

# Investigation of the Class of the Rational Difference Equations

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## Abstract

This paper is concerned with asymptotic behavior of the solution of a new class of rational Difference Equations. We consider the local and global stability of the solution. Moreover we investigate the new periodic character (periodic two) of solutions of these equations. Finally, we give some interesting counter examples in order to verify our strong results.

## Keywords

Difference Equation, Stability, Periodicity, Boundedness

## 1. Introduction

The objective of this work is to investigate the asymptotic behavior of the solutions of the following difference equation

$$\omega_{n+1} = a + \frac{b_0 \omega_{n-1}}{c_1 \omega_n + c_2 \omega_{n-1}} + \frac{b_1 \omega_{n-3}}{c_3 \omega_{n-2} + c_4 \omega_{n-3}}, n = 0, 1, 2, \dots \quad (1)$$

where  $a, b_0, b_1, c_1, c_2, c_3$  and  $c_4 \in [0, \infty)$  and the initial conditions  $\omega_{-3}, \dots, \omega_{-1}$  and  $\omega_0$  are arbitrary positive real numbers.

In recent years, there are a great interest in studying the rational difference equations. These equations describe real life situations in queuing theory, stochastic time series, combinatorial analysis, electrical network, number theory, genetics in biology, psychology, probability theory, statistical problems, economics, etc. The study of rational difference equations of high order (greater than one) is a big challenge. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one. Therefore, the study of such equations is so interesting. There has been a lot of work concerning the global asymptotic of solutions of ra-

tional difference equations [1]-[21].

In fact, there has been a lot of interest in studying the behavior of the nonlinear difference equation of the form

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}. \quad (2)$$

For multiple delay and order, see [3] [18] [19] [20] and their references for more results of this equation. In Theorems 4.7.1-4.7.5 in [12], Kulenovic investigated the asymptotic behavior of the solutions of the equation

$$x_{n+1} = \frac{\gamma x_{n-1}}{Bx_n + Cx_{n-1}}. \quad (3)$$

Metwally *et al.* in [14] established a global convergence result and then apply it to show that under appropriate hypotheses every positive solution of the difference equation

$$x_{n+1} = \sum_{i=0}^m \frac{A_i}{x_{n-2i}},$$

where  $A_i \in [0, \infty)$  for  $i = 0, 1, \dots, m$ . In [8], Elsayed studied the periodicity, the boundedness, and the global stability of the positive solution of the difference equation

$$x_{n+1} = \alpha + \beta \frac{x_n}{x_{n-1}} + \gamma \frac{x_{n-1}}{x_n},$$

where the parameters  $\alpha, \beta$  and  $\gamma$  are positive real numbers and the initial conditions  $x_{-1}, x_0$  are positive real numbers. Recently, Moaaz *et al.* [15] investigated some qualitative behavior of the following nonlinear difference equations

$$x_{n+1} = \alpha + \beta \frac{x_{n-l}}{x_{n-k}} + \gamma \frac{x_{n-l}}{x_{n-s}},$$

where the initial conditions  $x_{-r}, x_{-r+1}, \dots, x_0$  such that  $r = \max\{l, k, s\}$  are arbitrary real numbers and  $\alpha, \beta$  and  $\gamma$  are positive constants.

In this paper, in section 2, we state the sufficient condition for the asymptotic stability of Equation (1). Next, in section 3, we study the existence of periodic solutions of Equation (1). Finally, we study the boundedness nature of the solutions of Equation (1). Some numerical examples will be given to explicate our results.

During this study, we will need to many of the basic concepts. Before anything, the concept of equilibrium point is essential in the study of the dynamics of any physical system. A point  $\bar{\omega}$  in the domain of the function  $\Phi$  is called an **equilibrium point** of the equation

$$\omega_{n+1} = \Phi(\omega_n, \omega_{n-1}, \dots, \omega_{n-k}), \quad n = 0, 1, 2, \dots \quad (4)$$

if  $\bar{\omega}$  is a fixed point of  $\Phi$  [ $\Phi(\bar{\omega}, \bar{\omega}, \dots, \bar{\omega}) = \bar{\omega}$ ]. For a stability of equilibrium point, equilibrium point  $\bar{\omega}$  of Equation (4) is said to be **locally stable** if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that, if  $\omega_{-v} \in (0, \infty)$  for  $v = 0, 1, \dots, k$  with

