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A Characterization of Graphs with Rank No More Than 5

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Abstract

The rank of a graph is defined to be the rank of its adjacency matrix. In this paper, the Matlab was used to explore the graphs with rank no more than 5; the performance of the proposed method was compared with former methods, which is simpler and clearer; and the results show that all graphs with rank no more than 5 are characterized.

Keywords

Graph, Matrix, Rank, Nullity

1. Introduction

In this paper only consider simple graph of finite and unordered. $G=(V(G),E(G))$ is a graph, $V(G)=\{v_1,v_2,\cdots,v_n\}$ is vertices set of a graph G, the adjacency matrix $A(G)$ of a graph G is the $n\times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij}=1$ if v_i is adjacent to v_j, and $a_{ij}=0$ otherwise. Obviously, $A(G)$ is a real symmetric matrix, and all eigenvalues are real number, denoted by eigenvalues of a graph G. The rank of a graph G, written as $r(G)$, is defined to be the number of the rank of matrix $A(G)$. The nullity of a graph G is the multiplicity of the zero eigenvalues of matrix $A(G)$ and denoted by $\eta(G)$. Clearly, $\eta(G)+r(G)=|V(G)|$. In chemistry, the nullity is correlated with the stability of hydrocarbon that a graph G represented (see [1]-[6]). Collatz and Sinogowitz [1] posed the problem of characterizing all non- singular graphs, which is required to describe the issue of all nullity greater than zero; although this problem is very hard, still a lot of literature research it (see [5] [7] [8]). It is known that the rank $r(G)$ of a graph G is equal to 0 if and only if G is a null graph (*i.e.* a graph without edges), and there is no graph with rank 1. The graph G with the rank $r(G)$ is equal to 2 or 3, which is completely characterized in [8]. The graph G with the rank $r(G)$ is equal to 4, which is completely characterized in [9]. Although in [10], the graphs with rank 5 are characterized by using forbidden subgraph. In the paper, we completely characterize the graphs with rank no more than 5 by using Matlab. Compared to the method in [10], the method of this paper is simpler and clearer.

For a vertex x in G, the set of all vertices in G that are adjacent to x is denoted by $N_G(x)$. The distance between u and v, denoted by $\text{dist}_G(u,v)$, is the length of a shortest u, v-path in graph G. The distance between a vertex u and a subgraph H of G, denoted by $\text{dist}_G(u,H)$, is defined to be the value $\min\{\text{dist}_G(u,v): v \in V(H)\}$. Given a subset $S \subseteq V(G)$, the subgraph of G induced by S, is written as $G[S]$. The n-path, the n-cycle and the n-complete graph are denoted by P_n, C_n and K_n, respectively.

A subset $I \subseteq V(G)$ is called an independent set of G if the subgraph $G[I]$ is a null graph. Next we define a graph operation (see page 53 of [6]). Give a graph G with $V(G)=\{v_1,v_2,\cdots,v_n\}$. Let $\boldsymbol{m}=(m_1,m_2,\cdots,m_n)$ be a vector of positive integers. Denoted by $G \circ \boldsymbol{m}$ the graph is obtained from G by replacing each vertex v_i of G with an independent set of m_i vertices $v_i^1, v_i^2, \cdots, v_i^{m_i}$ and joining v_i^s with v_j^t if and only if v_i and v_j are adjacent in G. The resulting graph $G \circ \boldsymbol{m}$ is said to be obtained from G by multiplication of vertices. For graphs $G_1, G_2, \cdots, G_k$, we denote by $\mathcal{M}(G_1, G_2, \cdots, G_k)$ the class of all graphs that can be obtained from one of the graphs in $\{G_1, G_2, \cdots, G_k\}$ by multiplication of vertices.

2. Preliminaries

Lemma 2.1. [9] *Suppose that* G *and* H *are two graphs. If* $G \in \mathcal{M}(H)$, *then* $r(G)=r(H)$.

