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Daftardar-Jafari Method for Fractional Heat-Like and Wave-Like Equations with Variable Coefficients

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Abstract

In this paper, Daftardar-Gejji and Jafari method is applied to solve fractional heat-like and wave-like models with variable coefficients. The method is proved for a variety of problems in one, two and three dimensional spaces where analytical approximate solutions are obtained. The examples are presented to show the efficiency and simplicity of this method.

Keywords

Daftardar-Jafari Method, Heat-Like Equations, Wave-Like Equations, Fractional Calculus

1. Introduction

Various phenomena in engineering physics, chemistry, other sciences can be described very successfully by models using mathematical tools from fractional calculus, *i.e.*, the theory of derivatives and integrals of fractional non-integer order [1] [2] [3] [4]. A survey of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [5]. Oldham and Spanier [6], Miller and Ross [7] provide the history and a comprehensive treatment of this subject.

The fractional calculus has a three-century and two decades long history. The idea appeared in a letter by Leibniz to L'Hôpital in 1695. The subject of fractional calculus has gained importance during the past three decades due mainly to its demonstrated applications in different area of physics and engineering. Indeed it provides several potentially useful tools for solving differential and integral equations. It is important to solve time fractional partial differential equations. It was found that fractional time derivatives arise generally as infinitesimal generators of the time evolution when taking a long-time scaling limit. Hence, the importance of investigating fractional equations arises from the necessity to sharpen the concepts of equilibrium, stability states, and time evolution in the long-time limit (for more details, see [8] [9] [10]).

In this work, we will consider the fractional heat-like and wave-like equations of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + h(x,y,z)u_{zz}, \quad 0 < x < a,\ 0 < y < b,\ 0 < z < c,\ 0 < \alpha \le 2,\ t > 0 \tag{1}$$

subject to the Neumann boundary conditions

$$\begin{aligned} u(0,y,z,t) &= f_1(y,z,t), \quad u_x(a,y,z,t) = f_2(y,z,t), \\ u(x,0,z,t) &= g_1(x,z,t), \quad u_y(x,b,z,t) = g_2(x,z,t), \\ u(x,y,0,t) &= h_1(x,y,t), \quad u_z(x,y,c,t) = h_2(x,y,t), \end{aligned} \tag{2}$$

and the initial conditions

$$u(x,y,z,0) = \psi(x,y,z), \quad u_t(x,y,z,0) = \phi(x,y,z), \tag{3}$$

where α is a parameter describing the fractional derivative and u_t is the rate of change of temperature at a point over time, $u = u(x,y,z,t)$ is temperature as a function of time and space, while u_{xx}, u_{yy} and u_{zz} are the second spatial derivatives (thermal conductions) of temperature in *x*, *y*, and *z* directions, respectively. Finally, $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ are any functions in *x*, *y*, and *z*.

In the case of $0 < \alpha \le 1$, then Equation (1) reduces to a fractional heat-like equation with variable coefficients. And in the case of $1 < \alpha \le 2$, then Equation (1) reduces to a fractional wave-like equation which models anomalous diffusive and subdiffusive systems, description of fractional random walk, unification of diffusion and wave propagation phenomena [11] [12] [13] [14].

Recently, Molliq *et al.* [15] used the Variational iteration method (VIM) for solving Equation (1). The same equation was solved by Momani [16] utilizing the Adomian decomposition method (ADM). Xu and Cang [17], and Xu *et al.* [18] applied the homotopy analysis method (HAM) to solve it.

The Daftardar-Jafari method (DJM) developed in 2006 has been extensively used by many researchers for the treatment of linear and nonlinear ordinary and partial differential equations of integer and fractional order [19] [20] [21] [22] [23]. The method converges to the exact solution if it exists through successive approximations. For concrete problems, a few number of approximations can be used for numerical purposes with high degree of accuracy. The DJM does not require any restrictive assumptions for nonlinear terms as required by some existing techniques.

The main objective of this paper is to apply DJM to obtain fractional solutions for different models of Equation (1). While the VIM [15] requires the determination of Lagrange multiplier in its computational algorithm, DJM is independent of any such requirements. Moreover, unlike the ADM [16], where the calculation of the tedious Adomian polynomials is needed to deal with nonlienar terms, DJM handles linear and nonlinear terms in a simple and straightforward manner without any additional requirements.

