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Boundedness of Calderón-Zygmund Operator and Their Commutator on Herz Spaces with Variable Exponent

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Abstract

The aim of this paper is to study the boundedness of Calderón-Zygmund operator and their commutator on Herz Spaces with two variable exponents $p(.), q(.)$. By applying the properties of the Lebesgue spaces with variable exponent, the boundedness of the Calderón-Zygmund operator and the commutator generated by BMO function and Calderón-Zygmund operator is obtained on Herz space.

Keywords

Calderón-Zygmund Operator, Commutator, Herz Spaces with Variable Exponent, BMO Spaces

1. Introduction

Definition 1.1. Let T be a bounded linear operator from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$ (see [1], [2]). T is called a standard operator if T satisfies the following conditions:

1) T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$.

2) There exists a function $K(x,y)$ defined by $\{(x,y)\in(\mathbb{R}^n)\times(\mathbb{R}^n); x\neq y\}$ satisfies

$$|K(x,y)| \leq C/|x-y|^n, \quad (1.1)$$

where $C>0$.

3) $\langle Tf, g\rangle = \int_{(\mathbb{R}^n)}\int_{(\mathbb{R}^n)} K(x,y) f(y) g(x) \mathrm{d}x\mathrm{d}y$, for $f,g\in S(\mathbb{R}^n)$ with $\operatorname{supp}(f)\cap\operatorname{supp}(g)=\varnothing$

A standard operator T is called a γ-Calderón-Zygmund operator if K is a standard kernel satisfies:

$$\left|K(x,y)-K(z,y)\right| \le C\left|x-z\right|^{\gamma} \big/ \left|x-y\right|^{n+\gamma}; \tag{1.2}$$

$$\left|K(y,x)-K(y,z)\right| \le C\left|x-z\right|^{\gamma} \big/ \left|x-y\right|^{n+\gamma}, \tag{1.3}$$

if $|x-z| < \frac{1}{2}|x-y|$ for some $0 < \gamma \le 1$.

The bounded mean oscillation BMO space and BMO norm are defined, respectively, by

$$BMO\left(\mathbb{R}^n\right) = \left\{ b \in L^1_{loc}\left(\mathbb{R}^n\right) : \|b\|_{BMO\left(\mathbb{R}^n\right)} < \infty \right\}, \tag{1.4}$$

$$\|b\|_{BMO\left(\mathbb{R}^n\right)} = \sup_{B:\text{ball}} 1\big/|B| \int_B \left|b(x)-b_B\right| \mathrm{d}x. \tag{1.5}$$

The commutator of the Calderón-Zygmund operator is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x). \tag{1.6}$$

In 1983, J.-L. Jouné proved γ-Calderón-Zygmund operator is bounded on $L^p\left(\mathbb{R}^n\right)$ in [3]. Coifman, Rochberg and Weiss proved that commutator [b,T] is bounded on $L^p\left(\mathbb{R}^n\right)(1 < p < 1)$ (see [4]).

Kovácik and Rákosník introduced Lebesgue spaces and Sobolev spaces with variable exponents (see [5]). The function spaces with variable exponent has been recently obtained an increasing interest by a number of authors since many applications are found in many different fields, for example, in fluid dynamics (see [6]), image restoration (see [7] [8] [9]) and differential equations.

Herz spaces play an important role in harmonic analysis. After they were introduced in [10], the boundedness of some operators and some characterizations of Herz spaces with variable exponents were studied extensively (see [11]-[16]). In 2015, Wang and Tao introduced the Herz spaces with two variable exponents $p(.), q(.)$, and studied the parameterized Littlewood-Paley operators and their commutators on Herz spaces with variable exponents in [17].

In this paper, we will discuss the boundedness of the Calderón-Zygmund operator T and their commutator $[b,T]$ are bounded on Herz spaces with two variable exponents $p(.), q(.)$.

2. Definitions of Function Spaces with Variable Exponent

In this section we recall some definitions. Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We firstly recall the definition of the Lebesgue spaces with variable exponent.