By Lemma 2.1, we know that the rank of a graph doesn't change by multiplication of vertices. Let G be a graph, if exists a graph $H(\ncong G)$ such that $G \in \mathcal{M}(H)$, we call G is a non-basic graph. Otherwise, G is called a basic graph. The following we need find all basic graphs with rank no more than 5.

Lemma 2.2. [3] (1) *Let* $G = H_1 \cup H_2$, *where* H_1 *and* H_2 *be two graphs. Then* $r(G)=r(H_1)+r(H_2)$.

(2) Let H be an induced subgraph of G. Then $r(H) \le r(G)$.

Lemma 2.3. *Let* G *be a connected graph with rank* $k\,(\ge 2)$. *Then there exists an induced subgraph* H *(of* G*) on* k *vertices such that* $r(H)=k$, *and* $\text{dist}_G(u,H) \le 1$ *for each vertex* u *of* G.

Proof. Without loss of generality, suppose the previous k row vectors of $A(G)$ are linear independence, and the rest of the row vectors of $A(G)$ are linear combination of the previous k row vectors. Since $A(G)$ is a symmetrical matrix, we know that the rest of the column vectors of $A(G)$ are linear combination of the previous k column vectors. Therefore we can obtain the following matrix by using elementary transformation for $A(G)$,

$$\begin{bmatrix} A(H) & 0 \\ 0 & 0 \end{bmatrix}$$

where H is the induced subgraph (of G) with the k vertices which is correspondent to the previous k vectors, and $r(H)=r(G)=k$.

Suppose $v \in V(G)$ satisfying $\mathrm{dist}_G(v,H)=2$. Then there exists an induced subgraph F of G such that

$$A(F)=\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & x_1 & x_2 & \dots & x_k \\ 0 & x_1 & & & & \\ 0 & x_2 & & & & \\ \vdots & \vdots & & & A(H) & \\ 0 & x_k & & & & \end{bmatrix},$$

where $x_i \in \{0,1\}, i=1,2,\cdots,k$. Obviously, $r(F)=r(H)+2=k+2$, this contradicts to $r(G)=k$. □

Let H be an induced subgraph of G. For a vertex subset U of $V(H)$, denote by S_U^H (Abbreviated as S_U) the set
$\{x \in V(G) \setminus V(H) \mid N_G(x) \cap V(H)=U\}$.

3. Main Result

Let G be a graph with n vertices, $v_1,v_2,\cdots,v_n$ be ordered vertices of G. $u \in V(G)$, n-dimensional column vector $\alpha_u=(x_1,x_2,\cdots,x_n)^\top$ is called adjacency vector of u, where $x_i=1$ if u is adjacent to v_i, and $x_i=0$ otherwise.

For obtaining all connected basic graphs with rank r, we have two steps.

Step 1. Find out all graphs with rank r which have exactly r vertices. Denote them by $G_1,G_2,\cdots,G_s$.

Step 2. Find out all connected graphs with rank r which have more than r vertices. Let G be a graph with rank r. By Lemma 2.3, we know that G contains an induced subgraph G_i $(i \in \{1,2,\cdots,s\})$ with rank r and
$\mathrm{dist}_G(u,G_i) \le 1$ for each vertices u of G. Therefore, we consider the adjacent relation between u and the vertices of G_i. Let

$$B=\begin{bmatrix} A(G_i) & \alpha \\ \alpha^\top & 0 \end{bmatrix},$$

satisfying

$$r(B)=r(A(G_i))=r, \qquad (*)$$

where $A(G_i)$ is adjacency matrix of G_i, $\alpha=(x_1,x_2,\cdots,x_r)$, $x_i \in \{0,1\}$ $(i=1,2,\cdots,r)$ is a $|V(G_i)|$-dimensional column vector. We calculate all vectors α satisfying condition $(*)$ by MATLAB.