2. Fractional Calculus

In this section we will introduce some definitions and properties of the fractional calculus to enable us to follow the solutions of the problem given in this paper. These definitions include, Riemann-Liouville, Weyl, Reize, Compos, Caputo, and Nashimoto fractional operators.

Definition 1. Let $\alpha \in \mathbb{R}_+$. The operator J_a^α defined on the usual Lebesque space $L_1[a,b]$ by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)\mathrm{d}s,$$
$$J_a^0 f(x) = f(x),$$

for $a \le x \le b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J^α can be found in [3] [4], we mention the following:

For $f \in L_1[a,b]$, $\alpha, \beta \ge 0$ and $\gamma > -1$,

1. J_a^α exists for almost every $x \in [a,b]$,
2. $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$,
3. $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$,
4. $J_a^\alpha (x-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differential operator D^α proposed by *M.* Caputo in his work on the theory of viscoelasticity [24]. For more information about Caputo definition and properties see [3] [5] [24].

Definition 2. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} f^{(m)}(s)\mathrm{d}s,$$

for $m-1 < \alpha \le m,\ m \in \mathbb{N},\ x > 0.$

Also, we need here two of its basic properties.

Lemma 1. If $m-1 < \alpha \le m$ and $f \in L_1[a,b]$, then

$$D_a^\alpha J_a^\alpha f(x) = f(x),$$

and

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!},\ x > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the three-dimensional time fractional heat-like and wave-like Equation (1.1), where the unknown function $u = u(x, y, z, t)$ is a assumed to be a causal function of time, *i.e.*, vanishing for $t < 0$, and the fractional derivative is taken in Caputo sense to be:

Definition 3. For m to be the smallest integer that exceeds α, the Caputo fractional derivative of order $\alpha > 0$ is defined as

$$D^{\alpha}u(x,y,z,t) = \frac{\partial^{\alpha}u(x,y,z,t)}{\partial t^{\alpha}}$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_a^t (t-s)^{m-\alpha-1}\frac{\partial^m u(x,y,z,t)}{\partial t^m}\mathrm{ds}, & \text{for} \quad m-1<\alpha\le m, \\ \dfrac{\partial^m u(x,y,z,t)}{\partial t^m}, & \text{for} \quad \alpha = m \in \mathbb{N}. \end{cases}$$

For mathematical properties of fractional derivatives and integrals one can consult the above mentioned references.

3. The DJ Method

Consider the following general functional equation

$$u(\bar{x}) = f(\bar{x}) + N(u(\bar{x})), \tag{4}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function, $\bar{x} = (x_1, x_2, \cdots, x_n)$. We are looking for a solution u of Equation (4) having the series form

$$u(\bar{x}) = \sum_{i=0}^{\infty} u_i(\bar{x}). \tag{5}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{6}$$

From Equations (5) and (6), Equation (4) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{7}$$

We define the recurrence relation

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{n+1} = N\left(\displaystyle\sum_{j=0}^{n} u_j\right) - N\left(\displaystyle\sum_{j=0}^{n-1} u_j\right), n \in \mathbb{N}. \end{cases} \tag{8}$$

Then

$$u_1 + \cdots + u_{n+1} = N(u_0 + u_1 + \cdots + u_n), \quad n \in \mathbb{N} \tag{9}$$

and

$$\sum_{i=0}^{\infty} u_i = f + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{10}$$

The n-term approximate solution of Equation (4) is given by $u = u_0 + u_1 + \cdots + u_{n-1}$. In what follows, we apply DJM to six physical models to demonstrate the strength of the method and to establish exact solutions of these models.

4. Fractional Heat-Like Equations

In this section, we illustrate our analysis by examining the following three fractional heat-like equations.

