$$

If the family of coordinate transformations (5.1) with $\alpha \in \mathbb{R}$ forms an one-parameter group then so does the corresponding family of unitary transformations $\{U_f(\alpha) : \alpha \in \mathbb{R}\}$. The generator of the latter group [16] [17] is the complete vector field,

$$X = \left.\frac{\partial f_k}{\partial \alpha}\right|_{\alpha=0} \frac{\partial}{\partial x_k} \equiv \eta_k(x) \frac{\partial}{\partial x_k} \ (\text{sum on } k) \tag{5.3}$$

There corresponds to the vector field X [14] [16] [18] [19] [20] a classical momentum $P=\eta_k(x)p_k$, where p_k is the generalized momentum conjugate to x_k, and the quantum momentum

$$\hat{P}=-i\hbar\left(X+\frac{1}{2}divX\right), \tag{5.4}$$

which is essentially self-adjoint on the domain [14] [16] [18] [19]

$$D_p=\left\{f:f\in C^1\left(E^3\right),f,\hat{P}f\in L^2\left(E^3\right)\right\}, \tag{5.5}$$

where $C^1\left(E^3\right)$ is the set of continuously differentiable functions on the space.

We shall confine our attention to spatial symmetries of a physical system, which can be an electron, a hydrogen atom, a pendulum, a planet, a solar system, etc. We may imagine moving the system (a body) with respect to a coordinate frame S and in no time changing its position and orientation. The new hypothetical configuration can be achieved through a rotation by an angle φ about some axis Δ followed by a translation by some vector $\boldsymbol{b}$, and a particle of the body with coordinate $\boldsymbol{r}\in S$ is displaced instantly to the position $\overline{\boldsymbol{r}}=R(\varphi,\Delta)\boldsymbol{r}+\boldsymbol{b}\in S$. As opposed to the "active view" in which $\boldsymbol{r}$ and $\overline{\boldsymbol{r}}$ are the coordinates of the same particle in one frame of reference S, the "passive view" considers $\boldsymbol{r}$ and $\overline{\boldsymbol{r}}$ the coordinate of the same particle in two frames S and $\overline{S}$, where $\overline{S}$ is defined by the latter relation, or equivalently by $\boldsymbol{r}=R(-\varphi,\Delta)(\overline{\boldsymbol{r}}-\boldsymbol{b})$.

A physical observable, or an operator A in $L^2\left(E^3\right)$, is transformed under the coordinate transformation (5.1) to the operator $\overline{A}(\boldsymbol{r})=U_fA(\boldsymbol{r})U_f^{-1}=A\left(f^{-1}\boldsymbol{r}\right)$ [21] [22]. The observable A is said to be invariant under the transformation (5.1) if $\overline{A}(\boldsymbol{r})=A(\boldsymbol{r})$, which is equivalent to A commuting with U_f, $\left[A,U_f\right]=0$.

A physical system possesses a symmetry transformation, $\overline{\boldsymbol{r}}=\boldsymbol{f}(\boldsymbol{r},\alpha)$, if the system and its effective environment are indistinguishable from themselves before and after the transformation. This implies that the transformation must preserve the metric as well as the system's potential energy, which means that, the geometry of the surroundings and the prevailing forces are invariant under the transformation. The latter statements lead to an important conclusion: the transformation (5.1) is a symmetry transformation of a physical system if and only if its kinetic energy T and potential energy V are separately invariant under the transformation. Equivalently

$$\left[T,U_f\right]=0,\left[V,U_f\right]=0 \tag{5.6}$$

This does not mean of course that either T or V is conserved. In general neither T nor V is a constant of motion although their sum is, provided V does not depend on time. The invariance of the kinetic and potential energies follows from the invariance of the metric and the effective field of force respectively. The requirements (5.6) are more stringent than the commonly accepted fact that the transformation (5.1) is a symmetry transformation of the physical system if

$$\left[H,U_f\right]=0 \tag{5.7}$$

The latter equation results of course by summing the Equations (5.6).

Combining the result (5.6) with the fact that translations and rotations are the only continuous transformations under which the Laplace operator is invariant, and confining our consideration to unconstrained motion, we conclude that: the set of spatial continuous symmetries of any physical system is a subset of the symmetries of the space; it consists therefore of translations, rotations, and their compositions.