Obviously, $\alpha=(0,0,\cdots,0)^\top$ and adjacency vectors of any vertex v in G_i satisfy $(*)$; this implies that u is not adjacent to G_i or $u \in S^{G_i}_{N_{G_i}(v)}$. This is not the connected basic graphs that we need to find. Therefore, these $r+1$ vectors are called trivial vectors and the rest of the vectors (if it is exist) are non- trivial vectors. If there exist non-trivial vectors $\alpha_1,\alpha_2,\cdots,\alpha_t$ such that $(*)$ holds, then for any vector α_j $(j=1,2,\cdots,t)$, we can obtain a basic graph G_{ij} on $r+1$ vertices; its adjacency matrix is

$$B = \begin{bmatrix} A(G_i) & \alpha_j \\ \alpha_j^{\top} & 0 \end{bmatrix},$$

$$r(G_{ij}) = r(G_i) = r$$

(In fact, suppose G_{ij} is not a basic graph. Then it is obtained from some graph $H(\ncong G_{ij})$ by multiplication of vertices. Thus there are two vertices v_s and v_t which are not adjacent in G_{ij}; the adjacent relation between v_s and any vertex of G_{ij} and the adjacent relation between v_t and any vertex of G_{ij} are the same. Since α_j is non-trivial vector, we have $u \notin \{v_s, v_t\}$. Hence the adjacent relation between v_s and any vertex of $G_{ij} \setminus u$ and the adjacent relation between v_t and any vertex of $G_{ij} \setminus u$ are the same. (where $G_{ij} \setminus u = G_i$ is the graph obtained from G_{ij} by removing the vertex u and all edges associated with u). Note $G_{ij} \setminus u = G_i$, we have $r(G_i) < r$, a contradiction.)

Repeat the above process for G_{ij}, it will obtain a family of basic graphs. Continue to repeat the above process for these basic graphs until every basic graph does not produce non-trivial vectors. We can find out all basic graphs with rank r. Now we give two examples.

Example 3.1. Let G be a connected graph and $r(G) = 2$, then $G \in \mathcal{M}(K_2)$.

In fact, K_2 is a unique graph [7] with rank 2 which have exactly two xertice. Calculating by MATLAB, have three and only three vectors $\alpha = (x_1, x_2)^{\top} = (0,0)^{\top}, (1,0)^{\top}$ and $(1,0)^{\top}$ satisfying that the rank of matrix B is 2, and they are trivial.

$$B = \begin{bmatrix} 0 & 1 & x_1 \\ 1 & 0 & x_2 \\ x_1 & x_2 & 0 \end{bmatrix}$$

Hence, K_2 is unique basic graph with rank 2, thus $G \in \mathcal{M}(K_2)$.

Example 3.2. Let G be a connected graph and $r(G) = 3$, then $G \in \mathcal{M}(K_3)$.

In fact, K_3 is a unique graph [7] with rank 3 which have exactly three vertices. Calculating by MATLAB, have four and only four vectors $\alpha = (x_1, x_2, x_3)^{\top} = (0,0,0)^{\top}, (1,1,0)^{\top}, (1,0,1)^{\top}$ and $(1,1,0)^{\top}$ satisfying that the rank of matrix B is 3, and they are trivial.

$$B = \begin{bmatrix} 0 & 1 & 1 & x_1 \\ 1 & 0 & 1 & x_2 \\ 1 & 1 & 0 & x_3 \\ x_1 & x_2 & x_3 & 0 \end{bmatrix}$$

Hence, K_3 is unique basic graph with rank 3, thus $G \in \mathcal{M}(K_3)$.

The paper [9] has given all basic graphs with rank 4 (see **Figure 1**). It is easy to obtain these graphs with our method. We write the following theorem without proof.

Theorem 3.1. [9] *Let G be a graph. Then $r(G) = 4$ if and only if G can be obtained from one of the graphs shown in* **Figure 1** *by multiplication of vertices.*

Theorem 3.2. [7] *Suppose that G is a graph on 5 vertices. Then $r(G) = 5$ if and only if G is one of the graphs shown in* **Figure 2**.