Example 1. Firstly, let us consider the one-dimensional initial boundary value problems (IBVP)

$$D_t^\alpha u = \frac{1}{2} x^2 u_{xx}, \quad 0 < x < 1, \quad 0 < \alpha \le 1, \quad t > 0, \tag{11}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \mathrm{e}^t, \tag{12}$$

and the initial condition

$$u(x,0) = x^2. \tag{13}$$

Operating with $J^\alpha(\cdot) = J_0^\alpha(\cdot)$ on both sides of Equation (11) yields

$$u(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!} + \frac{1}{2} x^2 J^\alpha(u_{xx}).$$

For $0 < \alpha \le 1$ with $m = 1$ and using the initial condition (13), we set

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!} = x^2,$$

$$N(u) = \frac{1}{2} x^2 J^\alpha(u_{xx}).$$

Following the algorithm (8), the successive approximations are

$$u_1(x,t) = N(u_0) = \frac{1}{2} x^2 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\partial^2 u_0}{\partial^2 x} \mathrm{d}s = x^2 \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha,$$

$$u_2(x,t) = N(u_0 + u_1) - N(u_0) = x^2 \frac{\Gamma(1)}{\Gamma(2\alpha+1)} t^{2\alpha},$$

$$u_3(x,t) = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = x^2 \frac{\Gamma(1)}{\Gamma(3\alpha+1)} t^{3\alpha},$$

$$u_4(x,t) = N(u_0 + u_1 + u_2 + u_3) - N(u_0 + u_1 + u_2) = x^2 \frac{\Gamma(1)}{\Gamma(4\alpha+1)} t^{4\alpha},$$

$$\vdots$$

$$u_n(x,t) = N(u_0 + \cdots + u_n) - N(u_0 + \cdots + u_{n-1}) = x^2 \frac{\Gamma(1)}{\Gamma(n\alpha+1)} t^{n\alpha}, n \in \mathbb{N}.$$

Thus, the approximate solution in a series form is given by

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = x^2 \left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots \right).$$

So, the solution for the standard heat-like equation ($\alpha = 1$) is given by

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots\right).$$

This series has the closed form

$$u(x,t) = x^2 e^t,$$

which is the exact solution of the problem (11)-(13) compatible with VIM, ADM and HAM.

Example 2. Now, let us consider the two-dimensional IBVP

$$D_t^\alpha u = \frac{1}{2}\left(y^2 u_{xx} + x^2 u_{yy}\right), \quad 0 < x, y < 1, \quad 0 < \alpha \le 1, \quad t > 0, \tag{14}$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0,y,t) &= 0, \quad u_x(1,y,t) = 2\sinh t, \\ u_y(x,0,t) &= 0, \quad u_y(x,1,t) = 2\cosh t, \end{aligned} \tag{15}$$

and the initial condition

$$u(x,y,0) = y^2. \tag{16}$$

Operating with $J^\alpha(\cdot) = J_0^\alpha(\cdot)$ on both sides of Equation (14) yields

$$u(x,y,t) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,y,0^+)\frac{t^k}{k!} + \frac{1}{2} J^\alpha\left(y^2 u_{xx} + x^2 u_{yy}\right).$$

For $0 < \alpha \le 1$ with $m = 1$ and using the initial condition (16), we set

$$u_0(x,y,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,y,0^+)\frac{t^k}{k!} = y^2,$$

$$N(u) = \frac{1}{2} J^\alpha\left(y^2 u_{xx} + x^2 u_{yy}\right).$$

Using the algorithm (8), the successive approximations are

$$u_1(x,y,t) = N(u_0) = \frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(y^2\frac{\partial^2 u_0}{\partial^2 x} + x^2\frac{\partial^2 u_0}{\partial^2 y}\right)\mathrm{d}s = x^2\frac{\Gamma(1)}{\Gamma(\alpha+1)}t^\alpha,$$

$$u_2(x,y,t) = N(u_0 + u_1) - N(u_0) = y^2\frac{\Gamma(1)}{\Gamma(2\alpha+1)}t^{2\alpha},$$

$$u_3(x,y,t) = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = x^2\frac{\Gamma(1)}{\Gamma(3\alpha+1)}t^{3\alpha},$$

$$u_4(x,y,t) = N(u_0 + u_1 + u_2 + u_3) - N(u_0 + u_1 + u_2) = y^2\frac{\Gamma(1)}{\Gamma(4\alpha+1)}t^{4\alpha},$$

$$\vdots$$

$$\begin{aligned} u_n(x,y,t) &= N(u_0 + \cdots + u_n) - N(u_0 + \cdots + u_{n-1}) \\ &= \left[x^2\frac{1-(-1)^n}{2} + y^2\frac{1+(-1)^n}{2}\right]\frac{\Gamma(1)}{\Gamma(n\alpha+1)}t^{n\alpha}, n \in \mathbb{N}. \end{aligned}$$