Definition 2.1. [5] Let $p(\cdot): \Omega \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ f \text{ is measurable} : \int_\Omega \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0 \right\}. \tag{2.1}$$

For all compact $K \subset \Omega$, the space $L^{p(\cdot)}_{loc}(\Omega)$ is defined by

$$L_{loc}^{p(\cdot)}(\Omega)=\left\{ f \text{ is measurable}: f\in L^{p(\cdot)}(K)\right\}. \tag{2.2}$$

The Lebesgue spaces $L^{p(\cdot)}(\Omega)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\eta>0:\int_\Omega\left(\frac{|f(x)|}{\eta}\right)^{p(x)}\mathrm{d}x\le 1\right\}. \tag{2.3}$$

We denote $p_-=essinf\{p(x):x\in\Omega\}$, $p_+=ess\sup\{p(x):x\in\Omega\}$. Then $\mathcal{P}(\Omega)$ consists of all $p(\cdot)$ satisfying $p_->1$ and $p_+<\infty$. Let M be the Hardy-Littlewood maximal operator. We denote $\mathcal{B}(\Omega)$ to be the set of all function $p(\cdot)\in\mathcal{P}(\Omega)$ satisfying the M is bounded on $L^{p(\cdot)}(\Omega)$.

Definition 2.2. [18] Let $p(\cdot),q(\cdot)\in\mathcal{P}(\Omega)$. The mixed Lebesgue sequence space with variable exponent $\ell^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{f_j\}_{j=0}^\infty$ of the measurable functions on $\mathbb{R}^n$ such that

$$\left\|\{f_j\}_{j=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}=\inf\left\{\eta>0:Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\zeta}\right\}_{j=0}^\infty\right)\le 1\right\}<\infty,$$
$$Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^\infty\right)=\sum_{j=0}^\infty\inf\left\{\zeta_j>0;\int_{R^n}\left(\frac{|f_j(x)|}{\zeta_j^{\frac{1}{q(x)}}}\right)^{p(x)}\mathrm{d}x\le 1\right\}. \tag{2.4}$$

Let $B_k=\{x\in\mathbb{R}^n:|x|\le 2^k\}, C_k=B_k\setminus B_{k-1}, \chi_k=\chi_{C_k}$, $k\in\mathbb{Z}.$, for $q_+<\infty$, we have that

$$Q_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^\infty\right)=\sum_{j=0}^\infty\left\||f_j|^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}. \tag{2.5}$$

Let $B_k=\{x\in\mathbb{R}^n:|x|\le 2^k\}, C_k=B_k\setminus B_{k-1}, \chi_k=\chi_{C_k}$, $k\in\mathbb{Z}.$

Definition 2.3. [17] Let $\alpha\in\mathbb{R}^n, q(\cdot), p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)=\left\{f\in L_{loc}^{p(\cdot)}(\mathbb{R}^n\setminus\{0\}):\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)}<\infty\right\}.$$

Equipped the norm

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)}=\left\|\{2^{k\alpha}|f\chi_k|\}_{k=0}^\infty\right\|_{l^{q(\cdot)}(L^{p(\cdot)})}$$
$$=\inf\left\{\eta>0:\sum_{k=-\infty}^\infty\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}\le 1\right\}.$$

Remark 2.1. [17] Let $q_1(\cdot),q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying $(q_1)_+\le(q_2)_+$ and satisfy the following results:

1) $\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)\subset\dot{K}_{p(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$.

2) If $\frac{q_2(\cdot)}{q_1(\cdot)}\in\mathcal{P}(\mathbb{R}^n)$ and $\frac{q_2(\cdot)}{q_1(\cdot)}\ge 1$. For any $f\in\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$, by using

Lemma 3.7 and Remark 2.2, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{p_v}$$

$$\le \left\{\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{p_h}\right\}^{p_*} \le 1.$$

where

$$p_v = \begin{cases} \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_-, & \frac{2^{k\alpha}|f\chi_k|}{\eta} \le 1, \\ \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_+, & \frac{2^{k\alpha}|f\chi_k|}{\eta} > 1. \end{cases}$$

$$p_* = \begin{cases} \min_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v \le 1, \\ \max_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v > 1. \end{cases}$$

This implies that $\dot{K}_{p(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n) \subset \dot{K}_{p(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$.