$\sum_{i=0}^k |\omega_{-i} - \bar{\omega}| < \delta$ . As well,  $\bar{\omega}$  is said to be **locally asymptotically stable** if it is locally stable and there exists  $\gamma > 0$  such that, if  $\omega_{-\nu} \in (0, \infty)$  for  $\nu = 0, 1, \dots, k$  with  $\sum_{i=0}^k |\omega_{-i} - \bar{\omega}| < \gamma$ , then  $\lim_{n \rightarrow \infty} \omega_n = \bar{\omega}$ . Also,  $\bar{\omega}$  is said to be a **global attractor** if for every  $\omega_{-\nu} \in (0, \infty)$  for  $\nu = 0, 1, \dots, k$ , we have  $\lim_{n \rightarrow \infty} \omega_n = \bar{\omega}$ . On the other hand,  $\bar{\omega}$  is said to be unstable if it is not locally stable.

Finally, Equation (4) is called **permanent and bounded** if there exists numbers  $m$  and  $M$  with  $0 < m < M < \infty$  such that for any initial conditions  $\omega_{-\nu} \in (0, \infty)$  for  $\nu = 0, 1, \dots, k$  there exists a positive integer  $N$  which depends on these initial conditions such that  $m < \omega_n < M$  for all  $n \geq N$ .

The linearized equation of Equation (1) about the equilibrium point  $\hat{\omega}$  is

$$y_{n+1} = \sum_{i=0}^k p_i y_{n-i} \tag{5}$$

where

$$p_i = \frac{\partial F}{\partial \omega_{n-i}}(\hat{\omega}, \hat{\omega}, \dots, \hat{\omega}).$$

**Theorem 1.1** [12] *Assume that  $p_i \in \mathbb{R}$  for  $i = 0, 1, \dots, k$ . Then*

$$|p_0| + |p_1| + \dots + |p_k| < 1 \tag{6}$$

is a sufficient condition for the asymptotic stability of Equation(1).

## 2. The Stability of Solutions

The positive equilibrium point of Equation (1) is given by

$$\hat{\omega} = a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}.$$

Now, we define a function  $f \in C((0, \infty)^4, (0, \infty))$  such that

$$f(u, v, w, r) = a + \frac{b_0 v}{c_1 u + c_2 v} + \frac{b_1 r}{c_3 w + c_4 r}.$$

Therefore, we find

$$\frac{\partial f}{\partial u} = -\frac{b_0 c_1 v}{(c_1 u + c_2 v)^2}, \tag{7}$$

$$\frac{\partial f}{\partial v} = \frac{b_0 c_1 u}{(c_1 u + c_2 v)^2}, \tag{8}$$

$$\frac{\partial f}{\partial w} = -\frac{b_1 c_3 r}{(c_3 w + c_4 r)^2} \tag{9}$$

and

$$\frac{\partial f}{\partial r} = \frac{b_1 c_3 w}{(c_3 w + c_4 r)^2}. \tag{10}$$

**Theorem 2.1.** *Assume that  $\hat{\omega}$  be a +ve equilibrium point of Equation (1). If*

$$b_0c_1\mu + b_1c_3\lambda < a\lambda\mu + b_0c_2\mu + b_1c_4\lambda.$$

where  $\lambda = (c_1 + c_2)^2$  and  $\mu = (c_3 + c_4)^2$ , then  $\hat{w}$  is local stable.

*Proof.* From (7) to (10), we get

$$\frac{\partial f}{\partial u}(\hat{w}, \hat{w}, \hat{w}, \hat{w}) = -\frac{b_0c_1}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} = p_0,$$

$$\frac{\partial f}{\partial v}(\hat{w}, \hat{w}, \hat{w}, \hat{w}) = \frac{b_0c_1}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} = p_1,$$

$$\frac{\partial f}{\partial w}(\hat{w}, \hat{w}, \hat{w}, \hat{w}) = -\frac{b_1c_3}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} = p_2$$

and

$$\frac{\partial f}{\partial r}(\hat{w}, \hat{w}, \hat{w}, \hat{w}) = \frac{b_1c_3}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} = p_3.$$

Thus, the linearized equation of (1), is

$$y_{n+1} = p_0y_n + p_1y_{n-1} + p_2y_{n-2} + p_3y_{n-3}.$$

From Theorem 1.1, we have that Equation (1) is locally stable if

$$\left| -\frac{b_0c_1}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} \right| + \left| \frac{b_0c_1}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} \right| + \left| -\frac{b_1c_3}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} \right| + \left| \frac{b_1c_3}{\left(a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4}\right)(c_1 + c_2)^2} \right| < 1,$$

and hence,

$$2b_0c_1(c_3 + c_4)^2 + 2b_1c_3(c_1 + c_2)^2 < \left[ a + \frac{b_0}{c_1 + c_2} + \frac{b_1}{c_3 + c_4} \right] (c_1 + c_2)^2 (c_3 + c_4)^2.$$

