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Existence of Solutions of Three-Dimensional Fractional Differential Systems

Vadivel Sadhasivam, Jayapal Kavitha, Muthusamy Deepa

Post Graduate and Research Department of Mathematics, Thiruvalluvar Government Arts College (Affli. to Periyar University), Rasipuram, India
Email: ovsadha@gmail.com, kaviakshita@gmail.com, mdeepa.maths@gmail.com

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Abstract

In this article, we consider the three-dimensional fractional differential system of the form

$$D_{0^+}^{\alpha}u(t)=f_1\left(t,v(t),v'(t)\right),\ \ t\in(0,1),$$
$$D_{0^+}^{\beta}v(t)=f_2\left(t,w(t),w'(t)\right),\ \ t\in(0,1),$$
$$D_{0^+}^{\gamma}w(t)=f_3\left(t,u(t),u'(t)\right),\ \ t\in(0,1),$$

together with the Neumann boundary conditions,

$$u'(0)=u'(1)=0,\ \ v'(0)=v'(1)=0,\ \ w'(0)=w'(1)=0,$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma}$ are the standard Caputo fractional derivatives, $1<\alpha,\beta,\gamma\le 2$. A new result on the existence of solutions for a class of fractional differential system is obtained by using Mawhin's coincidence degree theory. Suitable examples are given to illustrate the main results.

Keywords

Fractional Differential Equations, Boundary Value Problem, Coincidence Degree Theory

1. Introduction

Fractional calculus is a very effective tool in the modeling of many phenomena like control of dynamical systems, porous media, electro chemistry, viscoelasticity, electromagnetic and so on. The fractional theory and its applications are mentioned by many papers and monographs, we refer [1]-[9]. For nonlinear fractional boundary value problem, many fixed point theorems were applied to investigate the existence of solutions as in references [10] [11] [12] [13]. On the other hand, there is another effective approach, Mawhin's coincidence theory, which proves to be very useful for determining the existence of solutions for fractional order differential equations. In recent years, boundary value problems for fractional differential equations at resonance have been studied in many papers (see [14]-[21]). The main motivation for investigating the fractional boundary value problem arises from fractional advection-dispersion equation.

Hu *et al.* [22] investigated the two-point boundary value problem for fractional differential equations of the following form

$$
\begin{aligned}
&D_{0^+}^{\alpha} x(t) = f\left(t, x(t), x'(t)\right),\ t \in [0,1], \\
&x(0) = 0,\ x'(0) = x'(1),
\end{aligned}
$$

where $D_{0^+}^{\alpha}$ is the Caputo fractional differential operator, $1 < \alpha \le 2$, and $f : [0,1] \times R^2 \to R$ is continuous.

In [23], Hu *et al.* extended the above boundary value problem to the existence of solutions for the following coupled system of fractional differential equations of the form

$$
\begin{aligned}
&D_{0^+}^{\alpha} u(t) = f\left(t, v(t), v'(t)\right),\ t \in (0,1), \\
&D_{0^+}^{\beta} v(t) = g\left(t, u(t), u'(t)\right),\ t \in (0,1), \\
&u(0) = v(0) = 0,\ u'(0) = u'(1),\ v'(0) = v'(1).
\end{aligned}
$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the Caputo fractional derivatives, $1 < \alpha \le 2$, $1 < \beta \le 2$, and $f, g : [0,1] \times R^2 \to R$ is continuous.

It seems that there has been no work done on the boundary value problem of system involving three nonlinear fractional differential equations. Motivated by the above observation, we investigate the following three-dimensional fractional differential system of the form

$$
\begin{aligned}
&D_{0^+}^{\alpha} u(t) = f_1\left(t, v(t), v'(t)\right),\ t \in (0,1), \\
&D_{0^+}^{\beta} v(t) = f_2\left(t, w(t), w'(t)\right),\ t \in (0,1), \\
&D_{0^+}^{\gamma} w(t) = f_3\left(t, u(t), u'(t)\right),\ t \in (0,1),
\end{aligned}
\tag{1}
$$

together with the Neumann boundary conditions,

$$
u'(0) = u'(1) = 0,\ v'(0) = v'(1) = 0,\ w'(0) = w'(1) = 0,
$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma}$ are the standard Caputo fractional derivatives, $1 < \alpha, \beta, \gamma \le 2$, and $f_1, f_2, f_3 : [0,1] \times R \times R \to R$ is continuous.

The main goal of this paper is to establish some new criteria for the existence of solutions of (1). The method is based on Mawhin's coincidence degree theory. The results in this paper are generalized of the existing ones.

2. Preliminaries

In this section, we give the definitions of fractional derivatives and integrals and some notations which are useful throughout this paper. There are several kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left sided definition on the half-axis R_+ and the Caputo fractional derivative.

Let X and Y be real Banach spaces and let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Y and there exist continuous projectors $P: X \to X, Q: Y \to Y$ such that

$$\operatorname{Im} P = \operatorname{Ker} L,\ \operatorname{Ker} Q = \operatorname{Im} L,\ X = \operatorname{Ker} L \oplus \operatorname{Ker} P,\ Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. Here K_P denotes the inverse of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$.

If Ω is an open bounded subset of X, and $\operatorname{dom} L \cap \bar{\Omega} \neq \phi$, then the map $N: X \to Y$ will be called L-compact on $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N: \bar{\Omega} \to X$ is compact, where I is the identity operator.

Lemma 1. [14] *Let* $L: \operatorname{dom} L \subset X \to Y$ *be a Fredholm operator with index zero and* $N: X \to Y$ *be L-compact on* $\bar{\Omega}$. *Assume that the following conditions are satisfied.*

1. $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;
2. $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;
3. $\deg(QN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) \neq 0$, where $Q: Y \to Y$ is a projection such that $\operatorname{Im} L = \operatorname{Ker} Q$.