Thus, the approximate solution in a series form is given by

$$u(x,y,t)=x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots+\frac{t^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}+\cdots\right)$$
$$+y^2\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots+\frac{t^{(2n)\alpha}}{\Gamma((2n)\alpha+1)}+\cdots\right).$$

So, the solution for the standard heat-like equation ($\alpha=1$) is given by

$$u(x,y,t)=\sum_{i=0}^{\infty}u_i(x,y,t)=x^2\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots+\frac{t^{2n+1}}{(2n+1)!}+\cdots\right)$$
$$+y^2\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots+\frac{t^{2n}}{(2n)!}+\cdots\right).$$

This series has the closed form

$$u(x,y,t)=x^2\sinh t+y^2\cosh t,$$

which is the exact solution of the problem (14)-(16) compatible with VIM, ADM and HAM.

Example 3. Let us consider the three-dimensional inhomogeneous IBVP

$$D_t^{\alpha}u=x^4y^4z^4+\frac{1}{36}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right),\quad 0<x,y,z<1,\ 0<\alpha\le 1,\ t>0, \tag{17}$$

subject to the boundary conditions

$$\begin{aligned} &u(0,y,z,t)=0,\quad u(1,y,z,t)=y^4z^4\left(e^t-1\right),\\ &u(x,0,z,t)=0,\quad u(x,1,z,t)=x^4z^4\left(e^t-1\right),\\ &u(x,y,0,t)=0,\quad u(x,y,1,t)=x^4y^4\left(e^t-1\right), \end{aligned} \tag{18}$$

and the initial condition

$$u(x,y,z,0)=0. \tag{19}$$

Operating with $J^{\alpha}(\cdot)=J_0^{\alpha}(\cdot)$ on both sides of Equation (17) yields

$$u(x,y,z,t)=\sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u\left(x,y,z,0^+\right)\frac{t^k}{k!}+J^{\alpha}\left(x^4y^4z^4\right)$$
$$+\frac{1}{36}J^{\alpha}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right).$$

For $0<\alpha\le 1$ with $m=1$ and using the initial condition (19), we set

$$u_0(x,y,z,t)=\sum_{k=0}^{0}\frac{\partial^k}{\partial t^k}u\left(x,y,z,0^+\right)\frac{t^k}{k!}+J^{\alpha}\left(x^4y^4z^4\right)=x^4y^4z^4\frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha},$$

$$N(u)=\frac{1}{36}J^{\alpha}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right).$$

Utilizing the algorithm (8), the successive approximations are

$$\begin{aligned} u_1(x,y,z,t) &= N(u_0) = \frac{1}{36}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(x^2\frac{\partial^2 u_0}{\partial^2 x}+y^2\frac{\partial^2 u_0}{\partial^2 y}+z^2\frac{\partial^2 u_0}{\partial^2 z}\right)\mathrm{d}s \\ &= x^4y^4z^4\frac{\Gamma(1)}{\Gamma(2\alpha+1)}t^{2\alpha}, \\ u_2(x,y,z,t) &= N(u_0+u_1)-N(u_0) = x^4y^4z^4\frac{\Gamma(1)}{\Gamma(3\alpha+1)}t^{3\alpha}, \\ &\vdots \\ u_n(x,y,z,t) &= N(u_0+\cdots+u_n)-N(u_0+\cdots+u_{n-1}) \\ &= x^4y^4z^4\frac{\Gamma(1)}{\Gamma[(n+1)\alpha+1]}t^{(n+1)\alpha},\ n\in\mathbb{N}. \end{aligned}$$

Thus, the approximate solution in a series form is given by

$$\begin{aligned} &u(x,y,z,t) \\ &= x^4y^4z^4\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots+\frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}+\cdots\right]. \end{aligned}$$

So, the solution for the standard heat-like equation ($\alpha=1$) is given by

$$\begin{aligned} u(x,y,z,t) &= x^4y^4z^4\left[t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots+\frac{t^{n+1}}{(n+1)!}+\cdots\right] \\ &= x^4y^4z^4\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots+\frac{t^n}{n!}+\cdots-1\right]. \end{aligned}$$

This series has the closed form

$$u(x,y,z,t) = x^4y^4z^4\left(\mathrm{e}^t-1\right),$$

which is the exact solution of the problem (17)-(19) compatible with VIM, ADM and HAM.