Remark 2.2. Let $v \in \mathbb{N}, a_v \ge 0, 1 \le p_v < \infty$. Then we have

$$\sum_{v=0}^{\infty} a_v \le \left(\sum_{v=0}^{\infty} a_h\right)^{p_*},$$

where

$$p_* = \begin{cases} \min_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v \le 1, \\ \max_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v > 1. \end{cases}$$

3. Properties and Lemmas of Variable Exponent

In this section, we recall some properties and some lemmas of variable exponent belonging to the class $\mathcal{B}(\mathbb{R}^n)$.

Proposition 3.1. [19] If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies

$$|p(x) - p(y)| \le \frac{-C}{\operatorname{Log}(|x-y|)}, |x-y| \le 1/2; \tag{3.1}$$

$$|p(x) - p(y)| \le \frac{C}{\operatorname{Log}(e+|x|)}, |y| \ge |x|. \tag{3.2}$$

Hence we have $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

Lemma 3.1. [5] Given $p(\cdot): \mathbb{R}^n \to [1,\infty)$ have that for all functions f and g,

$$\int_{\mathbb{R}^n} |f(x)g(x)| \mathrm{d}x \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{3.3}$$

where $C_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

Lemma 3.2. [5] Suppose that $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, for any $f \in L^{p_1(\cdot)}(\mathbb{R}^n), g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, when $\frac{1}{p(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{p_1(\cdot)}$, we get

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|g(x)\|_{L^{p_2}(\mathbb{R}^n)}\|f(x)\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}, \tag{3.4}$$

where $C_{p_1,p_2} = \left[1 + \frac{1}{p_{1-}} - \frac{1}{p_{1+}}\right]^{\frac{1}{p_-}}$.

Proposition 3.2. [20] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and T be a Calderón - Zygmund operator. Then we have

$$\|Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.5}$$

Lemma 3.3. [20] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n), b \in \mathrm{BMO}$ function and T be a Calderón - Zygmund operator.Then

$$\|[b,T]f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{3.6}$$

Lemma 3.4. [11] Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. If $i, j \in \mathbb{Z}$ with $i < j$, then we have

1. $C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$.
2. $\left\|(b - b_{B_i})\chi_{B_j}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C(j-i)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}$.

Lemma 3.5. [21] Let $p_u(\cdot) \in \mathcal{B}(\mathbb{R}^n)(u = 1,2)$, then there exist constants $0 < \iota_{u1}, \iota_{u2} < 1$, and $C > 0$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable subset $R \subset B$,

$$\frac{\|\chi_R\|_{L^{p_u(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_u(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{\iota_{u1}}, \frac{\|\chi_R\|_{L^{p'_u(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_u(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{\iota_{u2}}. \tag{3.7}$$

Lemma 3.6. [11] If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, there exist a constant $C > 0$ such that for any balls B in $\mathbb{R}^n$, we have

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C. \tag{3.8}$$

Lemma 3.7. [17] Suppose that $p(\cdot), q(\cdot) \in \mathcal{P}(\mathcal{B}^n)$. If $f \in L^{p(\cdot)q(\cdot)}$, then

$$\min\left(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}\right) \le \left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)}} \le \max\left(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}\right). \tag{3.9}$$

4. The Main Theorems and Their Proofs

Theorem 4.1. Suppose that $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n), q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. If $-n\iota_{12} < \alpha < n\iota_{11}$ with ι_{11}, ι_{12} as defined in Lemma 3.5, then the operator T is bounded from $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$.

Proof Let $h(x) \in \dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$. We write

$$h(x) = \sum_{j=-\infty}^{\infty} h(x)\chi_j = \sum_{j=-\infty}^{\infty} h_j(x).$$

By Definition 2.3, we have

$$\|T(h)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|T(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le 1\right\}. \quad (4.1)$$

Since

$$\begin{aligned}\left\|\left(\frac{2^{k\alpha}|T(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}&\le\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{\infty}T(h_j)\chi_k\right|}{\sum_{i=1}^{3}\eta_{1i}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\&\le\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T(h_j)\chi_k\right|}{\eta_{11}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\&\quad+\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T(h_j)\chi_k\right|}{\eta_{12}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\&\quad+\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T(h_j)\chi_k\right|}{\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}},\end{aligned} \quad (4.2)$$

where

$$\eta_{11}=\left\|\left\{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T(h_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})}, \quad (4.3)$$

$$\eta_{12}=\left\|\left\{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T(h_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})}, \quad (4.4)$$

$$\eta_{13}=\left\|\left\{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T(h_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

and

$$\eta=\sum_{i=1}^{3}\eta_{1i}.$$

Thus,

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|T(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le C.$$

We easily see that

$$\|T(h)\|_{\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}\le C\eta=C\sum_{i=1}^{3}\eta_{1i}. \quad (4.6)$$

This implies that we only need to prove $\eta_{11},\eta_{12},\eta_{13}\le C\|h\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$. Denote $\eta_{10}=\|h\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$.