Then the operator equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.

Definition 1. [6] *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $y: R_+ \to R$ *on the half-axis* R_+ *is given by*

$$\left(I_{0^+}^{\alpha} y\right)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} y(v) \mathrm{d}v \quad \text{for} \quad t > 0$$

provided the right hand side is pointwise defined on R_+.

Definition 2. [6] *Assume that* $x(t)$ *is* $(n-1)$*-times absolutely continuous function, the Caputo fractional derivative of order* $\alpha > 0$ *of x is given by*

$$\left(D_{0^+}^{\alpha} x\right)(t) := I_{0^+}^{n-\alpha} \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-v)^{n-\alpha-1} x^{(n)}(v) \mathrm{d}v \quad \text{for} \quad t > 0$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2. [6] *Let* $\alpha > 0$ *and* $n = -[-\alpha]$. *If* $x^{(n-1)} \in AC[0,1]$, *then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) := x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

where $c_i = -\dfrac{x^{(i)}(0)}{i!} \in R,\ i = 0,1,2,\cdots,n-1$, here n is the smallest integer greater than or equal to α.

In this paper, let us take $X = C^1[0,1]$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$ and $Y = C[0,1]$ with the norm $\|y\|_Y = \|y\|_\infty$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Then we denote $\bar{X} = X \times X \times X$ with the norm $\|(u,v,w)\|_{\bar{X}} = \max\{\|u\|_X, \|v\|_X, \|w\|_X\}$ and $\bar{Y} = Y \times Y \times Y$ with the norm $\|(x,y,z)\|_{\bar{Y}} = \max\{\|x\|_Y, \|y\|_Y, \|z\|_Y\}$. Clearly, both $\bar{X}$ and $\bar{Y}$ are Banach spaces.

Define the operators $L_i: \operatorname{dom} L \subset X \to Y,\ (i = 1,2,3)$ by

$$L_1u = D_{0^+}^{\alpha}u,\ L_2v = D_{0^+}^{\beta}v \text{ and } L_3w = D_{0^+}^{\gamma}w,$$

where

$$\text{dom } L_1 = \left\{u \in X \,\middle|\, D_{0^+}^{\alpha}u(t) \in Y, u'(0) = u'(1) = 0\right\},$$

$$\text{dom } L_2 = \left\{v \in X \,\middle|\, D_{0^+}^{\beta}v(t) \in Y, v'(0) = v'(1) = 0\right\},$$

and

$$\text{dom } L_3 = \left\{w \in X \,\middle|\, D_{0^+}^{\gamma}w(t) \in Y, w'(0) = w'(1) = 0\right\}.$$

Define the operator $L : \text{dom } L \subset \overline{X} \to \overline{Y}$ by

$$L(u,v,w) = (L_1u, L_2v, L_3w), \tag{2}$$

where $\text{dom } L = \left\{(u,v,w) \in \overline{X} \,\middle|\, u \in \text{dom } L_1, v \in \text{dom } L_2, w \in \text{dom } L_3\right\}$.

Let the Nemytski operator $N : \overline{X} \to \overline{Y}$ be defined as

$$N(u,v,w) = (N_1v, N_2w, N_3u),$$

where $N_1 : X \to Y$ is defined by

$$N_1v(t) = f_1(t, v(t), v'(t)),$$

$N_2 : X \to Y$ is defined by

$$N_2w(t) = f_2(t, w(t), w'(t)),$$

and $N_3 : X \to Y$ is defined by

$$N_3u(t) = f_3(t, u(t), u'(t)).$$

Then Neumann boundary value problem (1) is equivalent to the operator equation

$$L(u,v,w) = N(u,v,w),\ (u,v,w) \in \text{dom } L.$$

3. Main Results

In this section, we begin with the following theorem on existence of solutions for Neumann boundary value problem (1).

Theorem 1. *Let $f_1, f_2, f_3 : [0,1] \times R \times R \to R$ be continuous. Assume that*

(H1) there exist nonnegative functions $a_i, b_i, c_i \in C[0,1]$, $(i = 1,2,3)$ with

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) - (B_1 + C_1)(B_2 + C_2)(B_3 + C_3)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} > 0$$

such that for all $(u,v) \in R^2$, $t \in [0,1]$,

$$\left|f_i(t,u,v)\right| \le a_i(t) + b_i(t)|u| + c_i(t)|v| \quad \text{for } i = 1,2,3,$$

where $A_i = \|a_i\|_\infty$, $B_i = \|b_i\|_\infty$, $C_i = \|c_i\|_\infty$, $(i = 1,2,3)$;

(H2) there exists a constant $M > 0$ such that for all $t \in [0,1]$, $|u| > M$, $v \in R$ either

$$uf_1(t,u,v) > 0,\ uf_2(t,u,v) > 0,\ uf_3(t,u,v) > 0$$

or

$$uf_1(t,u,v)<0,\ uf_2(t,u,v)<0,\ uf_3(t,u,v)<0;$$

(H3) there exists a constant $M^*>0$ such that for every $m_1,m_2,m_3\in R$ satisfying $\min\{m_1,m_2,m_3\}>M^*$ either

$$m_1N_1(m_2)>0,\ m_2N_2(m_3)>0,\ m_3N_3(m_1)>0$$

or

$$m_1N_1(m_2)<0,\ m_2N_2(m_3)<0,\ m_3N_3(m_1)<0.$$

Then Neumann boundary value problem (1) has at least one solution.