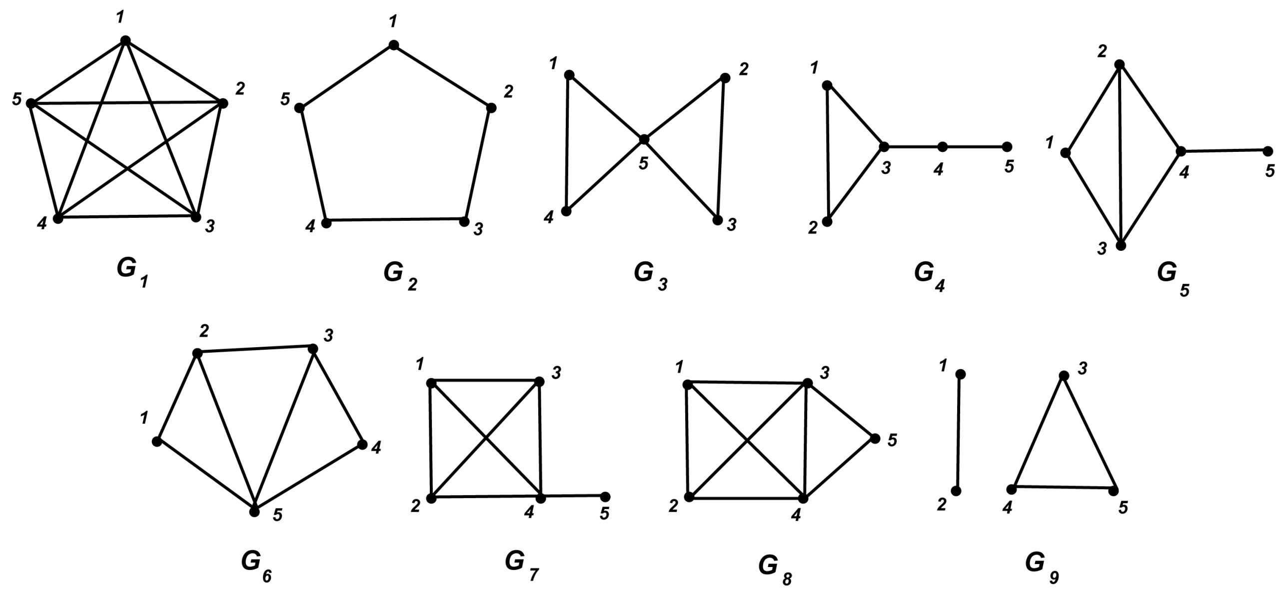

Figure 1. Basic graph with rank 4.