First, we consider η_{12}. By virtue of **Lemma 3.7**, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le \sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}^{\left(q_2^1\right)_k} \le \sum_{k=-\infty}^{\infty}\left(\left\|\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\right)^{\left(q_2^1\right)_k}, \tag{4.7}$$

where,

$$\left(q_2^1\right)_k=\begin{cases}\left(q_2\right)_-, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le 1,\\ \left(q_2\right)_+, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}> 1.\end{cases}$$

In the above, we use the Proposition 3.2 and Remark 2.2. Since $h(x)\in \dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}\left(\mathbb{R}^n\right)$, we have $\left\|\frac{2^{k\alpha}\left|h\chi_k\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\le 1$ and $\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|h\chi_k\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\le 1$, we get

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\ &\le C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-2}^{k+2}\left\|\frac{2^{k\alpha}\left|h_j\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\right)^{(q_2^1)_k}\\ &\le C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}\left|h\chi_k\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}^{\left(q_2^1\right)_k}\\ &\le C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|h\chi_k\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{\left(q_2^1\right)_k}{(q_1)_+}}\\ &\le C\left\{\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|h\chi_k\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right\}^{q_*}\\ &\le C.\end{aligned}$$

Here $(p_1)_+ \le (p_2)_- \le (q_2^1)_k$ and $q_* = \min_{k\in N} \frac{(q_2^1)_k}{(q_1)_+}$. That is

$$\eta_{12} \le C\eta_{10} \le C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}. \tag{4.8}$$

Let us now turn to estimate η_{11}. Noting that $x \in A_j$ and $j \le k-2$, by the generalized Hölder's inequality and the Minkowski's inequality, we get

$$\begin{aligned} |Th_j(x)| &\le \int_{A_j} |K(x,y)h_j(y)| \mathrm{d}y \\ &\le C\int_{A_j} |h_j(y)| / |x-y|^n \mathrm{d}y \\ &\le C2^{-kn} \int_{A_j} |h_j(y)| \mathrm{d}y \\ &\le C2^{-kn} \|h_j\|_{L^1(\mathbb{R}^n)}. \end{aligned} \tag{4.9}$$

By Lemmas 3.5-3.7 and the fact that $\left\| \frac{2^{j\alpha}|h\chi_j|}{\eta_{10}} \right\|_{L^{p_1(\cdot)q_1}} \le 1$, we easily see that

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-2} T(h_j)\chi_k \right|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\ &\le C \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-2} 2^{-kn} \|h_j\|_{L^1(\mathbb{R}^n)} \chi_k \right|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\ &\le C \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=}^{\infty} 2^{-kn} \|h_j\|_{L^1(\mathbb{R}^n)} \chi_k \right|}{\eta_{10}} \right)_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right\|^{(q_2^2)_k} \\ &\le C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=-\infty}^{k-2} 2^{-kn} \left\| \frac{h_j}{\eta_{10}} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^2)_k} \\ &\le C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=-\infty}^{k-2} 2^{-kn} \left\| \frac{h\chi_j}{\eta_{10}} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \left(|B_k| \|\chi_{B_k}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}^{-1} \right) \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^2)_k} \\ &\le C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=-\infty}^{k-2} 2^{-j\alpha} \left\| \frac{|2^{j\alpha} h\chi_j|}{\eta_{10}} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \right)^{(q_2^2)_k} \\ &\le C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha - n\iota_{11})} \left\| \left(\frac{|2^{j\alpha} h\chi_j|}{\eta_{10}} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)+}} \right\}^{(q_2^2)_k}, \end{aligned} \tag{4.10}$$

where

$$\left(q_2^2\right)_k=\begin{cases}\left(q_2\right)_-, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le 1,\\ \left(q_2\right)_+, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}>1.\end{cases}$$

Therefore, if $\left(q_1\right)_+<1$ and $\left(p_1\right)_+\le\left(p_2\right)_-\le\left(q_2^2\right)_k$, we can get

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\&\le C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha-n\eta_{11})}\right\}^{q_*}\\&\le C,\end{aligned}$$

where $q_*=\min_{k\in\mathbb{N}}\dfrac{\left(q_2^1\right)_k}{\left(q_1\right)_+}$.