Lemma 3. *Let L be defined by* (2). *Then*

$$\begin{aligned}\operatorname{Ker} L&=(\operatorname{Ker} L_1,\operatorname{Ker} L_2,\operatorname{Ker} L_3)\\&=\left\{(u,v,w)\in\overline{X}\,\middle|\,(u,v,w)=(u(0),v(0),w(0))\right\},\end{aligned}\tag{3}$$

and

$$\begin{aligned}\operatorname{Im} L&=(\operatorname{Im} L_1,\operatorname{Im} L_2,\operatorname{Im} L_3)\\&=\Big\{(x,y,z)\in\overline{Y}\,\Big|\,\int_0^1(1-s)^{\alpha-2}x(s)\mathrm{d}s\\&=\int_0^1(1-s)^{\beta-2}y(s)\mathrm{d}s=\int_0^1(1-s)^{\gamma-2}z(s)\mathrm{d}s=0\Big\}.\end{aligned}\tag{4}$$

Proof. By Lemma 2, $L_1u=D_{0^+}^{\alpha}u(t)=0$ has the solution

$$u(t)=u(0)+u'(0)t.$$

From the boundary conditions, we have

$$\operatorname{Ker} L_1=\{u\in X\,|\,u=u(0)\}.$$

For $x\in\operatorname{Im} L_1$, there exists $u\in\operatorname{dom} L_1$ such that $x=L_1u\in Y$. By using the Lemma 2, we get

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)\mathrm{d}s+u(0)+u'(0)t.$$

Then, we have

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}x(s)\mathrm{d}s+u'(0).$$

By the boundary value conditions of (1), we can get that x satisfies

$$\int_0^1(1-s)^{\alpha-2}x(s)\mathrm{d}s=0.$$

On the other hand, suppose $x\in Y$ and satisfies $\int_0^1(1-s)^{\alpha-2}x(s)\mathrm{d}s=0$. Let $u(t)=I_{0^+}^{\alpha}x(t)$, then $u\in\operatorname{dom} L_1$ and $D_{0^+}^{\alpha}u(t)=x(t)$. Hence, $x\in\operatorname{Im} L_1$. Then we get

$$\operatorname{Im} L_1=\left\{x\in Y\,\middle|\,\int_0^1(1-s)^{\alpha-2}x(s)\mathrm{d}s=0\right\}.$$

Similarly, we have

$$\operatorname{Ker} L_2=\{v\in X\,|\,v=v(0)\},\quad \operatorname{Im} L_2=\left\{y\in Y\,\middle|\,\int_0^1(1-s)^{\beta-2}y(s)\mathrm{d}s=0\right\},$$

and

$$\operatorname{Ker} L_3 = \left\{ w \in X \,\middle|\, w = w(0) \right\}, \quad \operatorname{Im} L_3 = \left\{ z \in Y \,\middle|\, \int_0^1 (1-s)^{\gamma-2} z(s)\,\mathrm{d}s = 0 \right\}.$$

□

Lemma 4. *Let L be defined by* (2). *Then L is a Fredholm operator of index zero,* $P : \bar{X} \to \bar{X}$ *and* $Q : \bar{Y} \to \bar{Y}$ *are the linear continuous projector operators can be defined as*

$$P(u,v,w) = (P_1 u, P_2 v, P_3 w) = (u(0), v(0), w(0)),$$

$$\begin{aligned} Q(x,y,z) &= (Q_1 x, Q_2 y, Q_3 z) \\ &= \Big((\alpha-1)\int_0^1 (1-s)^{\alpha-2} x(s)\,\mathrm{d}s, \\ &\quad (\beta-1)\int_0^1 (1-s)^{\beta-2} y(s)\,\mathrm{d}s, (\gamma-1)\int_0^1 (1-s)^{\gamma-2} z(s)\,\mathrm{d}s \Big). \end{aligned}$$

Further more, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by

$$K_P(x,y,z) = \left(I_{0^+}^{\alpha} x(t), I_{0^+}^{\beta} y(t), I_{0^+}^{\gamma} z(t) \right).$$

Proof. Clearly, $\operatorname{Im} P = \operatorname{Ker} L$ and $P^2(u,v,w) = P(u,v,w)$. It follows that $(u,v,w) = ((u,v,w) - P(u,v,w)) + P(u,v,w)$, we have $\bar{X} = \operatorname{Ker} P + \operatorname{Ker} L$. By using simple calculation, we get that $\operatorname{Ker} L \cap \operatorname{Ker} P = \{(0,0,0)\}$. Then we have

$$\bar{X} = \operatorname{Ker} P \oplus \operatorname{Ker} L.$$

For $(x,y,z) \in \bar{Y}$, we have

$$Q^2(x,y,z) = Q(Q_1 x, Q_2 y, Q_3 z) = \left(Q_1^2 x, Q_2^2 y, Q_3^2 z \right)$$

By the definition of Q_1, we get

$$Q_1^2 x = Q_1 x \cdot (\alpha-1)\int_0^1 (1-s)^{\alpha-2} x(s)\,\mathrm{d}s = Q_1 x.$$

Similarly, we can show that $Q_2^2 y = Q_2 y$ and $Q_3^2 z = Q_3 z$. Thus, we can get $Q^2(x,y,z) = Q(x,y,z)$.

Let

$$(x,y,z) = ((x,y,z) - Q(x,y,z)) + Q(x,y,z),$$

where $(x,y,z) - Q(x,y,z) \in \operatorname{Ker} Q$, $Q(x,y,z) \in \operatorname{Im} Q$. It follows that $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2(x,y,z) = Q(x,y,z)$, we get $\operatorname{Im} Q \cap \operatorname{Im} L = \{(0,0,0)\}$. It is clear that

$$\bar{Y} = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L.$$

Hence L is a Fredholm operator of index zero.