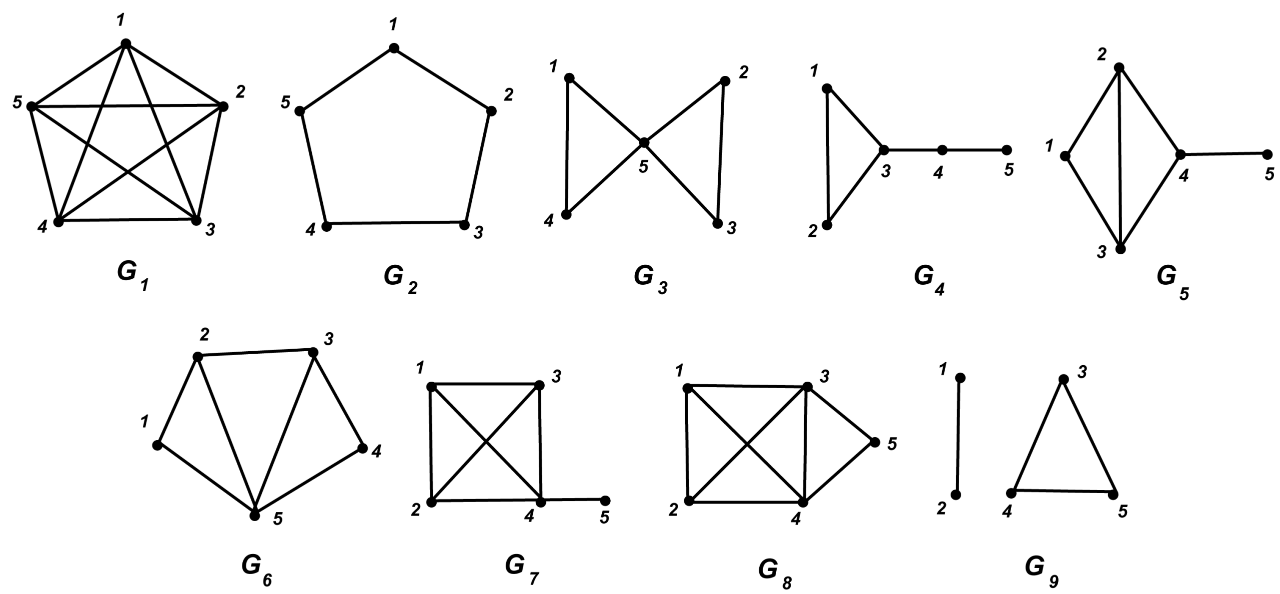

Figure 2. Graphs have exactly five vertices with rank 5.

Theorem 3.3. *Let* G *be a graph without isolated vertices. Then* $r(G)=5$ *if and only if* G *can be obtained from one of the graphs shown in* Figure 2 *and* Figure 3 *by multiplication of vertices.*

Proof. We now prove the necessary part. Assume that G is not connected, then $G=H_1\cup H_2$ and $r(H_1)=2$, $r(H_2)=3$, where H_1 and H_2 are two graphs. By the example 1 and example 2, we have $G\in\mathcal{M}(K_2\cup K_3)$. Now assume that G is connected. By Lemma 2.3, there exist induced subgraphs $H=G_i\ (i=1,2,\cdots,9)$ of G (see Figure 2) such that $\operatorname{dist}_G(u,H)\le 1$ for each vertex u of G. According to the differences of induced subgraphs that G contains, we consider the following Case 1-Case 5.