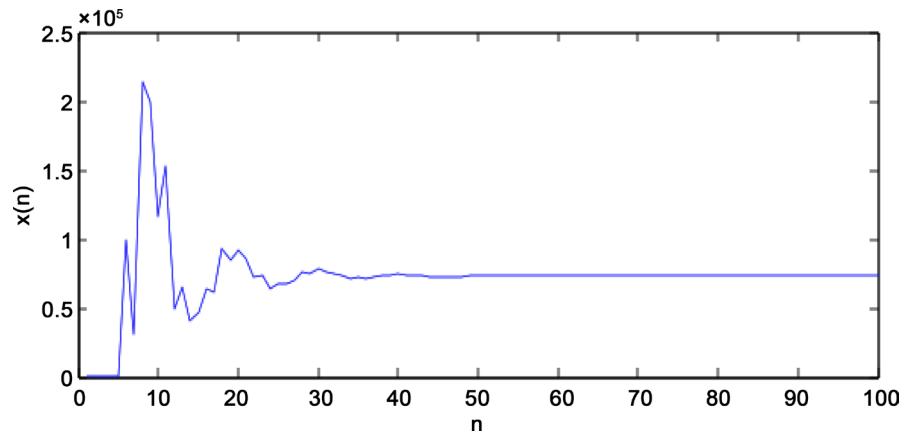
Then, we find

$$\begin{aligned} & b_0c_1(c_3 + c_4)^2 + b_1c_3(c_1 + c_2)^2 \\ & < a(c_1 + c_2)^2(c_3 + c_4)^2 + b_0c_2(c_3 + c_4)^2 + b_1c_4(c_1 + c_2)^2, \end{aligned}$$

This completes the proof of Theorem 2.1.  $\square$

**Example 2.1.** Figure 1 shows that Equation (1) has local stable solutions if  $a = 0.01$ ,  $b_0 = 200$ ,  $b_1 = 20$ ,  $c_1 = 0.002$ ,  $c_2 = 0.01$ ,  $c_3 = 0.002$  and  $c_4 = 0.001$ .

**Theorem 2.2.** If  $c_1 = c_2 = \alpha$  and  $c_3 = c_4 = \beta$ , then the equilibrium point  $\hat{w}$



**Figure 1.** The stable solution corresponding to difference Equation (1).

Equation (1) is global attractor.

*Proof.* We consider the following function

$$f(u, v; w, r) = a + \frac{b_0 v}{c_1 u + c_2 v} + \frac{b_1 r}{c_3 w + c_4 r} \tag{11}$$

Note that  $f$  non-decreasing for  $v, r$  and non-increasing for  $u, w$ . Let  $(\psi, \Psi)$  a solution of the system

$$\psi = f(\Psi, \psi, \Psi, \psi)$$

$$\Psi = f(\psi, \Psi, \psi, \Psi)$$

From (11), we have

$$(\Psi - a)(c_1 \psi + c_2 \Psi)(c_3 \psi + c_4 \Psi) = b_0 \Psi (c_3 \psi + c_4 \Psi) + b_1 \Psi (c_1 \psi + c_2 \Psi)$$

and

$$(\psi - a)(c_1 \Psi + c_2 \psi)(c_3 \Psi + c_4 \psi) = b_0 \psi (c_3 \Psi + c_4 \psi) + b_1 \psi (c_1 \Psi + c_2 \psi)$$

Thus, we get

$$\alpha \beta (\Psi - a)(\Psi + \psi) = \Psi b_0 \beta (\Psi + \psi) + \Psi b_1 \alpha (\Psi + \psi), \tag{12}$$

$$\alpha \beta (\Psi - a)(\Psi + \psi) = \psi b_0 \beta (\Psi + \psi) + \Psi b_1 \alpha (\Psi + \psi). \tag{13}$$

By subtracting (12) and (13), we have

$$(b_0 \beta + b_1 \alpha)(\Psi - \psi) = 0$$

Since  $b_0 \beta + b_1 \alpha \neq 0$ , we get

$$\Psi = \psi.$$

This completes the proof of this Theorem.  $\square$

### 3. Periodic Solution of Period Two

In this section, we investigate the existence of periodic solutions of Equation (1).