From the definitions of P and K_P, we will prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. Infact, for $(x,y,z) \in \operatorname{Im} L$, we have

$$LK_P(x,y,z) = \left(D_{0^+}^{\alpha}\left(I_{0^+}^{\alpha} x\right), D_{0^+}^{\beta}\left(I_{0^+}^{\beta} y\right), D_{0^+}^{\gamma}\left(I_{0^+}^{\gamma} z\right) \right) = (x,y,z). \tag{5}$$

Moreover, for $(u,v,w) \in \operatorname{dom} L \cap \operatorname{Ker} P$, we have $u(0) = v(0) = w(0) = 0$ and

$$\begin{aligned}K_P L(u,v,w) &= \left(I_{0^+}^{\alpha}\left(D_{0^+}^{\alpha}u(t)\right), I_{0^+}^{\beta}\left(D_{0^+}^{\beta}v(t)\right), I_{0^+}^{\gamma}\left(D_{0^+}^{\gamma}w(t)\right)\right)\\ &= \left(u(t)-u(0)-u'(0)t, v(t)-v(0)-v'(0)t, w(t)-w(0)-w'(0)t\right),\end{aligned}$$

which together with the boundary condition $u'(0)=v'(0)=w'(0)=0$ yields that

$$K_P L(u,v,w) = (u,v,w). \tag{6}$$

From (5) and (6), we get $K_P = \left(L|_{\operatorname{dom} L\cap \operatorname{Ker} P}\right)^{-1}$. □

Lemma 5. *Assume* $\Omega \subset \bar{X}$ *is an open bounded subset such that* $\operatorname{dom} L \cap \bar{\Omega} \neq \phi$, *then N is L-compact on* $\bar{\Omega}$.

Proof. By the continuity of f_1, f_2 and f_3, we can get $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are bounded. By the Arzela-Ascoli theorem, we will prove that $K_P(I-Q)N(\bar{\Omega}) \subset \bar{X}$ is equicontinuous.

From the continuity of f_1, f_2 and f_3, there exist constants $M_i > 0, (i=1,2,3)$ such that for all $(u,v,w) \in \bar{\Omega}$.

$$\left|(I-Q_1)N_1 v\right| \le M_1, \quad \left|(I-Q_2)N_2 w\right| \le M_2, \quad \left|(I-Q_3)N_3 u\right| \le M_3.$$

Furthermore, for $0 \le t_1 < t_2 \le 1, (u,v,w) \in \bar{\Omega}$, we have

$$\begin{aligned}&\left|\left(K_P(I-Q)N\left(u(t_2),v(t_2),w(t_2)\right)\right)-\left(K_P(I-Q)N\left(u(t_1),v(t_1),w(t_1)\right)\right)\right|\\ &=\left|K_P(I-Q)\left(N_1v(t_2),N_2w(t_2),N_3u(t_2)\right)-K_P(I-Q)\left(N_1v(t_1),N_2w(t_1),N_3u(t_1)\right)\right|\\ &=\left|\left(I_{0^+}^{\alpha}(I-Q_1)N_1v(t_2), I_{0^+}^{\beta}(I-Q_2)N_2w(t_2), I_{0^+}^{\gamma}(I-Q_3)N_3u(t_2)\right)\right.\\ &\quad\left.-\left(I_{0^+}^{\alpha}(I-Q_1)N_1v(t_1), I_{0^+}^{\beta}(I-Q_2)N_2w(t_1), I_{0^+}^{\gamma}(I-Q_3)N_3u(t_1)\right)\right|\\ &=\left|I_{0^+}^{\alpha}(I-Q_1)N_1v(t_2)-I_{0^+}^{\alpha}(I-Q_1)N_1v(t_1),\right.\\ &\quad I_{0^+}^{\beta}(I-Q_2)N_2w(t_2)-I_{0^+}^{\beta}(I-Q_2)N_2w(t_1),\\ &\quad\left. I_{0^+}^{\gamma}(I-Q_3)N_3u(t_2)-I_{0^+}^{\gamma}(I-Q_3)N_3u(t_1)\right|.\end{aligned}$$

By

$$\begin{aligned}&\left|I_{0^+}^{\alpha}(I-Q_1)N_1v(t_2)-I_{0^+}^{\alpha}(I-Q_1)N_1v(t_1)\right|\\ &\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q_1)N_1v(s)\,\mathrm{d}s-\int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q_1)N_1v(s)\,\mathrm{d}s\right|\\ &\le \frac{M_1}{\Gamma(\alpha)}\left[\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)\mathrm{d}s+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,\mathrm{d}s\right]=\frac{M_1}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\left(I_{0^+}^{\alpha}(I-Q_1)N_1v\right)'(t_2)-\left(I_{0^+}^{\alpha}(I-Q_1)N_1v\right)'(t_1)\right|\\ &\le \frac{\alpha-1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-2}(I-Q_1)N_1v(s)\,\mathrm{d}s-\int_0^{t_1}(t_1-s)^{\alpha-2}(I-Q_1)N_1v(s)\,\mathrm{d}s\right|\\ &\le \frac{M_1}{\Gamma(\alpha-1)}\left[\int_0^{t_1}\left((t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}\right)\mathrm{d}s+\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\,\mathrm{d}s\right]\\ &=\frac{M_1}{\Gamma(\alpha)}\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right).\end{aligned}$$