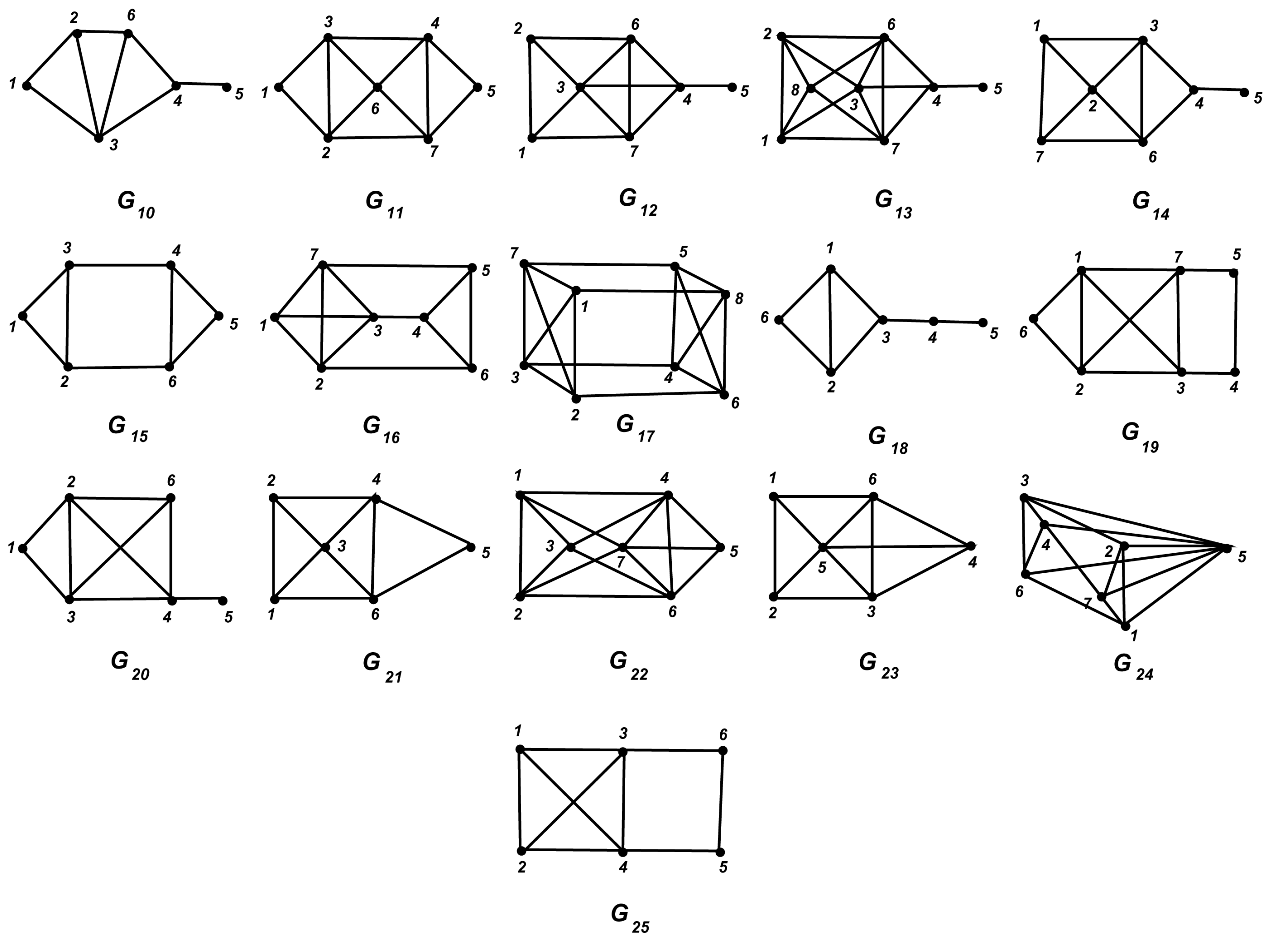

Figure 3. The basic graphs $G_i\left(i=10,11,\cdots,25\right)$ have more than five vertices with rank 5.

Case 1. G contains an induced subgraph $G_1=K_5$, $V\left(G_1\right)=\{1,2,3,4,5\}$,

$$A\left(G_1\right)=\begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{bmatrix}.$$

The following we first determine basic graph contain G_1. Let

$$B=\begin{bmatrix}A\left(G_1\right)&\alpha^{\top}\\\alpha&0\end{bmatrix}$$

where $\alpha=\left(x_1,x_2,x_3,x_4,x_5\right)$, $x_i\in\{0,1\}$ $\left(i=1,2,\cdots,5\right)$. For α satisfying $r\left(B\right)=r\left(A\left(G_1\right)\right)=5$ (or $\det\left(B\right)=0$), calculating by MATLAB, we obtain

$$\alpha=\left(0,0,0,0,0\right),\left(0,1,1,1,1\right),\left(1,0,1,1,1\right),\left(1,1,0,1,1\right),\left(1,1,1,0,1\right),\text{ or }\left(1,1,1,1,0\right).$$

This implies that not exist non-trivial vectors such that $r\left(B\right)=5$, hence G_1 is unique basic graph contain G_1 with rank 5, then $G\in\mathcal{M}\left(G_1\right)$.

Case 2. G contains an induced subgraph $G_2=C_5$, $V\left(G_2\right)=\{1,2,3,4,5\}$. Similar with Case 1, we know $G\in\mathcal{M}\left(G_2\right)$.

Case 3. G contains an induced subgraph G_3, $V(G_3)=\{1,2,3,4,5\}$. Similar with Case 1, we know $G\in\mathcal{M}(G_3)$.