5. Fractional Wave-Like Equations

In this section, we illustrate our analysis by examining the following three fractional wave-like equations.

Example 1. We first consider the one-dimensional IBVP

$$D_t^{\alpha}u = \frac{1}{2}x^2u_{xx},\quad 0<x<1,\quad 1<\alpha\le 2,\quad t>0, \tag{20}$$

subject to the boundary conditions

$$u(0,t)=0,\quad u(1,t)=1+\sinh t, \tag{21}$$

and the initial conditions

$$u(x,0)=x,\quad u_t(x,0)=x^2. \tag{22}$$

Operating with $J^{\alpha}(\cdot)=J_0^{\alpha}(\cdot)$ on both sides of Equation (20) yields

$$u(x,t)=\sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u\left(x,0^+\right)\frac{t^k}{k!}+\frac{1}{2}x^2J^{\alpha}\left(u_{xx}\right).$$

For $1<\alpha\le 2$ with $m=2$ and using the initial conditions (22), we set

$$u_0(x,t)=\sum_{k=0}^{1}\frac{\partial^k}{\partial t^k}u(x,0^+)\frac{t^k}{k!}=x+x^2t,$$

$$N(u)=\frac{1}{2}x^2J^\alpha(u_{xx}).$$

Following the algorithm (8), the successive approximations are

$$u_1(x,t)=N(u_0)=\frac{1}{2}x^2\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\frac{\partial^2u_0}{\partial^2x}\mathrm{d}s=x^2\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1},$$

$$u_2(x,t)=N(u_0+u_1)-N(u_0)=x^2\frac{\Gamma(2)}{\Gamma(2\alpha+2)}t^{2\alpha+1},$$

$$u_3(x,t)=N(u_0+u_1+u_2)-N(u_0+u_1)=x^2\frac{\Gamma(2)}{\Gamma(3\alpha+2)}t^{3\alpha+1},$$

$$u_4(x,t)=N(u_0+u_1+u_2+u_3)-N(u_0+u_1+u_2)=x^2\frac{\Gamma(2)}{\Gamma(4\alpha+2)}t^{4\alpha+1},$$

$$\vdots$$

$$u_n(x,t)=N(u_0+\cdots+u_n)-N(u_0+\cdots+u_{n-1})=x^2\frac{\Gamma(2)}{\Gamma(n\alpha+2)}t^{n\alpha+1},\ n\in\mathbb{N}.$$

Thus, the approximate solution in a series form is given by

$$u(x,t)=x+x^2\left(t+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\cdots+\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}+\cdots\right).$$

So, the solution for the standard heat-like equation ($\alpha=2$) is given by

$$u(x,t)=x+x^2\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots+\frac{t^{2n+1}}{(2n+1)!}+\cdots\right).$$

This series has the closed form

$$u(x,t)=x+x^2\sinh t,$$

which is the exact solution of the problem (20)-(22) compatible with VIM, ADM and HAM.

Example 2. We next consider the two-dimensional IBVP

$$D_t^\alpha u=\frac{1}{12}\left(x^2u_{xx}+y^2u_{yy}\right),\quad 0<x,y<1,\quad 1<\alpha\le 2,\quad t>0, \tag{23}$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0,y,t)&=0,\quad u_x(1,y,t)=4\cosh t,\\ u_y(x,0,t)&=0,\quad u_y(x,1,t)=4\sinh t, \end{aligned} \tag{24}$$

and the initial conditions

$$u(x,y,0)=x^4,\quad u_t(x,y,0)=y^4. \tag{25}$$

Operating with $J^\alpha(\cdot)=J_0^\alpha(\cdot)$ on both sides of Equation (23) yields

$$u(x,y,t)=\sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u(x,y,0^+)\frac{t^k}{k!}+\frac{1}{12}J^\alpha\left(x^2u_{xx}+y^2u_{yy}\right).$$