Orthogonal transformations preserve the norm of a vector [21] [22], and hence the invariance of the kinetic energy T under such transformations follows from the fact that, apart from a multiplicative constant, T is the square of the velocity vector. Explicitly, by (4.3), $\dot{\bar{x}}_i\dot{\bar{x}}_i = R_{ij}\dot{x}_j R_{ik}\dot{x}_k = \delta_{jk}\dot{x}_j\dot{x}_k = \dot{x}_j\dot{x}_j$. The invariance of the quantum operator of kinetic energy under rotation and translation follows from the invariance of Laplace operator under orthogonal transformation. Indeed, setting $\partial_i = \partial/\partial x_i$ and $\bar{\partial}_i = \partial/\partial\bar{x}_i$, we get from (4.3), $\partial_k = R_{ik}\bar{\partial}_i$, and hence

$$\nabla^2(\boldsymbol{r}) \equiv \partial_k\partial_k = R_{ik}R_{rk}\bar{\partial}_i\bar{\partial}_r = \delta_{ir}\bar{\partial}_i\bar{\partial}_r = \bar{\partial}_r\bar{\partial}_r \equiv \nabla^2(\bar{\boldsymbol{r}}) \equiv \nabla^2(f\boldsymbol{r}),$$

which prove our assertion.

The generator $X = \eta_k(x)\frac{\partial}{\partial x_k}$ of the 1PG (5.1), when the latter is a symmetry transformation, preserves the metric, and the volume element in particular, yielding $div X = 0$. The vanishing of the divergence of a Killing field can also be derived through multiplying Equations (4.3) by g^{ij} to get

$$\eta_k g^{ij} g_{ij,k} + 2\eta_{k,k} = 0 \to \frac{1}{2g} g_{,k}\eta_k + \eta_{k,k} = 0 \to \left(\sqrt{g}\eta_k\right)_{,k} = 0 \to div X = 0.$$

Here g is the determinant of the covariant metric tensor g_{ij}.

There corresponds to the Killing field X, the classical momentum $P = \eta_k p_k$ and the quantum momentum [14] [16] [18] [19]

$$\hat{P} = -i\hbar\eta_k(x)\frac{\partial}{\partial x_k} \tag{5.8}$$

which is essentially self-adjoint on the domain (5.5).

The one-parameter unitary group U_f acting in L^2 can be expressed as an exponential function in its infinitesimal generator X [17] [23],

$$U_f = \mathrm{e}^{\alpha X} \quad (\alpha \in \mathbb{R}). \tag{5.9}$$

In vicinity of the identity, $\alpha = 0$, we have

$$U_f = I + \alpha X + 0(\alpha^2) = S_\alpha + 0(\alpha^2) \tag{5.10}$$

where $S_\alpha = I + \alpha X$ is an infinitesimal symmetry transformation. The expression (5.10) of U_f in vicinity of the identity operator, I, shows that the potential energy commutes with U_f if and only if it commutes with U_f generator, X:

$$[V, U_f] = 0 \leftrightarrow [V, S_\alpha] = 0 \leftrightarrow [V, X] = 0 \tag{5.11}$$

Parallel relations hold for the kinetic energy T and the Hamiltonian H; they commute with U_f if and only if they commutes with *X*.

The kinetic energy is invariant under rotations and translation, and hence commute with *X*. The transformation (5.1) is therefore a symmetry transformation of a physical system if and only if its generator *X* commutes with the system potential energy. Noting that $[X,V]=0 \leftrightarrow X(V)=0$, we state:

Theorem: *The quantum* (*classical*) *momentum observable* $\hat{P}=-i\hbar\eta_k(x)\frac{\partial}{\partial x_k}$ ($P=\eta_k p_k$ *) is a constant of the motion if and only if the Lie derivative of the potential energy by the vector field X vanishes*:

$$\eta_k(x)\frac{\partial}{\partial x_k}V=0. \tag{5.12a}$$

When Cartesian coordinates are employed the latter relation can be written as $\eta_k V_{,k}=(\eta_1,\eta_2,\eta_3)\cdot(V_{,1},V_{,2},V_{,3})=0$, which is abbreviated by $X\cdot\nabla V=0$. This signifies that the vector field *X* is orthogonal to gradient *V* (*i.e.* to the force field), and hence, is tangent to the level surfaces of *V*.

Noting that the system mechanical energy $E=T+V$ is conserved if *V* is independent of time, we may adjoin to (5.12a) the energy conservation relation

$$\frac{\partial V}{\partial t}=0 \quad \leftrightarrow \text{Energy is conserved.} \tag{5.12b}$$

On the account of (5.12a) and (5.12b), a closed system of particles is invariant under translation, rotation, and translation in time. There follows that the system's total linear momentum, total angular momentum, internal energy (and of course translational kinetic energy) are conserved.