Case 4. G contains an induced subgraph G_4, $V(G_4)=\{1,2,3,4,5\}$,

$$A(G_4)=\begin{bmatrix}0&1&1&0&0\\1&0&1&0&0\\1&1&0&1&0\\0&0&1&0&1\\0&0&0&1&0\end{bmatrix}.$$

First considering basic graph contain G_4, let

$$B=\begin{bmatrix}A(G_4)&\alpha^{\top}\\\alpha&0\end{bmatrix}$$

where $\alpha=(x_1,x_2,x_3,x_4,x_5)$, $x_i\in\{0,1\}$ $(i=1,2,\cdots,5)$. For α satisfying $r(B)=5$, calculating by MATLAB, we obtain

$$\alpha=(0,0,0,0,0),(0,1,1,0,0),(1,0,1,0,0),(1,1,0,1,0),(0,0,1,0,1),(0,0,0,1,0),$$
$$(1,0,1,1,0),(0,1,1,1,0),(1,0,0,1,1),(0,1,0,1,1),(1,1,0,0,0).$$

we know the first six vectors is trivial.

Case 4.1. For non-trivial vector $\alpha=(0,1,1,1,0)$, (or $(1,0,1,1,0)$), then there exists a graph G_{10} (the adjacency matrix of G_{10} is B), it is a basic graph contain G_4 with rank 5. $V(G_{10})=\{1,2,3,4,5,6\}$, Same as above, calculating by MATLAB for G_{10}, we obtain 3 non-trivial vectors. $\alpha=(0,1,0,1,1,1),(1,0,1,1,0,1)$, (or $(1,1,0,0,0,1)$).

Case 4.1.1. For non-trivial vector $\alpha=(0,1,0,1,1,1)$, then there exists a graph G_{11} is a basic graph contain G_{10} with rank 5. Same as above, calculating by MATLAB for G_{11}, we obtain not exist non-trivial vectors. Hence G_{11} is a unique basic graph contain G_{11} with rank 5.

Case 4.1.2. For non-trivial vector $\alpha=(1,0,1,1,0,1)$, then there exists a graph G_{12} is a basic graph contain G_{10} with rank 5. Same as above, calculating by MATLAB for G_{12}, we obtain not exist non-trivial vectors $(1,1,0,0,0,1,1)$, the resulting produce a graph G_{13} is a basic graph contain G_{12} with rank 5. Calculating by MATLAB for G_{13}, we obtain not exist non-trivial vectors, Hence G_{13} is a unique basic graph contain G_{13} with rank 5.

Case 4.1.3. For non-trivial vector $\alpha=(1,1,0,0,0,1)$, then there exists a graph G_{14} is a basic graph contain G_{10} with rank 5. Calculating by MATLAB for G_{14}, we obtain exist a non-trivial vectors $\alpha=(1,0,1,1,0,1,1)$. The resulting produce a graph G_{13} is a basic graph contain G_{14} with rank 5. Similar with Case 4.1.2, G_{13} is a unique basic graph contain G_{13} with rank 5.

Case 4.2. For non-trivial vector $\alpha=(0,1,0,1,1)$,(or $(1,0,0,1,1)$), then there exists a graph G_{15} is a basic graph contain G_4 with rank 5. $V(G_{15})=1,2,3,4,5,6$, Same as above, calculating by MATLAB for G_{15}, we obtain 3 non-trivial vectors. $\alpha=(1,1,1,0,1,0),(1,0,0,1,1,1)$, (or $(0,1,1,1,0,1)$).

Case 4.2.1. For non-trivial vector $\alpha=(1,1,1,0,1,0)$, (or $(1,0,0,1,1,1)$), then there exists a graph G_{16} is a basic graph contain G_{15} with rank 5. Same as above, calculating by MATLAB for G_{16} exist a non-trivial vectors $(1,0,0,1,0,1,1)$, the resulting produce a graph G_{17} is a basic graph contain G_{16} with rank 5. Calculating with MATLAB for G_{17}, we obtain not exist non-trivial vectors. Hence G_{17} is a unique basic graph contain G_{17} with rank 5.

Case 4.2.2. For non-trivial vector $\alpha=(0,1,1,1,0,1)$, then there exists a graph G_{11} is a basic graph contain G_{15} with rank 5. Similar with Case 4.1.1, G_{11} is a unique basic graph contain G_{11} with rank 5.