For $1<\alpha\le 2$ with $m=2$ and using the initial conditions (25), we set

$$u_0(x,y,t)=\sum_{k=0}^{1}\frac{\partial^k}{\partial t^k}u\left(x,y,0^+\right)\frac{t^k}{k!}=x^4+y^4t,$$

$$N(u)=\frac{1}{12}J^\alpha\left(x^2u_{xx}+y^2u_{yy}\right).$$

Using the algorithm (8), the successive approximations are

$$\begin{aligned}
u_1(x,y,t)&=N(u_0)=\frac{1}{12}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(x^2\frac{\partial^2u_0}{\partial^2x}+y^2\frac{\partial^2u_0}{\partial^2y}\right)\mathrm{ds}\\
&=x^4\frac{\Gamma(2)}{\Gamma(\alpha+1)}t^\alpha+y^4\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1},\\
u_2(x,y,t)&=N(u_0+u_1)-N(u_0)=x^4\frac{\Gamma(2)}{\Gamma(2\alpha+1)}t^{2\alpha}+y^4\frac{\Gamma(2)}{\Gamma(2\alpha+2)}t^{2\alpha+1},\\
u_3(x,y,t)&=N(u_0+u_1+u_2)-N(u_0+u_1)\\
&=x^4\frac{\Gamma(2)}{\Gamma(3\alpha+1)}t^{3\alpha}+y^4\frac{\Gamma(2)}{\Gamma(3\alpha+2)}t^{3\alpha+1},\\
u_4(x,y,t)&=N(u_0+u_1+u_2+u_3)-N(u_0+u_1+u_2)\\
&=x^4\frac{\Gamma(2)}{\Gamma(4\alpha+1)}t^{4\alpha}+y^4\frac{\Gamma(2)}{\Gamma(4\alpha+2)}t^{4\alpha+1},\\
&\ \vdots\\
u_n(x,y,t)&=N(u_0+\cdots+u_n)-N(u_0+\cdots+u_{n-1})\\
&=x^4\frac{\Gamma(2)}{\Gamma(n\alpha+1)}t^{n\alpha}+y^4\frac{\Gamma(2)}{\Gamma(n\alpha+2)}t^{n\alpha+1},\ n\in\mathbb{N}.
\end{aligned}$$

Thus, the approximate solution in a series form is given by

$$\begin{aligned}
u(x,y,t)=&\,x^4\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}+\cdots\right)\\
&+y^4\left(t+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\cdots+\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}+\cdots\right).
\end{aligned}$$

So, the solution for the standard heat-like equation ($\alpha=2$) is given by

$$\begin{aligned}
u(x,y,t)=&\,x^4\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots+\frac{t^{2n}}{(2n)!}+\cdots\right)\\
&+y^4\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots+\frac{t^{2n+1}}{(2n+1)!}+\cdots\right).
\end{aligned}$$

This series has the closed form

$$u(x,y,t)=x^4\cosh t+y^4\sinh t,$$

which is the exact solution of the problem (23)-(25) compatible with VIM, ADM and HAM.

Example 3. Finally, we consider the three-dimensional inhomogeneous IBVP

$$D_t^\alpha u = \left(x^2+y^2+z^2\right)+\frac{1}{2}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right),\\ 0<x,y,z<1, 1<\alpha\le 2, t>0, \tag{26}$$

subject to the boundary conditions

$$\begin{aligned} &u(0,y,z,t)=y^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right), && u(1,y,z,t)=\left(1+y^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right),\\ &u(x,0,z,t)=x^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right), && u(x,1,z,t)=\left(1+x^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right),\\ &u(x,y,0,t)=\left(x^2+y^2\right)\left(e^t-1\right), && u(x,y,1,t)=\left(x^2+y^2\right)\left(e^t-1\right)+\left(e^{-t}-1\right), \end{aligned} \tag{27}$$

and the initial conditions

$$u(x,y,z,0)=0,\quad u_t(x,y,z,0)=x^2+y^2-z^2. \tag{28}$$

Operating with $J^\alpha(\cdot)=J_0^\alpha(\cdot)$ on both sides of Equation (26) yields

$$\begin{aligned} u(x,y,z,t) &= \sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u\left(x,y,z,0^+\right)\frac{t^k}{k!}+J^\alpha\left(x^2+y^2+z^2\right)\\ &\quad+\frac{1}{2}J^\alpha\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right). \end{aligned}$$