6. Examples

Example 1. Consider a particle with the Lagrangian

$$\mathcal{L}=\frac{1}{2}m\left(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2\right)-V\left(a_1x_1+a_2x_2+a_3x_3\right) \tag{6.1}$$

where (x_1,x_2,x_3) are rectangular Cartesian coordinates. Here the potential is constant on a plane Π, given by $a_1x_1+a_2x_2+a_3x_3=c$, and changes its value from a plane *c* to another c'. The symmetries consist of:

1) The group of translations $x_i \to x_i+b_i\ (i=1,2,3)$ in the plane Π. The components of the displacement vector $\boldsymbol{b}=(b_1,b_2,b_3)$ are not independent because it is in the plane Π. Indeed, in order *V* remains unchanged by the transformation, *i.e.*, is an invariant, we should have $\sum_{i=1}^{3}a_i(x_i+b_i)=c$, which yields $\sum_{i=1}^{3}a_ib_i=0$, or as to say $\boldsymbol{n}\cdot\boldsymbol{b}=0$.

2) Rotations in the plane Π about an axis defined by the normal $\boldsymbol{n}=(a_1,a_2,a_3)$ to Π and passing through any point.

When checking formula (5.12a) we should remember that a generator of symmetry can only be a linear combinations of ∂_i and $\frac{i}{\hbar}L_i$, and only those combination that satisfy (5.12a) are the possible generators of continuous spatial symmetries. It is clear that the results we obtain hold good equally for classical and quantum momenta.

1) Two independent generators of the group of translation in Π can be chosen as

$$X_1 = a_2\,\partial/\partial x_1 - a_1\,\partial/\partial x_2,\ \ X_2 = a_3\,\partial/\partial x_2 - a_2\,\partial/\partial x_3.$$

Any other generator of this group is a linear combination of X_1 and X_2. It is apparent that the Lie derivative of V by X_1 and X_2 vanishes, which give rise to the conserved momenta $P_1 = a_2 p_1 - a_1 p_2$ and $P_2 = a_3 p_2 - a_2 p_3$. We may replace X_2 by $X_2' = -a_1 a_3 \partial_1 - a_2 a_3 \partial_2 + \left(a_1^2 + a_2^2\right)\partial_3$ which is orthogonal to X_1.

2) The generator of the group of rotations is $X_3 = \frac{i}{\hbar}\boldsymbol{n}\cdot\boldsymbol{L} = \frac{i}{\hbar}a_i L_i \equiv \frac{i}{\hbar}P_3$, which is the component of the angular momentum vector $\boldsymbol{L} = \left(L_1, L_2, L_3\right)$ on $\boldsymbol{n}$. Any other conserved momentum is a linear combination of P_1, P_2 and P_3.

It is straightforward to verify that the derived momenta are constants of motion through showing that their commutation relations, or Poisson bracket, with the Hamiltonian

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + V\left(a_1 x_1 + a_2 x_2 + a_3 x_3\right)$$

vanish. For P_1, for instance, we have

In the quantum case: $\left[P_1, H\right] = \left[P_1, V\right] = -i\hbar\left[X_1, V\right] = -i\hbar X_1\left(V\right) = 0$

In the classical case:

$$\left\{P_1, H\right\} = \frac{\partial P_1}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial P_1}{\partial p_j}\frac{\partial H}{\partial x_j} = -\frac{\partial P_1}{\partial p_j}\frac{\partial H}{\partial x_j} = -\left(a_2\frac{\mathrm{d}V}{\mathrm{d}c}a_1 - a_1\frac{\mathrm{d}V}{\mathrm{d}c}a_2\right) = 0$$

Example 2. We apply (5.12a) to the following example which is given in [6]. Consider a one particle system with the Lagrangian given in cylindrical coordinates by

$$\mathcal{L} = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varnothing}^2 + \dot{z}^2\right) - V\left(\rho, a\varnothing + z\right)$$

It is clear that the Lie derivative of V by the vector field $X = \frac{\partial}{\partial\varnothing} - a\frac{\partial}{\partial z}$ vanishes. Moreover, X is an infinitesimal motion of the space because it is a linear combination of the infinitesimal motions $\frac{\partial}{\partial\varnothing}$ and $\frac{\partial}{\partial z}$. Since $X\left(V\right) = 0$, the momentum $P = p_{\varnothing} - a p_z$ is conserved.