If $\left(q_1\right)_+\ge 1$ and $\left(q_2^2\right)_k\ge\left(q_2\right)_-\ge\left(q_2\right)_+\ge 1$. By Remark 2.2 and applying the generalized Hölder's inequality, we obtain

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T\left(h_j\right)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\&\le C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-2}\left(k-j\right)2^{(k-j)(\alpha-n\eta_{11})(q_1)_+/2}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\right\}^{\frac{\left(q_2^2\right)_k}{(q_1)_+}}\\&\quad\times\left(\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha-n\eta_{11})\left((q_1)_+\right)'/2}\right)^{\frac{\left(q_2^2\right)_k}{\left((q_1)_+\right)'}}\\&\le C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha-n\eta_{11})(q_1)_+/2}\right\}^{q_*}\\&\le C,\end{aligned}$$

where $q_*=\min_{k\in\mathbb{N}}\dfrac{\left(q_2^2\right)_k}{\left(q_1\right)_+}$.

Hence, we see that

$$\eta_{11}\le C\eta_{10}\le C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}\left(\mathbb{R}^n\right)}. \tag{4.11}$$

Finally, we estimate η_{13}. Noting that for each $x\in A_j$ and $j\ge k+2$, we have

$$\left|Th_j(x)\right| \le \int_{A_j}\left|K(x,y)h_j(y)\right|\mathrm{d}y \le C\int_{A_j}\left|h_j(y)\right|/\left|x-y\right|^n \mathrm{d}y \le C2^{-jn}\left\|h_j\right\|_{L^1(\mathbb{R}^n)}. \quad (4.12)$$

By Lemma 3.7 and $\left\|\frac{2^{j\alpha}\left|h\chi_j\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)q_1(\cdot)}} \le 1$, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T(h_j)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\le C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}2^{-jn}\left\|h_j\right\|_{L^1(\mathbb{R}^n)}\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}2^{-jn}\left\|h_j\right\|_{L^1(\mathbb{R}^n)}\chi_k\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}^{(q_2^3)_k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{h_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^3)_k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{h\chi_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left(\left|B_j\right|\left\|\chi_{B_j}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{-1}\right)\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^3)_k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}\left\|\frac{h\chi_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{-1}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^3)_k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-j\alpha}\left\|\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{j_k}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\right)^{(q_2^3)_k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha+n\iota_{12})}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)+}}\right\}^{(q_2^3)_k}, \quad (4.13)$$

where

$$(q_2^3)_k = \begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T(h_j)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T(h_j)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1.\end{cases}$$

Then we have $\eta_{13} \le C\eta_{10} \le C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$, by using the same argument in η_{11}. Thus, we prove Theorem 4.1. □

Theorem 4.2. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that $p_1(\cdot) \in \mathcal{B}(\mathbb{R}^n), q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. If $-n\iota_{12} < \alpha < n\iota_{11}$ with ι_{11}, ι_{12} as defined in lemma 3.5, then the commutator $[b,T]$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$.

Proof Let $h(x)\in \dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n), b\in \mathrm{BMO}(\mathbb{R}^n)$.We write

$$h(x)=\sum_{j=-\infty}^{\infty}h(x)\chi_j=\sum_{j=-\infty}^{\infty}h_j(x)$$

By virtue of the definition of $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$, we have

$$\left\|[b,T](h)\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|[b,T](h)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le 1\right\}. \quad (4.14)$$

Since

$$\left\|\left(\frac{2^{k\alpha}\left|[b,T](h)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}[b,T](h_j)\chi_k\right|}{\sum_{i=1}^{3}\eta_{2i}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\le\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{\infty}[b,T](h_j)\chi_k\right|}{\eta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}+\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}[b,T](h_j)\chi_k\right|}{\eta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \quad (4.15)$$

$$+\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k\right|}{\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}.$$