For $1<\alpha\le 2$ with $m=2$ and using the initial conditions (28), we set

$$\begin{aligned} u_0(x,y,z,t) &= \sum_{k=0}^{1}\frac{\partial^k}{\partial t^k}u\left(x,y,z,0^+\right)\frac{t^k}{k!}+J^\alpha\left(x^2+y^2+z^2\right)\\ &= \left(x^2+y^2-z^2\right)t+\left(x^2+y^2+z^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)},\\ N(u) &= \frac{1}{2}J^\alpha\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right). \end{aligned}$$

Utilizing the algorithm (8), the successive approximations are

$$\begin{aligned} u_1(x,y,z,t) &= N(u_0)=\frac{1}{2}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(x^2\frac{\partial^2u_0}{\partial^2x}+y^2\frac{\partial^2u_0}{\partial^2y}+z^2\frac{\partial^2u_0}{\partial^2z}\right)\mathrm{d}s\\ &= \left(x^2+y^2\right)\left(\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}+\frac{\Gamma(2)}{\Gamma(2\alpha+1)}t^{2\alpha}\right)\\ &\quad+z^2\left(-\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}+\frac{\Gamma(2)}{\Gamma(2\alpha+1)}t^{2\alpha}\right),\\ u_2(x,y,z,t) &= N(u_0+u_1)-N(u_0)=\left(x^2+y^2\right)\left(\frac{\Gamma(2)}{\Gamma(2\alpha+2)}t^{2\alpha+1}+\frac{\Gamma(2)}{\Gamma(3\alpha+1)}t^{3\alpha}\right)\\ &\quad+z^2\left(-\frac{\Gamma(2)}{\Gamma(2\alpha+2)}t^{2\alpha+1}+\frac{\Gamma(2)}{\Gamma(3\alpha+1)}t^{3\alpha}\right),\\ &\ \ \vdots\\ u_n(x,y,z,t) &= N(u_0+\cdots+u_n)-N(u_0+\cdots+u_{n-1})\\ &= \left(x^2+y^2\right)\left(\frac{\Gamma(2)}{\Gamma(n\alpha+2)}t^{n\alpha+1}+\frac{\Gamma(2)}{\Gamma((n+1)\alpha+1)}t^{(n+1)\alpha}\right)\\ &\quad+z^2\left(-\frac{\Gamma(2)}{\Gamma(n\alpha+2)}t^{n\alpha+1}+\frac{\Gamma(2)}{\Gamma((n+1)\alpha+1)}t^{(n+1)\alpha}\right), \end{aligned}$$

$n \in \mathbb{N}$. Thus, the approximate solution in a series form is given by

$$u(x,y,z,t) = (x^2+y^2)\left(t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right)$$
$$+ z^2\left(-t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots\right).$$

So, the solution for the standard heat-like equation ($\alpha = 2$) is given by

$$u(x,y,z,t) = (x^2+y^2)\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right)$$
$$+ z^2\left(-t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\right).$$

This series has the closed form

$$u(x,y,z,t) = (x^2+y^2)(e^t - 1) + z^2(e^{-t} - 1),$$

which is the exact solution of the problem (26)-(28) compatible with VIM, ADM and HAM.

6. Conclusion

In this work, DJM has been successfully used to solve fractional heat-like and wave-like equations with variable coefficients giving it a wider applicability. The proposed scheme was applied directly without any need for transformation formulae or restrictive assumptions. Results have shown that the analytical approximate solution process of DJM is compatible with those methods in the literature providing analytical approximation such as VIM, ADM and HAM. The results obtained in all studied cases demonstrate the reliability and the efficiency of this method.

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