**Theorem 3.1.** Assume that  $c_1 = c_3 = \alpha$  and  $c_2 = c_4 = \beta$ . Equation (1) has positive prime period-two solutions if

$$(b_0 + b_1)(\alpha - \beta) > a(\alpha^2 + \beta^2) + 2a\alpha\beta. \tag{14}$$

*Proof.* Suppose that there exists a prime period-two solution

$$\dots, \sigma, \rho, \sigma, \rho, \sigma, \rho, \dots$$

of Equation (1). We will prove that condition (14) holds. We see from Equation (1) that if  $c_1 = c_3 = \alpha$  and  $c_2 = c_4 = \beta$ , then  $\omega_n = \omega_{n-2} = \rho$ ,  $\omega_{n+1} = \omega_{n-1} = \omega_{n-3} = \sigma$  and so,

$$\sigma = a + \frac{b_0\sigma}{c_1\rho + c_2\sigma} + \frac{b_1\sigma}{c_3\rho + c_4\sigma} \text{ and } \rho = a + \frac{b_0\rho}{c_1\sigma + c_2\rho} + \frac{b_1\rho}{c_3\sigma + c_4\rho}.$$

Thus, we have

$$\alpha\sigma\rho + \beta\sigma^2 = a\alpha\rho + a\beta\sigma + b_0\sigma + b_1\sigma \quad (15)$$

and

$$\alpha\sigma\rho + \beta\rho^2 = a\alpha\sigma + a\beta\rho + b_0\rho + b_1\rho. \quad (16)$$

By subtracting (15) and (16), we have

$$\beta(\sigma^2 - \rho^2) = -a(\sigma - \rho) + a\beta(\sigma - \rho) + b_0(\sigma - \rho) + b_1(\sigma - \rho),$$

then, we find

$$\sigma + \rho = \frac{-\alpha a + \beta a + b_0 + b_1}{\beta}. \quad (17)$$

By combining (15) and (16), we have

$$2\alpha\sigma\rho + \beta(\sigma^2 + \rho^2) = a\alpha(\sigma + \rho) + a\beta(\sigma + \rho) + b_0(\sigma + \rho) + b_1(\sigma + \rho)$$

and so,

$$\beta\sigma\rho(\alpha - \beta) = a\alpha(-\alpha a + a\beta + b_0 + b_1)$$

Since,  $\sigma^2 + \rho^2 = (\sigma + \rho)^2 - 2\sigma\rho$ , we obtain

$$\sigma\rho = \frac{a\alpha(-\alpha a + a\beta + b_0 + b_1)}{\beta(\alpha - \beta)} \quad (18)$$

Now, it is clear from (17) and (18) that  $\sigma$  and  $\rho$  are both two positive distinct roots of the quadratic equation

$$u^2 + (\sigma + \rho)u + \sigma\rho = 0. \quad (19)$$

Hence, we obtain

$$\frac{(-\alpha a + \beta a + b_0 + b_1)^2}{\beta^2} - \frac{4a\alpha(-\alpha a + a\beta + b_0 + b_1)}{\beta(\alpha - \beta)} > 0,$$