Case 4.3. For non-trivial vector $\alpha=(1,1,0,0,0)$, there exists a graph G_{18} is a basic graph contain G_4 with rank 5. Same as above, calculating by MATLAB for G_{18}, we obtain three non-trivial vectors $\alpha=(1,1,1,0,1,0),(0,1,1,1,0,1)$, (or $(1,0,1,1,0,1)$).

Case 4.3.1. For non-trivial vector $\alpha=(1,1,1,0,1,0)$, there exists a graph G_{19} is a basic graph contain G_{18} with rank 5. Same as above, calculating by MATLAB for G_{19}, we obtain not exist non-trivial vectors. Hence G_{19} is a unique basic graph contain G_{19} with rank 5.

Case 4.3.2. For non-trivial vector $\alpha=(0,1,1,1,0,1)$,(or $\alpha=(1,0,1,1,0,1)$), there exists a graph G_{14} is a basic graph included G_{18} with rank 5. Similar with Case 4.1.3, we obtain G_{13} and G_{14} it is only one basic graph contain G_{14} with rank 5.

In a word, basic graph contain G_4 with rank 5 are G_4, $G_i\ (i=10,11,\cdots,19)$. Let G be a graph contain G_4 with rank 5, then it must be a multiplication of vertices graph of one of G_4, $G_i\ (i=10,11,\cdots,19)$, thus
$G\in\mathcal{M}(G_i)(i=4,10,11,\cdots,19)$.

Case 5. G contains an induced subgraph which is $G_i\ (i=5,6,7,8)$, similar with Case 4, we first find basic graphs contain G_i with rank 5. The result and logic levels below in Figure 4 and process is omitted.

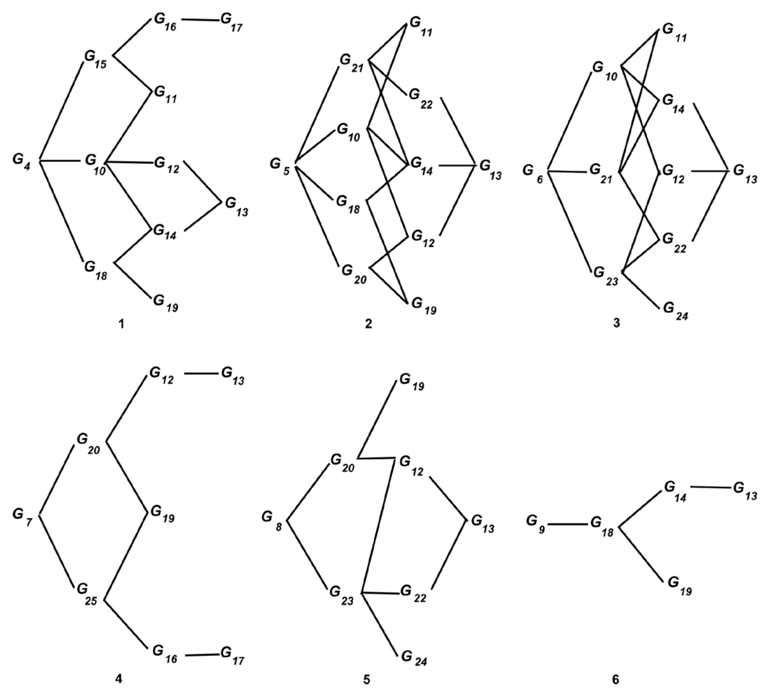

Figure 4. The level indicate figure of basic graphs contain $G_i\ (i=4,5,\cdots,9)$ with rank 5.

Summarize the previous cases, we can obtain $G \in \mathcal{M}(G_i)(i = 1, 2, \cdots, 25)$.

Sufficiency is obvious by the proof process of the necessity. The proof is completed.

By Examples 3.1, 3.2 and Theorems 3.1-3.3, we immediately get the following Theorem

Theorem 3.4. *Let* G *be a graph, then* $r(G) \le 5$ *if and only if* $G \in \mathcal{M}(H)$, *where* H *is an induced subgraph of* $G_1, G_2, G_3, G_{11}, G_{13}, G_{17}, G_{19}$ *and* G_{24} (see Figure 2 and Figure 3).

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