Let

$$\eta_{21}=\left\|\left\{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}[b,T](h_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})}, \quad (4.16)$$

$$\eta_{22}=\left\|\left\{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}[b,T](h_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})}, \quad (4.17)$$

$$\eta_{23}=\left\|\left\{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})}, \quad (4.18)$$

and

$$\eta=\sum_{i=1}^{3}\eta_{2i}.$$

Therefore, we can obtain

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|[b,T](h)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le C.$$

Thus it follows that,

$$\left\|[b,T](h)\right\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)}\le C\eta=C\sum_{i=1}^{3}\eta_{1i}. \quad (4.20)$$

Hence $\eta_{21},\eta_{22},\eta_{23}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$. Denoting $\eta_{10}=C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$, firstly we estimate η_{22} as in Theorem 4.1. Applying Lemma 3.3, we imme- diately arrive at

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-2}^{k+2}[b,T](h_j)\chi_k\right|}{\eta_{10}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le C.$$

So we can get that

$$\eta_{21}\le C\eta_{10}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}. \tag{4.21}$$

Next we estimate η_{21}, Let $x\in A_j$, $j\le k-2$.

$$\begin{aligned}\left|[b,T]h_j\right| &\le \int_{A_j}\left|K(x,y)(b(x)-b(y))h_j(y)\right|\mathrm{d}y\\ &\le C\int_{A_j}\left|(b(x)-b(y))h_j(y)\right|/|x-y|^n\,\mathrm{d}y\\ &\le C2^{-nk}\left|b(x)-b_{B_j}\right|\int_{A_j}\left|h_j(y)\right|\mathrm{d}y+\int_{A_j}\left|b_{B_j}-b(y)\right|\left|h_j(y)\right|\mathrm{d}y\\ &\le C2^{-nk}\left|b(x)-b_{B_j}\right|\left\|h_j\right\|_{L^1(\mathbb{R}^n)}+\left\|b(\cdot)-(b_{B_j})h_j\right\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{4.22}$$

Thus, from Lemmas 3.4-3.7, We obtain that

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}[b,T](h_j)\chi_k\right|}{\eta_{10}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\\
&\le C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}2^{-nk}\left|b(x)-b_{B_j}\right|\left\|h_j\right\|_{L^1(\mathbb{R}^n)}\chi_k\right|}{\eta_{10}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{(q_2^2)_k}\\
&\quad+C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}2^{-nk}\left\|(b(\cdot)-b_{B_j})h_j\right\|_{L^1(\mathbb{R}^n)}\chi_k\right|}{\eta_{10}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{(q_2^2)_k}\\
&\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-kn}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{-1}\left\|(b(x)-b_{B_j})\chi_{B_j}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}\\
&\quad+C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-kn}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}(k-j)\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}\\
&\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-kn}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}(k-j)\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}\\
&\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}(k-j)2^{-j\alpha}\left\|\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}\right)^{(q_2^2)_k}.\end{aligned}$$

Therefore, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{\infty}[b,T](h_j)\chi_k\right|}{\eta_{10}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\le C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)(\alpha-n\eta_{11})}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)+}}\right\}^{\left(q_2^2\right)_k}, \quad (4.23)$$

where

$$\left(q_2^2\right)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}[b,T](h_j)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\le 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}[b,T](h_j)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}>1.\end{cases}$$

This, for $(q_1)_+<1$, $(p_1)_+\le(p_2)_-\le\left(q_2^2\right)_k$, along with Remark 2.2, tells us that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}[b,T](h_j)\chi_k\right|}{\eta_{10}\|b\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\le C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\sum_{k=j+2}^{\infty}(k-j)2^{(k-j)(\alpha-n\eta_{11})}\right\}^{q_*}\le C,$$

where $q_*=\min_{k\in N}\frac{\left(q_2^2\right)_k}{(q_1)_+}$.