7. Symmetries and Potentials

The symmetries of a free particle are the symmetries of the space E_3. The generators of these symmetries can be chosen the following six linearly independent Killing vector fields:

$$X_i = \partial_i,\ \ Y_i = \epsilon_{ijk} x_j \partial_k\ \ \left(i = 1, 2, 3\right) \tag{7.1}$$

where ϵ_{ijk} is the permutation (Levi-Civita) symbol. Any other infinitesimal generator of symmetry is a linear combination of the Killing vectors (7.1). The constants of motions corresponding to (7.1) are the components of the momentum and angular momentum:

$$p_i \text{ and } L_i = \epsilon_{ijk} x_j p_k \quad (i = 1,2,3) \tag{7.2}$$

It is easy to check that these momenta commute (have a vanishing Poisson bracket) with the Hamiltonian $H = (2m)^{-1} p_i p_i$.

The 1-PGs of symmetry transformations whose generators are the Killing fields (7.1) are: The three 1-PGs of translations along the coordinate axes:

$$\overline{x}_i = x_i + \alpha_i,\ \overline{x}_{i+1} = x_{i+1},\ \overline{x}_{i+2} = x_{i+2} \tag{7.3a}$$

And the three 1-PGs of rotations about the coordinate axes (no sum on repeated induces in the rest of this section)

$$\overline{x}_i = x_i \cos\beta_i + x_{i+1}\sin\beta_i,\ \overline{x}_{i+1} = -x_i \sin\beta_i + x_{i+1}\cos\beta_i,\ \overline{x}_{i+2} = x_{i+2} \tag{7.3b}$$

where $(i = 1,2,3)$ and $i+1$ stands for $(i+1)\bmod 3$. *I.e.*, $i+1 = 1, i+2 = 2$ for $i = 3$, and $i+2 = 1$ for $i = 2$. Any other symmetry transformation is a composite of the these six independent symmetry transformations.

The 1-PGs of unitary transformations of $L^2(E^3)$ induced by the 1-PGs of symmetry transformations of E^3 are

$$U(\alpha_k) = \mathrm{e}^{-\alpha_k \partial_k},\ U(\beta_k) = \mathrm{e}^{\frac{i\beta_k}{\hbar} L_k} \tag{7.4}$$

As an example we calculate $U(\beta_1)$ in which we drop the index 1, and call the coordinates (x, y, z)

$$U(\beta)\psi(\mathbf{r}) = \psi\left(R^{-1}(\beta)\mathbf{r}\right) = \psi(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, z) \tag{7.5}$$

In vicinity of the identity, $\beta = 0$,

$$\begin{aligned} U(\beta)\psi(\mathbf{r}) &= \psi(x - \beta y, y + \beta x, z) \\ &= \left[1 + \beta(x\partial_y - y\partial_x) + \frac{1}{2}\beta^2 (x\partial_y - y\partial_x)^2 - \cdots\right]\psi(\mathbf{r}) \\ &= \mathrm{e}^{\frac{i\beta}{\hbar} L_z}\psi(\mathbf{r}) \end{aligned} \tag{7.6}$$

We list here the symmetries that occur according to the functional form of the potential energy. In all cases below the total energy is conserved as V does not depend on time.

1) If the potential energy depends only on *x*, then the constants of motion are p_y, p_z, L_x, and any function in them. Parallel statements hold for $V = V(y)$ and $V = V(z)$.

2) If the potential energy is of the form $V = V(x, y)$ then p_z is conserved. Moreover, if $V(x, y) = V(x^2 + y^2)$ then $L_z V = 0$ and L_z is conserved. If $V = V(ax + by)$ then p_z and $bp_x - ap_y$ are conserved as well as any function in them. Parallel statements hold for a cyclic permutation in the coordinates.

3) If the potential energy is of the form $V(x, y, z) = V(x^2 + y^2 + z^2)$ then $\boldsymbol{L} = (L_x, L_y, L_z)$ is conserved. If $V(x, y, z) = V(ax + by + cz)$ then $bp_x - ap_y$, $cp_y - bp_z$ and $aL_x + bL_y + cL_z$ are conserved, as well as any function in them.

8. Conclusion

The set of symmetries of a mechanical system is a subset of the symmetries of the space. A space's symmetry is admitted as a symmetry of a system if the directional derivative of the potential energy by its infinitesimal generator vanishes. The found results provide a scheme to specify the sought symmetries, and shed new insight in the inspiring and beautiful Noether's theorem.

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