Similarly, we can show that

$$\left|I_{0^+}^{\beta}(I-Q_2)N_2w(t_2)-I_{0^+}^{\beta}(I-Q_2)N_2w(t_1)\right|\le\frac{M_2}{\Gamma(\beta+1)}\left(t_2^{\beta}-t_1^{\beta}\right),$$

$$\left|\left(I_{0^+}^{\beta}(I-Q_2)N_2w\right)'(t_2)-\left(I_{0^+}^{\beta}(I-Q_2)N_2w\right)'(t_1)\right|\le\frac{M_2}{\Gamma(\beta+1)}\left(t_2^{\beta-1}-t_1^{\beta-1}\right),$$

$$\left|I_{0^+}^{\gamma}(I-Q_3)N_3u(t_2)-I_{0^+}^{\gamma}(I-Q_3)N_3u(t_1)\right|\le\frac{M_3}{\Gamma(\gamma+1)}\left(t_2^{\gamma}-t_1^{\gamma}\right),$$

$$\left|\left(I_{0^+}^{\gamma}(I-Q_3)N_3u\right)'(t_2)-\left(I_{0^+}^{\gamma}(I-Q_3)N_3u\right)'(t_1)\right|\le\frac{M_3}{\Gamma(\gamma+1)}\left(t_2^{\gamma-1}-t_1^{\gamma-1}\right).$$

Since t^{α}, $t^{\alpha-1}$, t^{β}, $t^{\beta-1}$, t^{γ} and $t^{\gamma-1}$ are uniformly continuous on [0, 1], we have $K_P(I-Q)N(\bar{\Omega})\subset X$ is equicontinuous. Thus $K_P(I-Q)N:\bar{\Omega}\to X$ is compact. □

Lemma 6. *Assume that* (H_1), (H_2) *hold, then the set*

$$\Omega_1=\left\{(u,v,w)\in \mathrm{dom}L\setminus \mathrm{Ker}L \,\middle|\, L(u,v,w)=\lambda N(u,v,w),\lambda\in(0,1)\right\}$$

is bounded.

Proof. Let $(u,v,w)\in\Omega_1$, then $N(u,v,w)\in \mathrm{Im}\,L$. By (4), we get

$$\int_0^1(1-s)^{\alpha-2}f_1(s,v(s),v'(s))\mathrm{d}s=0,$$

$$\int_0^1(1-s)^{\beta-2}f_2(s,w(s),w'(s))\mathrm{d}s=0$$

and

$$\int_0^1(1-s)^{\gamma-2}f_3(s,u(s),u'(s))\mathrm{d}s=0.$$

Then, by integral mean value theorem, there exist constants $\xi,\eta,\zeta\in(0,1)$ such that $f_1(\xi,v(\xi),v'(\xi))=0,\ f_2(\eta,w((\eta),w'(\eta)))=0$ and $f_3(\zeta,u(\zeta),u'(\zeta))=0.$ Then we get

$$v(\xi)f_1(\xi,v(\xi),v'(\xi))=0,\ w(\eta)f_2(\eta,w(\eta),w'(\eta))=0$$
$$\text{and}\ \ u(\zeta)f_3(\zeta,u(\zeta),u'(\zeta))=0.$$

From (H_2), we get $|v(\xi)|\le M$, $|w(\eta)|\le M$ and $|u(\zeta)|\le M$. Hence we have

$$|u(t)|=\left|u(\zeta)+\int_{\zeta}^{t}u'(s)\mathrm{d}s\right|\le M+\|u'\|_{\infty}. \tag{7}$$

We obtain

$$\|u\|_{\infty}\le M+\|u'\|_{\infty}. \tag{8}$$

Similarly, we can show that

$$\|v\|_{\infty}\le M+\|v'\|_{\infty} \tag{9}$$

and

$$\|w\|_{\infty}\le M+\|w'\|_{\infty}. \tag{10}$$

By $L(u,v,w)=\lambda N(u,v,w)$, we get

$$u(t)=\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\ f_1(s,v(s),v'(s))\mathrm{d}s+u(0),$$

$$v(t)=\frac{\lambda}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\ f_2(s,w(s),w'(s))\mathrm{d}s+v(0)$$

and

$$w(t)=\frac{\lambda}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}\ f_3(s,u(s),u'(s))\mathrm{d}s+w(0).$$

Then

$$u'(t)=\frac{\lambda}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\ f_1(s,v(s),v'(s))\mathrm{d}s,$$

$$v'(t)=\frac{\lambda}{\Gamma(\beta-1)}\int_0^t(t-s)^{\beta-2}\ f_2(s,w(s),w'(s))\mathrm{d}s$$

and

$$w'(t)=\frac{\lambda}{\Gamma(\gamma-1)}\int_0^t(t-s)^{\gamma-2}\ f_3(s,u(s),u'(s))\mathrm{d}s.$$

So,

$$\begin{aligned}\|u'\|_\infty &=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\left|f_1(s,v(s),v'(s))\right|\mathrm{d}s\\ &\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\left[a_1(s)+b_1(s)|v(s)|+c_1(s)|v'(s)|\right]\mathrm{d}s\\ &\le\frac{1}{\Gamma(\alpha-1)}\left[A_1+B_1M+(B_1+C_1)\|v'\|_\infty\right]\int_0^t(t-s)^{\alpha-2}\,\mathrm{d}s\\ &\le\frac{1}{\Gamma(\alpha)}\left[A_1+B_1M+(B_1+C_1)\|v'\|_\infty\right].\end{aligned}\tag{12}$$