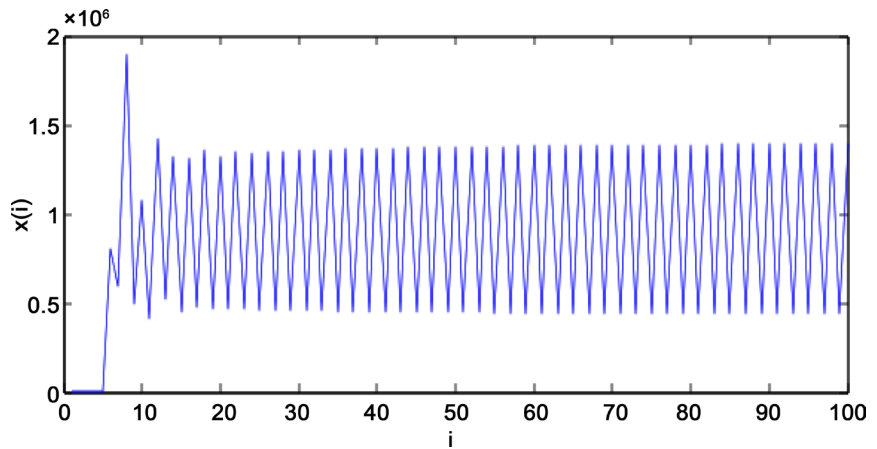
which is equivalent to

$$(b_0 + b_1)(\alpha - \beta) > a(\alpha^2 + \beta^2) + 2a\alpha\beta.$$

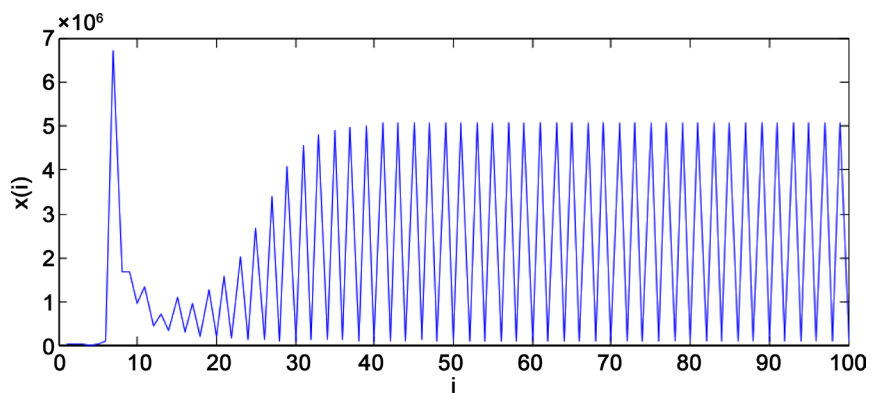
This completes the proof of Theorem 3.1.  $\square$

**Example 3.1.** Consider Equation (1) with  $c_1 = c_4 = \beta = 0.001$ ,  $c_3 = c_4 = \alpha = 0.002$ ,  $a = 0.01$ ,  $b_0 = 200$  and  $b_1 = 20000$ . By Theorem 3.1, Equation (1) has prime period two solution (see **Figure 2**).

**Example 3.2.** Consider Equation (1) with  $c_1 = c_4 = \beta = 0.0003$ ,  $c_3 = c_4 =$



**Figure 2.** Prime period two solution of Equation (1).



**Figure 3.** Prime period two corresponding to differences Equation (1).

$\alpha = 0.005$ ,  $a = 0.8$ ,  $b_0 = 500$  and  $b_1 = 2000$ . By Theorem 3.1, Equation (1) has prime period two solution (see Figure 3).

#### 4. Boundedness of the Solutions

In this section, we study the characteristic task of boundedness of the positive solutions of Equation (1).

**Theorem 4.1.** *Every solution of (1) is bounded and persists.*

*Proof.* From Equation (1), we have

$$\begin{aligned} \omega_{n+1} &= a + \frac{b_0 \omega_{n-1}}{c_1 \omega_n + c_2 \omega_{n-1}} + \frac{b_1 \omega_{n-3}}{c_3 \omega_{n-2} + c_4 \omega_{n-3}} \\ &\leq a + \frac{b_0 \omega_{n-1}}{c_2 \omega_{n-1}} + \frac{b_1 \omega_{n-3}}{c_4 \omega_{n-3}} \\ &= a + \frac{b_0}{c_2} + \frac{b_1}{c_4} \end{aligned}$$

then,

$$0 < \omega_n \leq a + \frac{b_0}{c_2} + \frac{b_1}{c_4} = M$$

Hence, the proof is completed.  $\square$

**Conclusion 1.** This work is concerned with studying a dynamics and behavior of solutions of a new class of difference Equation (1). Our results extend and generalize to previous studies, for example, Equation (2) (if  $b_1 = c_1 = 0$ ) and Equation (3) (if  $a = b_1 = 0$ ). Furthermore, we obtain the following results:

- The +ve equilibrium point  $\hat{\omega}$  of equation (1) is local stable if  $b_0c_1\mu + b_1c_3\lambda < a\lambda\mu + b_0c_2\mu + b_1c_4\lambda$ , where  $\lambda = (c_1 + c_2)^2$  and  $\mu = (c_3 + c_4)^2$ . Also, if  $c_1 = c_2$  and  $c_3 = c_4$ , then  $\hat{\omega}$  is global attractor.
- Equation (1) has a prime period-two solutions if  $c_1 = c_3 = \alpha$ ,  $c_2 = c_4 = \beta$  and  $(b_0 + b_1)(\alpha - \beta) > a(\alpha^2 + \beta^2) + 2\alpha\beta$ .
- Every solution of (1) is bounded and persists.

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