If $(q_1)_+\le 1$, it is follows from Remark 2.2 and Hölder's inequality that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}[b,T](h_j)\chi_k\right|}{\eta_{10}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\le C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)(\alpha-n\eta_{11})(q_1)_+/2}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\right\}^{\frac{\left(q_2^2\right)_k}{(q_1)+}}$$

$$\times\left(\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)(\alpha-n\eta_{11})((q_1)_+)'/2}\right)^{\frac{\left(q_2^2\right)_k}{((q_1)+)'}}$$

$$\le C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{\left|2^{j\alpha}h\chi_j\right|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\sum_{k=j+2}^{\infty}(k-j)2^{(k-j)(\alpha-n\eta_{11})(q_1)_+/2}\right\}^{q_*}$$

$$\le C,$$

where $q_* = \min_{k \in N} \frac{\left(q_2^2\right)_k}{\left(q_1\right)_+}$.

This implies that

$$\eta_{21} \le C\eta_{10} \|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)} \le C\|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)} \|h\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}\left(\mathbb{R}^n\right)}. \tag{4.24}$$

Finally we estimate η_{23}, for any $x \in A_j$, $j \ge k+2$, by the same way to argument in η_{21}, we obtain that

$$\begin{aligned} \left|[b,T]h_j\right| &\le \int_{A_j} \left|K(x,y)\left(b(x)-b(y)\right)h_j(y)\right| \mathrm{d}y \\ &\le C\int_{A_j} \left|\left(b(x)-b(y)\right)h_j(y)\right| / |x-y|^n \,\mathrm{d}y \\ &\le C2^{-nj} \left|b(x)-b_{B_k}\right| \int_{A_j} \left|h_j(y)\right| \mathrm{d}y + \int_{A_j} \left|b_{B_k}-b(y)\right| \left|h_j(y)\right| \mathrm{d}y \\ &\le C2^{-nj} \left|b(x)-b_{B_j}\right| \left\|h_j\right\|_{L^1\left(\mathbb{R}^n\right)} + \left\|b(\cdot)-\left(b_{B_j}\right)h_j\right\|_{L^1\left(\mathbb{R}^n\right)}, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k+2}^{\infty} [b,T]\left(h_j\right)\chi_k \right|}{\eta_{21} \|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} \left| \sum_{j=k+2}^{\infty} 2^{-nj} \left|b(x)-b_{B_j}\right| \left|h_j\right|_{L^1\left(\mathbb{R}^n\right)} \chi_k \right|}{\eta_{10} \|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}} \right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}^{\left(q_2^3\right)_k} \\ &+ C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} \left| \sum_{j=k+2}^{\infty} 2^{-nj} \left\| \left(b(\cdot)-b_{B_j}\right)h_j \right\|_{L^1\left(\mathbb{R}^n\right)} \chi_k \right|}{\eta_{10} \|b\|_{\mathrm{BMO}\left(\mathbb{R}^n\right)}} \right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}^{\left(q_2^3\right)_k} \\ &\le C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=-\infty}^{k-2} 2^{-jn} \left\| \frac{\left|h_j\right|}{\eta_{10}} \right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)} (k-j) \left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)} \left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}\left(\mathbb{R}^n\right)} \right)^{\left(q_2^3\right)_k} \\ &\le C \sum_{k=-\infty}^{\infty} \left(2^{k\alpha} \sum_{j=-\infty}^{k-2} (j-k) 2^{-j\alpha} \left\| \frac{\left|2^{j\alpha} h\chi_j\right|}{\eta_{10}} \right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)} \frac{\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}}{\left\|\chi_{B_j}\right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}} \right)^{\left(q_2^3\right)_k} \\ &\le C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{k-2} (j-k) 2^{(k-j)(\alpha+n\delta_2)} \left\| \left(\frac{\left|2^{j\alpha} h\chi_j\right|}{\eta_{10}} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)q_1(\cdot)}\left(\mathbb{R}^n\right)}^{\frac{1}{(q_1)+}} \right\}^{\left(q_2^3\right)_k}, \end{aligned} \tag{4.26}$$

where

$$\left(q_2^3\right)_k = \begin{cases} \left(q_2\right)_-, & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-2} [b,T]\left(h_j\right)\chi_k \right|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le 1, \\ \left(q_2\right)_+, & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-2} [b,T]\left(h_j\right)\chi_k \right|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

Hence, we arrive at that $\eta_{23} \le C\eta_{10} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|h\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$ by the similar argument in the proof Theorem 4.1.

This completes the proof of Theorem 4.2. □

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