Similarly, we have

$$\|v'\|_\infty\le\frac{1}{\Gamma(\beta)}\left[A_2+B_2M+(B_2+C_2)\|w'\|_\infty\right]\tag{12}$$

and

$$\|w'\|_\infty\le\frac{1}{\Gamma(\gamma)}\left[A_3+B_3M+(B_3+C_3)\|u'\|_\infty\right].\tag{13}$$

Combining (13) with (12), we get

$$\begin{aligned}\|v'\|_\infty\le\frac{1}{\Gamma(\beta)\Gamma(\gamma)}&\left[(A_2+B_2M)\Gamma(\gamma)+(B_2+C_2)(A_3+B_3M)\right.\\ &\left.+(B_2+C_2)(B_3+C_3)\|u'\|_\infty\right].\end{aligned}\tag{14}$$

Combining (14) with (11), we get

$$\begin{aligned}\|u'\|_\infty\le\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}&[(A_1+B_1M)\Gamma(\beta)\Gamma(\gamma)+(B_1+C_1)(A_2+B_2M)\Gamma(\gamma)\\ &+(B_1+C_1)(B_2+C_2)(A_3+B_3M)+(B_1+C_1)(B_2+C_2)(B_3+C_3)\|u'\|_\infty].\end{aligned}$$

Thus, from $\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)-(B_1+C_1)(B_2+C_2)(B_3+C_3)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}>0$ and (14), we get

$$\begin{aligned}\|u'\|_\infty &\le \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)-(B_1+C_1)(B_2+C_2)(B_3+C_3)}\big[(A_1+B_1M)\Gamma(\beta)\Gamma(\gamma)\\ &\quad +(B_1+C_1)(A_2+B_2M)\Gamma(\gamma)+(B_1+C_1)(B_2+C_2)(A_3+B_3M)\big]\\ &:= K_1\end{aligned}$$

$$\begin{aligned}\|v'\|_\infty &\le \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\big[(A_2+B_2M)\Gamma(\gamma)+(B_2+C_2)(A_3+B_3M)\\ &\quad +(B_2+C_2)(B_3+C_3)K_1\big]:=K_2\end{aligned}$$

and

$$\|w'\|_\infty \le \frac{1}{\Gamma(\gamma)}\big[A_3+B_3M+(B_3+C_3)K_1\big]:=K_3.$$

From (8), (9) and (10), we have

$$\|(u,v,w)\|_{\bar{X}} \le \max\{K_1+M, K_2+M, K_3+M\}:=K.$$

Hence Ω_1 is bounded. □

Lemma 7. *Assume that* (H_3) *holds, then the set*

$$\Omega_2=\{(u,v,w)|(u,v,w)\in \operatorname{Ker} L, N(u,v,w)\in \operatorname{Im} L\}$$

is bounded.

Proof. For $(u,v,w)\in\Omega_2$, we have $(u,v,w)=(m_1,m_2,m_3)$, $m_1,m_2,m_3\in R$. Then from $N(u,v,w)\in \operatorname{Im} L$,

$$\int_0^1(1-s)^{\alpha-2} f_1(s,m_2,0)\mathrm{d}s=0,$$

$$\int_0^1(1-s)^{\beta-2} f_2(s,m_3,0)\mathrm{d}s=0$$

and

$$\int_0^1(1-s)^{\gamma-2} f_3(s,m_1,0)\mathrm{d}s=0.$$

From (H_3) imply that $|m_1|,\ |m_2|,\ |m_3|\le M^*$. Thus, we get

$$\|(u,v,w)\|_{\bar{X}}\le M^*.$$

Therefore Ω_2 is bounded. □

Lemma 8. *Assume that the first part of* (H_3) *holds, then the set*

$$\Omega_3=\{(u,v,w)\in \operatorname{Ker} L|\lambda(u,v,w)+(1-\lambda)QN(u,v,w)=(0,0,0),\lambda\in[0,1]\}$$

is bounded.

Proof. For $(u,v,w)\in\Omega_3$, we have $(u,v,w)=(m_1,m_2,m_3)$, $m_1,m_2,m_3\in R$ and

$$\lambda m_1+(1-\lambda)(\alpha-1)\int_0^1(1-s)^{\alpha-2} f_1(s,m_2,0)\mathrm{d}s=0, \tag{15}$$

$$\lambda m_2+(1-\lambda)(\beta-1)\int_0^1(1-s)^{\beta-2} f_2(s,m_3,0)\mathrm{d}s=0 \tag{16}$$

and

$$\lambda m_3 + (1-\lambda)(\gamma-1)\int_0^1 (1-s)^{\gamma-2} f_3(s, m_1, 0)\,\mathrm{d}s = 0. \tag{17}$$

If $\lambda = 0$, then by (H_3), we get $|m_1|,\ |m_2|,\ |m_3| \le M^*$. If $\lambda = 1$, then $m_1 = m_2 = m_3 = 0$. For $\lambda \in (0,1]$, we obtain $|m_1|,\ |m_2|,\ |m_3| \le M^*$. Otherwise, if $|m_1|$ or $|m_2|$ or $|m_3| > M^*$, from (H_3), one has

$$\lambda m_1^2 + (1-\lambda)(\alpha-1)\int_0^1 (1-s)^{\alpha-2} m_1 f_1(s, m_2, 0)\,\mathrm{d}s > 0$$

or

$$\lambda m_2^2 + (1-\lambda)(\beta-1)\int_0^1 (1-s)^{\beta-2} m_2 f_2(s, m_3, 0)\,\mathrm{d}s > 0$$

or

$$\lambda m_3^2 + (1-\lambda)(\gamma-1)\int_0^1 (1-s)^{\gamma-2} m_3 f_3(s, m_1, 0)\,\mathrm{d}s > 0,$$

which contradict to (15) or (16) or (17). Hence, Ω_3 is bounded.

Remark 1 *Suppose the second part of* (H_3) *holds, then the set*

$$\Omega_3' = \left\{(u,v,w) \in \operatorname{Ker} L \,\middle|\, -\lambda(u,v,w) + (1-\lambda)QN(u,v,w) = (0,0,0), \lambda \in [0,1]\right\}$$

is bounded. □

Proof of the Theorem 1: Set $\Omega = \left\{(u,v,w) \in \bar{X} \,\middle|\, \|(u,v,w)\|_{\bar{X}} < \max\{K, M^*\} + 1\right\}$.

From the Lemma 4 and Lemma 5 we can get L is a Fredholm operator of index zero and N is L-compact on $\bar{\Omega}$. By Lemma 6 and Lemma 7, we obtain

(1) $L(u,v,w) \ne \lambda N(u,v,w)$ for every $((u,v,w),\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L$ for every $(u,v,w) \in \operatorname{Ker} L \cap \partial\Omega$.

Choose

$$H((u,v,w),\lambda) = \pm\lambda(u,v,w) + (1-\lambda)QN(u,v,w).$$

By Lemma 8 (or Remark 1), we get $H((u,v,w),\lambda) \ne 0$ for $(u,v,w) \in \operatorname{Ker} L \cap \partial\Omega$. Therefore

$$\begin{aligned}
\deg\left(QN|_{\operatorname{Ker}L}, \operatorname{Ker}L \cap \Omega, 0\right) &= \deg\left(H(.,0), \operatorname{Ker}L \cap \Omega, 0\right) \\
&= \deg\left(H(.,1), \operatorname{Ker}L \cap \Omega, 0\right) \\
&= \deg\left(\pm I, \operatorname{Ker}L \cap \Omega, 0\right) \ne 0.
\end{aligned}$$

Thus, the condition (3) of Lemma 1 is satisfied. By Lemma 1, we obtain $L(u,v,w) = N(u,v,w)$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$. Hence Neumann boundary value problem (1) has at least one solution. This completes the proof.

4. Examples

In this section, we give two examples to illustrate our main results.

Example 1. *Consider the following Neumann boundary value problem of fractional differential equation of the form*

$$\begin{aligned}
&D_{0^+}^{\frac{5}{4}} u(t) = \frac{1}{8}(v(t)-6) + \frac{t^2}{8}(1+v'(t))^{\frac{1}{2}},\ t \in (0,1),\\
&D_{0^+}^{\frac{3}{2}} v(t) = \frac{1}{6}(w(t)-4) + \frac{t^3}{6}\cos^2 w'(t),\ t \in (0,1),\\
&D_{0^+}^{\frac{7}{4}} w(t) = \frac{1}{10}(u(t)-8) + \frac{t^3}{10}\sin^2 u'(t),\ t \in (0,1),\\
&u'(0) = u'(1) = 0,\ v'(0) = v'(1) = 0,\ w'(0) = w'(1) = 0.
\end{aligned} \tag{18}$$

Here $\alpha = \frac{5}{4},\ \beta = \frac{3}{2},\ \gamma = \frac{7}{4}$. Moreover,

$$f_1(t, v(t), v'(t)) = \frac{1}{8}(v(t)-6) + \frac{t^2}{8}(1+v'(t))^{\frac{1}{2}},$$

$$f_2(t, w(t), w'(t)) = \frac{1}{6}(w(t)-4) + \frac{t^3}{6}\cos^2 w'(t),$$

$$f_3(t, u(t), u'(t)) = \frac{1}{10}(u(t)-8) + \frac{t^3}{10}\sin^2 u'(t).$$

Now let us compute $a_1(t),\ b_1(t),\ c_1(t)$ from $f_1(t, v(t), v'(t))$.

$$\begin{aligned}
f_1(t, v(t), v'(t)) &= \frac{1}{8}(v(t)-6) + \frac{t^2}{8}(1+v'(t))^{\frac{1}{2}}\\
&= \frac{1}{8}(v(t)-6) + \frac{t^2}{8}\left(1 + \frac{1}{2}v'(t) + \cdots\right)\\
&\le \frac{1}{8}(v(t)-6) + \frac{t^2}{8}.
\end{aligned}$$

$$|f_1(t, v(t), v'(t))| \le \frac{1}{8}|v(t)| + \frac{7}{8}.$$

From the above inequality, we get $a_1(t) = \frac{7}{8},\ b_1(t) = \frac{1}{8},\ c_1(t) = 0.$ Also,

$$f_2(t, w(t), w'(t)) = \frac{1}{6}(w(t)-4) + \frac{t^3}{6}\cos^2 w'(t)$$

$$|f_2(t, w(t), w'(t))| \le \frac{1}{6}|w(t)| + \frac{5}{6}.$$

Here, $a_2(t) = \frac{5}{6},\ b_2(t) = \frac{1}{6},\ c_2(t) = 0$. Finally,

$$f_3(t, u(t), u'(t)) = \frac{1}{10}(u(t)-8) + \frac{t^3}{10}\sin^2 u'(t).$$

$$|f_3(t, u(t), u'(t))| \le \frac{1}{10}|u(t)| + \frac{8}{10}.$$

We get, $a_3(t) = \frac{4}{5},\ b_3(t) = \frac{1}{10},\ c_3(t) = 0$. And we get, $B_1(t) = \frac{1}{8}$, $B_2(t) = \frac{1}{6}$, $B_3(t) = \frac{1}{10}$, $C_1(t) = C_2(t) = C_3(t) = 0$. Choose $M = M^* = 8$. Also,

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)-(B_1+C_1)(B_2+C_2)(B_3+C_3)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$
$$=\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{4}\right)-(B_1B_2B_3)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{4}\right)}$$
$$\approx 0.73605543>0,$$

where $\Gamma\left(\frac{1}{4}\right)\approx 3.625$, $\Gamma\left(\frac{3}{4}\right)\approx 1.225416702$ and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. All the conditions of Theorem 1 are satisfied. Hence, boundary value problem (18) has at least one solution.

Example 2. *Consider the Neumann boundary value problem of fractional differential equation of the following form*

$$\begin{aligned}
&D_{0^+}^{\frac{4}{3}}u(t)=\frac{1}{17}(v(t)-15)+\frac{t^4}{20}\log(1+v'(t)),\ t\in(0,1),\\
&D_{0^+}^{\frac{5}{4}}v(t)=\frac{1}{9}(w(t)-7)+\frac{t^6}{25}(1+w'(t))^{\frac{1}{3}},\ t\in(0,1),\\
&D_{0^+}^{\frac{3}{2}}w(t)=\frac{1}{13}(u(t)-11)+\frac{t^7}{15}\arctan u'(t),\ t\in(0,1),\\
&u'(0)=u'(1)=0,\ v'(0)=v'(1)=0,\ w'(0)=w'(1)=0.
\end{aligned}\tag{19}$$

Here $\alpha=\frac{4}{3},\ \beta=\frac{5}{4},\ \gamma=\frac{3}{2}$. Moreover,

$$f_1(t,v(t),v'(t))=\frac{1}{17}(v(t)-15)+\frac{t^4}{20}\log(1+v'(t)),$$

$$f_2(t,w(t),w'(t))=\frac{1}{9}(w(t)-7)+\frac{t^6}{25}(1+w'(t))^{\frac{1}{3}},$$

$$f_3(t,u(t),u'(t))=\frac{1}{13}(u(t)-11)+\frac{t^7}{15}\arctan u'(t).$$

Now let us compute $a_1(t),\ b_1(t),\ c_1(t)$ from $f_1(t,v(t),v'(t))$.

$$\begin{aligned}
f_1(t,v(t),v'(t))&=\frac{1}{17}(v(t)-15)+\frac{t^4}{20}\log(1+v'(t))\\
&=\frac{1}{17}(v(t)-15)+\frac{t^4}{20}\left(v'(t)-\frac{(v'(t))^2}{2!}+\cdots\right)
\end{aligned}$$

$$|f_1(t,v(t),v'(t))|\le\frac{1}{17}|v(t)|+\frac{15}{17}+\frac{1}{20}|v'(t)|$$

From the above inequality, we get $a_1(t)=\frac{15}{17},b_1(t)=\frac{1}{17},c_1(t)=\frac{1}{20}$. Also,

$$\begin{aligned}
f_2(t,w(t),w'(t))&=\frac{1}{9}(w(t)-7)+\frac{t^6}{25}(1+w'(t))^{\frac{1}{3}}\\
&=\frac{1}{9}(w(t)-7)+\frac{t^6}{25}\left(1+\frac{1}{3}w'(t)+\cdots\right)
\end{aligned}$$

$$\left|f_2\left(t,w(t),w'(t)\right)\right| \le \frac{1}{9}\left|w(t)\right| + \frac{7}{9} + \frac{1}{25}\left(1+\frac{1}{3}\left|w'(t)\right| + \cdots\right)$$
$$\le \frac{1}{9}\left|w(t)\right| + \frac{184}{225} + \frac{1}{75}\left|w'(t)\right|.$$

Here, $a_2(t) = \frac{184}{225}, b_2(t) = \frac{1}{9}, c_2(t) = \frac{1}{75}$. Similarly,

$$f_3\left(t,u(t),u'(t)\right) = \frac{1}{13}\left(u(t)-11\right) + \frac{t^7}{10}\arctan u'(t).$$
$$= \frac{1}{13}\left(u(t)-11\right) + \frac{t^7}{15}\left(u'(t) - \frac{\left(u'(t)^3\right)}{3} + \cdots\right).$$

$$\left|f_3\left(t,u(t),u'(t)\right)\right| \le \frac{1}{13}\left|u(t)\right| + \frac{11}{13} + \frac{1}{15}u'(t).$$

Here, $a_3(t) = \frac{11}{13}$, $b_3(t) = \frac{1}{13}$, $c_3(t) = \frac{1}{15}$. We get, $B_1(t) = \frac{1}{17}$, $B_2(t) = \frac{1}{9}$, $B_3(t) = \frac{1}{13}$, $C_1(t) = \frac{1}{20}$, $C_2(t) = \frac{1}{75}$, $C_3(t) = \frac{1}{15}$. Choose $M = M^* = 15$.

Also,

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma) - (B_1+C_1)(B_2+C_2)(B_3+C_3)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$
$$= \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{2}\right) - \left(\left(\frac{1}{17}+\frac{1}{20}\right)\left(\frac{1}{9}+\frac{1}{75}\right)\left(\frac{1}{13}+\frac{1}{15}\right)\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{2}\right)}$$
$$\approx 0.96872 > 0,$$

where $\Gamma\left(\frac{1}{4}\right) \approx 3.625$, $\Gamma\left(\frac{1}{3}\right) \approx 0.1924$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Hence all the conditions of Theorem 1 are satisfied. Therefore, boundary value problem (19) has at least one solution.

5. Conclusion

We have investigated some existence results for three-dimensional fractional differential system with Neumann boundary condition. By using Mawhin's coincidence degree theory, we established that the given boundary value problem admits at least one solution. We also presented examples to illustrate the main results.

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